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Threefolds with Kodaira Dimension 0 or 3

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Summary. – Using the theory of adjoints and pluricanonical adjoints, we construct three nonsingular threefolds, as desingularizations of degree six hypersurfaces in $\mathbb{P}^4$, having the irregularities $q_1 = q_2 = 0$ and the following periodical sequences of plurigenera respectively

$$(p_y, P_2, P_3, \ldots, P_m, \ldots) = (0, 0, 1, 0, 0, 1, \ldots), (0, 0, 0, 1, 0, 0, 1, \ldots), (0, 0, 0, 1, 0, 0, 0, 1, \ldots).$$

In the Appendix, starting from the second above-mentioned example, we construct a threefold of general type with $q_1 = q_2 = 0$, $p_y = 1$, $P_2 = 2$ whose $m$-canonical transformation is birational if and only if $m \geq 11$.

Introduction.

L. Godeaux constructed nonsingular, algebraic threefolds $Y_2, Y_3, Y_5$ such that $2K_{Y_i} \equiv 0$ and $K_{Y_2} \not\equiv 0, 3K_{Y_3} \equiv 0$ and $K_{Y_3} \not\equiv 0, 5K_{Y_5} \equiv 0$ and $K_{Y_5} \not\equiv 0$, where $K_{Y_i}$ is a canonical divisor on $Y_i$, $i = 2, 3, 5$, and “$\equiv$” denotes linear equivalence (cf. [G1, G2, G3]).

By the Riemann-Roch theorem, it is not difficult to see that the first irregularity is $q_1(Y_i) = \dim_k H^1(Y_i, \mathcal{O}_{Y_i}) > 0$ for the three examples, $i = 2, 3, 5$. The Kodaira dimension of these threefolds is zero. As for the $m$-genus $P_m(Y_i) = \dim_k H^0(Y_i, \mathcal{O}_{Y_i}(mK_{Y_i})), i = 2, 3, 5$, of the above threefolds, we find that
\[ P_{2j}(Y_2) = 1, \forall j, \text{ and } P_n(Y_2) = 0 \text{ for } n \neq 2j; \]
\[ P_{3j}(Y_3) = 1, \forall j, \text{ and } P_n(Y_3) = 0 \text{ for } n \neq 3j; \]
\[ P_{5j}(Y_5) = 1, \forall j, \text{ and } P_n(Y_5) = 0 \text{ for } n \neq 5j. \]

This prompts the question of whether nonsingular threefolds \( Z_2, Z_3, Z_5 \)
having the first irregularity \( q_1(Z_i) = 0, i = 2, 3, 5 \) and the above respective
plurigenera actually exist, i.e.

\[ P_{2j}(Z_2) = 1, \forall j, \text{ and } P_n(Z_2) = 0 \text{ for } n \neq 2j; \]
\[ P_{3j}(Z_3) = 1, \forall j, \text{ and } P_n(Z_3) = 0 \text{ for } n \neq 3j; \]
\[ P_{5j}(Z_5) = 1, \forall j, \text{ and } P_n(Z_5) = 0 \text{ for } n \neq 5j. \]

It should be noted that they again have Kodaira dimension zero.

Among many other constructions, in [U] K. Ueno presented a threefold \( Z_2 \)
with the above properties, as well as \( q_2(Z_2) = \dim_k H^2(Z_2, O_{Z_2}) = 0, \)
thus answering in this way to the above question relating to the existence of the first
threefold. Other threefolds \( Z_2, \) with the same properties as Ueno’s example,
were subsequently constructed by S. Chiaruttini and M. C. Ronconi (cf. [CR]),
and by M. C. Ronconi (cf. ICM-1998, Short Comm.).

In the present paper, we affirmatively answer the question of whether
threefolds \( Z_3 \) and \( Z_5 \) with the above properties and also having \( q_2(Z_i) = 0, i = 3, 5 \)
exist. In addition, we fill the gap between \( Z_3 \) and \( Z_5 \) by constructing a non-
singular threefold \( Z_4 \) having \( q_1(Z_4) = q_2(Z_4) = 0, P_{4j}(Z_4) = 1, \forall j, \) and \( P_n(Z_4) = 0 \)
for \( n \neq 4j, \) which is missing in the parallel constructions by Godeaux. Moreover,
to our knowledge, there are no examples in the literature of nonsingular
threefolds \( Y_4 \) with \( 4K_{Y_4} = 0 \) and \( 2K_{Y_4} \neq 0. \)

Concerning the above examples \( Z_3, Z_4 \) and \( Z_5, \) there remains the problem of
how to establish explicitly their structure in terms of the existence of Iitaka fibrations
on them.

The only explicit result, with regard to Iitaka fibrations, is the existence of a
net of elliptic curves on \( Z_3 \) (see section 7).

On the matter of the existence of Iitaka fibrations, M. C. Ronconi is studying
which properties of Enriques surfaces have analogues among the threefolds with
\( q_1 = q_2 = p_g = 0, P_{2i} = 1 \) and \( P_{2i+1} = 0 \) (cf. ICM-1998, Short Comm.).

In the construction of our threefolds \( Z_3, Z_4 \) and \( Z_5, \) we use the theory of
adjoints and pluricanonical adjoints developed in [S_1]. Said theory enables us to
construct the three nonsingular threefolds as desingularizations of degree six
hypersurfaces in \( \mathbb{P}^4 \) endowed with suitable singularities. We can apply said
theory because the singularities on the three hypersurfaces satisfy the hypo-
theses of [S_1], i.e. it must be possible to resolve the singularities on the
threefolds with local blow-ups along linear affine subspaces; moreover, the de-
gree six hypersurfaces in \( \mathbb{P}^4 \) must have singularities of codimension \( \geq 2 \) (that is
the hypersurfaces must be normal). In the construction, we have to solve two
problems: the first is to find suitable singularities such that we have \( p_q = P_2 = 0 \) and \( P_3 = 1, \ p_g = P_2 = P_3 = 0 \) and \( P_4 = 1, \ p_g = P_2 = P_3 = P_4 = 0 \) and \( P_5 = 1 \) respectively; the second is to prove that the above sequences of plurigenera are periodical (see sections 5, 12 and 18).

The ground field \( k \) is an algebraically closed field of characteristic zero, which we may assume to be the field of complex numbers.

The example \( Z_5 \) partially answers the following question on nonsingular threefolds: which is the minimum integer \( m_0 \) such that \( q_1 = q_2 = p_g = P_2 = P_3 = \cdots = P_{m_0} = 0 \Rightarrow P_m = 0, \forall m \)? That is to say, the example tells us that \( m_0 \geq 6 \). The solution to the above problem is still unknown; for instance, we do not know whether a threefold with \( q_1 = q_2 = p_g = 0 \) and \( P_{6i} = 1, \ P_n = 0 \) if \( n \neq 6i, \ i \geq 1 \) exists. All the examples we have constructed in this direction (either published or not) satisfy the implication \( q_1 = q_2 = p_g = P_2 = \cdots = P_5 = 0 \Rightarrow P_m = 0, \forall m \).

In the Appendix, with a construction similar to that of \( Z_4 \), but imposing only four of the five singularities imposed on the hypersurface in \( \mathbb{P}^4 \), we construct a nonsingular threefold \( X \) of general type having \( q_1(X) = q_2(X) = 0, \ p_g(X) = 1, \quad P_2(X) = 2 \), whose \( m \)-canonical transformation is birational if and only if \( m \geq 11 \). From a result provided by M. Chen [C], we know that a threefold, with the bi-genus \( P_2 \geq 2 \), has the \( m \)-canonical transformation which is birational for \( m \geq 16 \). As a consequence, the sharpest limitation, for the birationality of the \( m \)-canonical transformation for threefolds with \( P_2 = 2 \), is now between 11 and 16.

We note that the threefolds constructed here have no analogues among surfaces, in the sense that there are no regular nonsingular surfaces having one of the above sequences of plurigenera; in fact, according to Castelnuovo’s criterion of rationality, a regular nonsingular surface with the bi-genus \( P_2 = 0 \) is rational. Moreover, to the best of the author’s knowledge, no examples of threefolds with the same above birational invariants of \( Z_3, Z_4, Z_5 \) are available in the literature.

This paper is organized as follows: we construct \( Z_3 \) in sections 1-7, \( Z_4 \) in sections 8-13 and \( Z_5 \) in sections 14-19.

Construction of the first threefold \( Z_3 \).

1. – Imposing singularities on a degree six hypersurface \( V \) in \( \mathbb{P}^4 \).

Let \((x_0, x_1, x_2, x_3, x_4)\) be homogeneous coordinates in \( \mathbb{P}^4 \) and let us indicate as \( f_6(X_0, X_1, X_2, X_3, X_4) \) a form (homogeneous polynomial) of degree 6, in the variables \( X_0, X_1, X_2, X_3, X_4 \), defining a hypersurface \( V \subset \mathbb{P}^4 \) of degree six. We impose a triple point on \( V \) at each of the three vertices \( A_0 = (1, 0, 0, 0, 0), \ A_1 = (0, 1, 0, 0, 0) \) and \( A_3 = (0, 0, 1, 0, 0) \), and an ordinary 4-ple (quadruple) point at each of the remaining
two vertices $A_2 = (0, 0, 1, 0, 0, 0), A_4 = (0, 0, 0, 0, 1)$ of the fundamental tetrahedron $X_0X_1X_2X_3X_4 = 0$.

The equation for $V$, with the imposed singularities, is of the following type

$$V : f_0(X_0, X_1, X_2, X_3, X_4)$$

$$= X_0^3(a_{33000}X_1^3 + ...) + X_1^3(a_{23100}X_0^2X_2 + ...) + X_2^3(...) + X_3^3(...) + X_4^3(...)$$

$$+ a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + ... + a_{00022}X_0X_1X_2^2X_3^2X_4^2 = 0,$$

where $a_{ijkhl} \in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^lX_4^m$.

We impose an infinitely near triple line $r_i$ at the point $A_i$, $i = 0, 1, 3, 4$, in the first neighbourhood. We follow the same method as in [S1], section 5, and impose the same triple line $r_0$ infinitely near $A_0$. To be more precise, let us consider the affine open set $U_0 \ni A_0$ in $\mathbb{P}^4$ given by $X_0 \neq 0$ of affine coordinates

$$\left( \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0} \right).$$

The affine equations of $V \cap U_0$ is given by $f_0(1, x, y, z, t) = 0$, where $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}, z = \frac{X_3}{X_0}, t = \frac{X_4}{X_0}$.

The affine coordinates of $A_0$ are $(0, 0, 0, 0)$. So, the blow-up of $\mathbb{P}^4$ at the point $A_0$ is locally given by the formulas:

$$B_{x_1} : \begin{cases} x = x_1 \\ y = x_1y_1 \\ z = x_1z_1 \\ t = x_1t_1 \end{cases} \quad B_{y_2} : \begin{cases} x = x_2y_2 \\ y = y_2 \\ z = y_2z_2 \\ t = y_2t_2 \end{cases} \quad B_{z_3} : \begin{cases} x = x_3z_3 \\ y = y_3z_3 \\ z = z_3 \\ t = z_3t_3 \end{cases} \quad B_{t_4} : \begin{cases} x = x_4t_4 \\ y = y_4t_4 \\ z = z_4t_4 \\ t = t_4 \end{cases}$$

and we consider $B_{t_4}$. The strict (or proper) transform $V'$ of $V$ with respect to the local blow-up $B_{t_4}$ has an affine equation given by

$$\frac{1}{t_4^3} f_0(1, x_4t_4, y_4t_4, z_4t_4, t_4) = a_{31200}x_4y_4^2 + ... + a_{00022}y_4^2z_4^2t_4^2 = 0.$$

On this threefold $V'$ we impose the triple line given affinely by

$$\begin{cases} x_4 = 0 \\ y_4 = 0 \quad (i.e. \quad t_4 = 0) \end{cases}$$

we make this line a locus of triple points on $V'$). Therefore, the conditions on the coefficients $a_{ijkhl}$, for $V$ to have the triple line $r_0$ infinitely near are the same as in [S1], p. 176, given by

$$a_{32010} = 0 \quad a_{30201} = 0 \quad a_{30003} = 0 \quad a_{20013} = 0$$
$$a_{32001} = 0 \quad a_{30120} = 0 \quad a_{21030} = 0 \quad a_{10023} = 0$$
$$a_{31110} = 0 \quad a_{30111} = 0 \quad a_{20130} = 0 \quad a_{12021} = 0$$
$$a_{31101} = 0 \quad a_{30102} = 0 \quad a_{20031} = 0 \quad a_{21012} = 0$$
$$a_{31020} = 0 \quad a_{30030} = 0 \quad a_{10032} = 0 \quad a_{20121} = 0$$
$$a_{31010} = 0 \quad a_{30021} = 0 \quad a_{21003} = 0 \quad a_{20112} = 0$$
$$a_{31002} = 0 \quad a_{30012} = 0 \quad a_{21013} = 0 \quad a_{20022} = 0$$
$$a_{30210} = 0$$
To tell the truth, some of these coefficients are already zero, having imposed two 4-ple points on $V$; they are the coefficients of the type $a_{ijkhl}$ or $a_{ijklh}$. Nevertheless, we have written said coefficients here to ensure that we have correct results in the following rotations of indices and variables.

Again according to $[S_1]$, we impose an infinitely near triple line $r_i$ at $A_i$, for $i=1,3$, with suitable rotations of the indices $ijkhl$ of the coefficient $a_{ijkhl}$ and of the corresponding variables $a_{ijklh}X_i^0X_j^1X_k^2X_h^3X_i^4$.

The rotations of the indices and variables passing from $A_0$ to $A_1$ and from $A_1$ to $A_3$, are as follows.

\[ \begin{align*}
\text{Rotations of indices (and variables)} \\
A_0 & \mapsto A_1 \mapsto A_3 \\
ijklh & \mapsto ijkhl
\end{align*} \]

We give the final equation for our hypersurface $V$, after imposing all the above-mentioned singularities.

\[
V : f_6(X_0, X_1, X_2, X_3, X_4) = \\
a_{3120}X_0^3X_1X_2^2 + \\
a_{6210}X_1^3X_2^2X_3 + \\
a_{10203}X_0X_2X_3^3 + \\
a_{22200}X_0^2X_1X_2^2X_3 + a_{221110}X_0^2X_1X_2^2X_3 + a_{22020}X_0^2X_1^2X_2^2X_3 + a_{212110}X_0^2X_1X_2^2X_3 + \\
a_{212120}X_0^2X_1X_2^3X_4 + a_{211211}X_0^2X_1X_2^2X_3X_4 + a_{210202}X_0^2X_1^2X_2^2X_3 + a_{210210}X_0^2X_1X_2^2X_3 + \\
a_{20202}X_0^2X_1^2X_2^2X_3 + a_{202110}X_0^2X_1^2X_2X_3 + a_{20202}X_0^2X_1^2X_2^2X_3 + a_{122110}X_0^2X_1^2X_2X_3 + \\
a_{122120}X_0^2X_1X_2X_3^3 + a_{121120}X_0X_1^2X_2^2X_3 + a_{111111}X_0^2X_1X_2X_3X_4 + a_{111112}X_0^2X_1X_2X_3X_4 + \\
a_{101212}X_0^2X_1^2X_2^2X_3 + a_{101212}X_0X_1^2X_2^2X_3 + a_{100220}X_0X_1^2X_2^2X_3 + a_{100220}X_0X_1^2X_2^2X_3 + \\
a_{022111}X_0^2X_1X_2X_3X_4 + a_{022020}X_0^2X_1X_2X_3X_4 + a_{022112}X_0^2X_1X_2X_3X_4 + a_{022112}X_0^2X_1X_2X_3X_4 + \\
a_{011212}X_0^2X_1X_2X_3X_4 + a_{002220}X_0X_1X_2X_3X_4 = 0.
\]

Several coefficients can be chosen as equal to zero because they are inessential for the computation of the birational invariants of a desingularization $Z_3$ of $V$. The shortest equation with the essential coefficients is

\[
V : f_6(X_0, X_1, X_2, X_3, X_4) = \\
a_{3120}X_0^3X_1X_2^2 + \\
a_{6210}X_1^3X_2^2X_3 + \\
a_{10203}X_0X_2X_3^3 + \\
a_{22200}X_0^2X_1X_2^2X_3 + a_{221110}X_0^2X_1X_2^2X_3 + a_{22020}X_0^2X_1^2X_2^2X_3 + a_{212110}X_0^2X_1X_2^2X_3 + \\
a_{212120}X_0^2X_1X_2^3X_4 + a_{211211}X_0^2X_1X_2^2X_3X_4 + a_{210202}X_0^2X_1^2X_2^2X_3 + a_{210210}X_0^2X_1X_2^2X_3 + \\
a_{20202}X_0^2X_1^2X_2^2X_3 + a_{202110}X_0^2X_1^2X_2X_3 + a_{20202}X_0^2X_1^2X_2^2X_3 + a_{122110}X_0^2X_1^2X_2X_3 + \\
a_{122120}X_0^2X_1X_2X_3^3 + a_{121120}X_0X_1^2X_2^2X_3 + a_{111111}X_0^2X_1X_2X_3X_4 + a_{111112}X_0^2X_1X_2X_3X_4 + \\
a_{101212}X_0^2X_1^2X_2^2X_3 + a_{101212}X_0X_1^2X_2^2X_3 + a_{100220}X_0X_1^2X_2^2X_3 + a_{100220}X_0X_1^2X_2^2X_3 + \\
a_{022111}X_0^2X_1X_2X_3X_4 + a_{022020}X_0^2X_1X_2X_3X_4 + a_{022112}X_0^2X_1X_2X_3X_4 + a_{022112}X_0^2X_1X_2X_3X_4 + \\
a_{011212}X_0^2X_1X_2X_3X_4 + a_{002220}X_0X_1X_2X_3X_4 = 0.
\]
From here on, \( V \) denotes this last hypersurface defined by the above last form \( f_0(X_0, X_1, X_2, X_3, X_4) \) for a generic choice of the parameters \( a_{ijkl} \).

2. – Imposed and unimposed singularities of \( V \): the actual singularities.

We consider the hypersurface \( V \) at the end of section 1.

New singularities appear on the generic \( V \) close to the singularities imposed on \( V \); they are actual or infinitely near singularities. We call a singularity on \( V \) actual to distinguish it from those infinitely near.

There are seven actual double (straight) lines on \( V \) given by \( A_0A_1, A_0A_3, A_0A_4, A_1A_3, A_1A_4, A_2A_4 \) and \( A_2A_4 \), according to the following picture, where the double lines are drawn in bold type.

![Diagram of singularities](image)

The generic \( V \) has no other actual singularities. It follows that the generic \( V \) is reduced, irreducible and normal.

3. – The infinitely near singularities of \( V \).

In section 2, we described the actual singularities on \( V \); in the present section, we describe the infinitely near singularities (whether they are imposed or not). To do so, we need the

**Resolution of singularities of \( V \)**

The desingularization of \( V \) is very long, but also very easy. In section 1, we imposed a triple line infinitely near the triple point \( A_0 \). Said computation can be interpreted as the beginning of a desingularization of \( A_0 \) and of the singularities infinitely near \( A_0 \).

As an example, we solve only the singularities of \( V \) belonging to the affine open set \( U_2 = \{ X_2 \neq 0 \} \).
Blow-up of $\mathbb{P}^4$ at $A_2$

Here, we can assume that the blow-up of $A_2$ is the first that we perform. So, let $\pi : \mathbb{P}_1 \to \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ at $A_2$ and let $U_2 \supset A_2$ be the affine open set in $\mathbb{P}^4$ given by $X_2 \neq 0$ of affine coordinates $\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_4}{x_2}\right)$.

We use $V_{U_2}$ to indicate the affine threefold $V \cap U_2$ of the affine equation $f_0(x, y, 1, z, t) = 0$, where $x = \frac{X_0}{X_2}, y = \frac{X_1}{X_2}, z = \frac{X_3}{X_2}, t = \frac{X_4}{X_2}$.

The point $A_2$ has affine coordinates $(0, 0, 0, 0)$.

The blow-up $\pi$ at the point $A_2$ is locally given by the same formulas as in section 1: $B_{x_1}, B_{y_2}, B_{z_3}, B_{t_4}$.

- **The strict (or proper) transform of $V_{U_2}$, with respect to $B_{x_1}$, is given by**

  $V_{x_1} : \frac{1}{x_1} f_0(x_1, x_1 y_1, 1, x_1 z_1, x_1 t_1)$

  $= a_{31200} y_1 + a_{03210} y_1^3 z_1 + a_{10230} z_1^3 + a_{22220} y_1^2 + \cdots + a_{20202} t_1^2 + \cdots = 0$.

We are interested in the singularities infinitely near $A_2$, i.e. we focus on the singularities on $V_{x_1}$ belonging to the exceptional divisor $E_1$ of the blow-up $\pi$.

Locally, an equation of $E_1$ is given by $x_1 = 0$.

The (incomplete) linear system defining $V_{x_1}$ has the base points on $x_1 = 0$ given by the unique point $O = (0, 0, 0, 0)$. According to Bertini, the singularities on $x_1 = 0$ of the (generic) $V_{x_1}$ are only on the base points of the linear system, i.e. only $O$ can be a singular point. But we have

$$\left(\frac{\partial V_{x_1}}{\partial y_1}\right)_O = a_{31200} \neq 0.$$ 

So, there are no singularities on $V_{x_1}$ infinitely near $A_2$.

The base points outside the exceptional divisor $x_1 = 0$ are given by the line

$$\begin{cases} y_1 = 0 \\ z_1 = 0 \\ t_1 = 0 \end{cases}.$$ 

According to Bertini, the possible singularities on $V_{x_1}$ belong to this line. But we have

$$\left(\frac{\partial V_{x_1}}{\partial y_1}\right)_{y_1 = z_1 = t_1 = 0} = a_{31200} \neq 0.$$ 

In conclusion, $V_{x_1}$ is nonsingular.

Likewise, we see that

- the strict transform of $V_{U_2}$ with respect to $B_{y_2}$ is again nonsingular;
- the strict transform of $V_{U_2}$ with respect to $B_{z_3}$ is also nonsingular.

Next, we find that
the strict transform of $V_{U_2}$ with respect to $B_t$ is given by

\[ V_{t_i} : \frac{1}{t_4^4} f_6(x_4 t_4, y_4 t_4, 1, z_4 t_4, t_4) \]

\[ = a_{312003} x_4^2 y_4 + a_{032100} y_4^3 z_4 + a_{102300} x_4^3 z_4 + a_{222000} x_4^2 y_4^2 + \cdots + a_{220200} x_4^2 + \cdots + a_{002222} z_4^2 = 0 \]

and it has the double singular line $\ell : \begin{cases} x_4 = 0 \\ y_4 = 0 \text{ as unique singularity.} \\ z_4 = 0 \end{cases}$

is the strict transform of the actual line $A_2 A_4 \cap U_2$ on $V_{U_2}$.

It is easy to check that, thanks to the presence in the equation of the addenda $a_{220200} x_4^2$, $a_{002222} z_4^2$, the line $\ell$ is resolved with only one blow-up.

By patching $V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4}$ together, from general blow-up theory, we obtain a threefold $V' \cap U'$ defined on $U' = \pi^{-1}(U_2)$, where $V'$ is the strict transform of $V$ with respect to the blow-up $\pi : \mathbb{P}_1 \to \mathbb{P}^4$ of $\mathbb{P}^4$ at $A_2$. We can cover $V' \cap U'$ with $V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4}$. Thus, locally, we can blow up $\ell$ on $V_{t_4}$.

**CONCLUSION.** We have resolved the singularities of $V_{U_2} = V \cap U_2$.

The tree of the blow-ups is as follows

```
     V_{U_2}
     /      \
    /        \
 V_{x_1}   V_{y_2} V_{z_3} V_{t_4}
      \     /    \
       /   ns  ns  ns
      /    /    \
     V_{x_{41}} V_{y_{42}} V_{z_{43}}
           ns  ns  ns
```

where “ns” means “nonsingular”.

By doing long and tedious calculations similar to those above and in [S.,], we obtain a desingularization of $V \subset \mathbb{P}^4$. In this desingularization, we can see that new unimposed infinitely near singularities also appear on $V$ among the imposed infinitely near singularities. They are only double singular curves and isolated double points. So, none of the unimposed singularities affect the birational invariants of a desingularization $Z_3$ of $V$, such as the irregularities and the plur-
igenera of $Z_3$. This means that, in calculating these invariants, we can assume that there are only the imposed singularities on $V$.

Having said as much, we consider the desingularization of $V$ as achieved.

4. – The $m$-canonical adjoints to $V \subset \mathbb{P}^4$

Let

$$P_r \xrightarrow{\pi_r} \ldots \xrightarrow{\pi_3} P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 = \mathbb{P}^4$$

be a sequence of blow-ups solving the singularities of $V$.

If we call $V_i \subset P_i$ the strict transform of $V_{i-1}$ with respect to $\pi_i$, then we obtain from the above sequence

$$Z_3 = V_r \xrightarrow{\pi'_r} \ldots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_i = \pi_i|_{V_i} : V_i \rightarrow V_{i-1}$ and $\sigma_{Z_3} : Z_3 \rightarrow V$, $\sigma = \pi_r \circ \ldots \circ \pi_1$, is a desingularization of $V \subset \mathbb{P}^4$.

Let us assume that $\pi_i$ is a blow-up along a subvariety $Y_{i-1}$ of $\mathbb{P}_{i-1}$, of dimension $j_{i-1}$, which can be either a singular or a nonsingular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. $Y_{i-1}$ is a locus of singular or simple points of $V_{i-1}$). Let $m_{i-1}$ be the multiplicity of the variety $Y_{i-1}$ on $V_{i-1}$.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, \ldots, r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$ in $\mathbb{P}^4$ is an $m$-canonical adjoint to $V$ (with respect to the sequence of blow-ups $\pi_1, \ldots, \pi_r$) if the restriction to $Z_3$ of the divisor

$$D_m = \pi'_r \{ \pi'_{r-1} \ldots \pi'_1(\Phi_{m(d-5)}) - mn_0E_1 \ldots - mn_{r-2}E_{r-1} \} - mn_{r-1}E_r$$

is effective, i.e. $D_m|_{Z_3} \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of $\pi_i$ and $\pi'_i : Div(P_{i-1}) \rightarrow Div(P_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S1], sections 1,2).

An $m$-canonical adjoint $\Phi_{m(d-5)}$ is a global $m$-canonical adjoint to $V$ (with respect to $\pi_1, \ldots, \pi_r$) if the divisor $D_m$ is effective on $P_r$, i.e. $D_m \geq 0$ (loc. cit.).

Note that, if $\Phi_{m(d-5)}$ is an $m$-canonical adjoint to $V$, then $D_m|_{Z_3} \equiv mK$, where ‘$\equiv$’ denotes linear equivalence and $K$ denotes a canonical divisor on $Z_3$.

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that the blow-up $\pi_1$ is the blow-up at the 4-ple point $A_2$, $\pi_2$ is the blow-up at the 4-ple point $A_4$, $\pi_3$ is the blow-up at the triple point $A_0$, $\pi_4$ is the blow-up along the triple curve infinitely near $A_0$, $\pi_5$ is the blow-up at the triple point $A_1$, $\pi_6$ is the blow-up along the triple curve infinitely near $A_1$, $\pi_7$ is the blow-up at the triple point $A_3$, and $\pi_8$ is the blow-up along the triple curve infinitely near $A_3$.

The example $V$ has degree $d = 6$ and $D_m$ is given by:

\[ (*) \quad D_m = \pi'_{r-1} \{ \pi'_{r-2}(\Phi_m) - mE_1 \ldots - mE_{r-2} \} - mE_r \]
where $E_i$ is the exceptional divisor of the blow-up $\pi_i$ and, to be more specific, $E_1$ is the exceptional divisor of the blow-up $\pi_1$ at the 4-fold point $A_2$, ... and $E_8$ is the exceptional divisor of the blow-up $\pi_8$ along the triple curve infinitely near $A_3$.

No other exceptional divisors are subtracted in $D_m$ because, as we said, the unimposed singularities are either actual or infinitely near double singular curves or isolated double points on our (generic) $V$. Put more precisely, the exceptional divisors of the blow-ups along the curves appear with coefficient $n_h = 0$ in the above expression of $D_m$, whereas the exceptional divisors of the blow-ups at double points appear with coefficient $n_h = -1$: we have indicated these divisors as $\sum mE$. In addition, note that the exceptional divisor of a blow-up at a triple point also appears with coefficient $n_h = 0$ in $D_m$. From here on, we omit writing $\sum mE$, because they are not essential to the computation of the birational invariants that we shall consider.

5. – The plurigenera of a desingularization $Z_3$ of $V$.

Let $\sigma_{\underline{z_3}}: Z_3 \longrightarrow V$ be a desingularization of our hypersurface $V \subset P^4$, where $\sigma = \pi_r \circ \cdots \circ \pi_1$ (section 4).

**Proposition 1.** – The plurigenera of $Z_3$ are given by $P_{3i} = 1$, $\forall i \geq 1$, and $P_m = 0$ if $m \neq 3i$.

**Proof.** – Let us consider the equation of $V$: $f_6(X_0, X_1, X_2, X_3, X_4) = 0$ at the end of section 1, and we arrange the form $f_6$ according to the powers of $X_2$.

$$f_6 = \varphi(X_0, X_1, X_3, X_4)X_2^2 + \varphi_2(X_0, X_1, X_3, X_4)X_2 + \varphi_6(X_0, X_1, X_3, X_4) = 0,$$

where $\varphi_i(X_0, X_1, X_3, X_4)$ is a form of degree $i$ in $X_0, X_1, X_3, X_4$.

Next, let us consider the hypersurface $\Phi_m$, appearing in (*) section 4 and assume that its equation is $F_m(X_0, X_1, X_2, X_3, X_4) = 0$, of degree $m$. Arranging the form $F_m$ according to the powers of $X_2$, we can write

$$F_m(X_0, X_1, X_2, X_3, X_4) = \psi_s(X_0, X_1, X_3, X_4)X_2^{m-s} + \psi_{s+1}(X_0, X_1, X_3, X_4)X_2^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_3, X_4),$$

where $\psi_j(X_0, X_1, X_3, X_4)$ is a form of degree $j$ in $X_0, X_1, X_3, X_4$ and $s$ is an integer satisfying $0 \leq s \leq m$. So, $\Phi_m$ has at $A_2$ an $s$-ple point, with $0 \leq s \leq m$.

We need the preliminary result, concerning $\Phi_m$, given by the following

**Lemma 1.** – If $\Phi_m$ is an $m$-canonical adjoint (either global or not), then, modulo $V : f_6 = 0$, we can assume $s \geq m - 1$; i.e. if $\Phi_m$ is an $m$-canonical
adjoint, then we can assume that its equation, modulo \( f_0 = 0 \), is defined by the form

\[
F_m = \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).
\]

The idea for the proof is due to M. C. Ronconi [CR], [Ro].

**Proof of Lemma 1.** For \( m = 1 \), the lemma is trivial, so we assume \( m \geq 2 \). For the same reason, considering \( \Phi_m : \psi_x X_2^{m-s} + \psi_{s+1} X_2^{m-s-1} + \cdots + \psi_m \), we assume \( s \leq m - 2 \).

Let us consider the affine open set \( U_2 \) given by \( X_2 \neq 0 \) of affine coordinates \((x_0, x_1, x_2, x_3, x_4)\). Let us write the affine equation of \( V \cap U_2 = U_{U_2} \) as follows:

\[
U_{U_2} : f_0(x, y, 1, z, t) = \phi_4(x, y, z, t) + \phi_5(x, y, z, t) + \phi_6(x, y, z, t) = 0
\]

and the local equation of \( \Phi_m \):

\[
\Phi_{mU_2} : F_m(x, y, 1, z, t) = \psi_x(x, y, z, t) + \cdots + \psi_m(x, y, z, t) = 0,
\]

where \( x = \frac{X_0}{X_2}, y = \frac{X_1}{X_2}, z = \frac{X_3}{X_2}, t = \frac{X_4}{X_2} \).

Let us consider the first blow-up \( \pi_1 \) at \( A_2 \). One of the local expressions of \( \pi_1 \) is given by \( B_{z_3} : \begin{cases} x = x_3z_3 \\ y = y_3z_3 \\ z = z_3 \\ t = t_3z_3 \end{cases} \). The strict transform \( V_{z_3} \) of \( U_{U_2} \) with respect to \( B_{z_3} \) is given by

\[
V_{z_3} : 1\frac{f_0(x_3z_3, y_3z_3, 1, z_3, t_3)}{z_3} = f_{U_2}(x_3, y_3, z_3, t_3)
\]

\[
= \phi_4(x_3, y_3, 1, t_3) + \phi_5(x_3, y_3, 1, t_3)z_3 + \phi_6(x_3, y_3, 1, t_3)z_3^2 = 0
\]

Now, let us consider the total transform of \( \Phi_{mU_2} \) with respect to \( B_{z_3} \). We note that \( B_{z_3} \) locally coincides, up to isomorphisms, with its total transform with respect to the desingularization \( \pi_{z_3} \) (because \( V_{z_3} \) is nonsingular, see the tree of blow-ups in section 3). This total transform of \( \Phi_{mU_2} \) is

\[
\Phi_{mU_2}^* : z_3^s[\psi_x(x_3, y_3, 1, t_3) + \cdots + \psi_m(x_3, y_3, 1, t_3)z_3^{m-s}] \]

Next, if \( \Phi_m \) is an \( m \)-canonical adjoint to \( V \), then, from (*) and the definition of \( m \)-canonical adjoint to \( V \) (section 4), we can deduce in particular

\[
[\Phi_{mU_2} - m(1E_{1|V_{z_3}})]_{V_{z_3}} \geq 0,
\]

where \( E_1 \) is the exceptional divisor of \( \pi_1 \). This inequality, translated in terms of
polynomials, gives us
\[ z_3^s[\psi_s(x_3, y_3, 1, t_3) + \cdots + \psi_m(x_3, y_3, 1, t_3) z_3^{m-s}] + B(x_3, y_3, z_3, t_3)[\varphi_4(x_3, y_3, 1, t_3) + \varphi_5(x_3, y_3, 1, t_3) z_3 + \varphi_6(x_3, y_3, 1, t_3) z_3^2] = z_3^m (\cdots), \]
where \( B(x_3, y_3, z_3, t_3) \) is a suitable polynomial. In this equality of polynomials, we have \( B(x_3, y_3, z_3, t_3) = z_3^3 B'(x_3, y_3, z_3, t_3) \), so, we can simplify \( z_3^3 \) and we put \( z_3 = 0 \) in the remaining equality, obtaining the equality of polynomials
\[(*) \quad \psi_s(x_3, y_3, 1, t_3) = B'(x_3, y_3, z_3, 0) \varphi_4(x_3, y_3, 1, t_3). \]

Multiplying both sides of (*) by \( z_3^s \), from the equalities in \( B_{z_3} \), we obtain
\[ \psi_s(x, y, z, t) = B''(x, y, z, t) \varphi_4(x, y, z, t); \]
and by homegenizing, we return to the forms
\[(***) \quad \psi_s(X_0, X_1, X_3, X_4) = B''(X_0, X_1, X_3, X_4) \varphi_4(X_0, X_1, X_3, X_4), \]
where \( B''(X_0, X_1, X_3, X_4) \) is a form of degree \( s - 4 \) in \( X_0, X_1, X_3, X_4 \).

At this point, we consider \([S_1], \) Corollary 8, section 3: if \( V \) is normal, there is an isomorphism of projective spaces for any \( m \geq 1 \)
\[ \left( \begin{array}{c} \text{linear system of} \\ \text{m-canonical adjoints to V} \end{array} \right)_{\mid V} \longrightarrow |mK_{Z_3}| \]
\[ \Phi_{mV} \longrightarrow D_{mV}. \]

Bearing in mind that our purpose is to compute the \( m \)-canonical genus \( P_m = \dim |mK_{Z_3}| + 1 = \dim \left( \text{linear system of m-canonical adjoints} \right)_{\mid V} + 1 \), we can substitute \( \Phi_m \) with \( \Phi'_m \) if \( \Phi'_m \mid V = \Phi_m \mid V \). This is equivalent to saying that the form \( F'_m \) defining \( \Phi'_m \) must be of the type \( F'_m = F_m + A f_6 \), where \( A \) is a form of degree \( m - 6 \), \( F_m \) is the form defining \( \Phi_m \) and \( f_6 \) is the form defining \( V \). To be more precise, we can take \( F'_m \) given by
\[(*)' \quad F'_m = F_m - B''(X_0, X_1, X_3, X_4) X_2^{m-s-2} f_6, \]
where \( B''(X_0, X_1, X_3, X_4) \) is the form in (***)-1. Of course, we need \( m - s - 2 \geq 0 \) for this substitution. Note that we can assume \( m - s - 2 \geq 0 \); otherwise, if \( m - s - 2 < 0 \), then the Lemma is true. From (***) we obtain
\[ F'_m : \psi_{s+1}(X_0, X_1, X_3, X_4) X_2^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_3, X_4) \]
\[ - B''(X_0, X_1, X_3, X_4)(\varphi_5 X_2^{m-s-1} + \varphi_6 X_2^{m-s-2}) = 0. \]
\( \Phi'_m : F'_m = 0 \) has an \((s + 1)\)-ple point at \( A_2 \). So we can iterate the process and substitute \( \Phi'_m \) with a new \( m \)-canonical adjoint \( \Phi''_m \) having an \((s + 2)\)-ple point at
$A_2$, and so on. Since the inequality $m - s - 2 \geq 0$ in (**) must hold, the iteration stops when $s > m - 2$.

This proves Lemma 1.

**Proof of Proposition 1 (continuation) (see also [CR]).** In Lemma 1, we established that we can assume $\Phi_m : \psi_m - 1 X_2 + \psi_m = 0$. Starting from this $\Phi_m$, and using the same arguments as in the proof of Lemma 1, we obtain a formula similar to (**), i.e.

\[
\psi_m - 1(X_0, X_1, X_3, X_4) = B''(X_0, X_1, X_3, X_4)\psi_4(X_0, X_1, X_3, X_4),
\]

where $B''(X_0, X_1, X_3, X_4)$ is a form of degree $m - 5$. In particular, the equality (**) tells us that $\psi_m - 1(X_0, X_1, X_3, X_4)$ can be divided by $\varphi_4(...)$.

Next, we order the forms $f_0$ and $\Phi_m$ according to the powers of $X_4$,

\[
f_0 = \varphi_0^2(X_0, X_1, X_2, X_3)X_4^2 + \varphi_0^2(X_0, X_1, X_2, X_3)X_4 + \varphi_0^2(X_0, X_1, X_2, X_3),
\]

where $\varphi_0^2(X_0, X_1, X_3, X_4)$ is a form of degree $i$ in $X_0, X_1, X_2, X_3$.

$\Phi_m : \psi_j(X_0, X_1, X_2, X_3)X_4^{m-i} + \psi_{i+1}(X_0, X_1, X_2, X_3)X_4^{m-i-1} + \cdots + \psi_m(X_0, X_1, X_2, X_3),
\]

where $\psi_j(X_0, X_1, X_2, X_3)$ is a form of degree $j$ in $X_0, X_1, X_2, X_3$.

Then we apply Lemma 1 to the $m$-canonical adjoint $\Phi_m$ and we consider its behaviour under the blow-up $\pi_2$ at the point $A_4$. In this case, if we change $X_2$ with $X_4$, the result of Lemma 1 or, to be more precise, the analogous equality of (**), tells us that the form $\psi_j(X_0, X_1, X_2, X_3)$ must be divisible by $\varphi_4(X_0, X_1, X_2, X_3)$. But $\varphi_4(X_0, X_1, X_2, X_3)$ contains $X_2^2$, whereas $\Phi_m : \psi_m - 1 X_2 + \psi_m = 0$ contains $X_2$ to the power of 1. Iterating the process, this implies that $t = m$. Comparing

\[
\Phi_m : B''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0
\]

and $\Phi_m : \psi'_m(X_0, X_1, X_2, X_3) = 0$ [since $\varphi_4(X_0, X_1, X_3, X_4)$ contains $X_4$], we obtain

\[
\Phi_m : \psi''_m(X_0, X_1, X_3) + Af_0 = 0,
\]

where $\psi''_m(X_0, X_1, X_3)$ is a form of degree $m$ in the variables $X_0, X_1, X_3$.

We continue the proof considering the blow-up at $A_0$ performed at the beginning of section 1 and let $U_0 \ni A_0$ be the affine open set in $\mathbb{P}^4$ given by $X_0 \neq 0$.

We consider the local blow-up of $\mathbb{P}^4$ at the point $A_0$ given by $B_{z_3} : \begin{cases} x = x_3 z_3 \\ y = y_3 z_3 \\ z = z_3 \\ t = z_3 t_3 \end{cases}$.

For the sake of brevity, we consider this blow-up as the first in solving the singularities of $V_1$; we can do so because the blow-ups $\pi_i$ and $\pi_j$ are interchangeable. The total transform of $\Phi_m$ with respect to $B_{z_3}^*$ is given by $B_{z_3}^*(\Phi_m) : \psi''_m(1, x_3 z_3, z_3) = 0$. Now we consider the affine triple line infinitely
near $A_0$, given by \[
\begin{align*}
x_3 &= 0 \\
y_3 &= 0 \\
z_3 &= 0
\end{align*}
\] and we consider the local blow-up given by \[
\begin{align*}
x_3 &= x_{31} \\
y_3 &= x_{31}y_{31} \\
z_3 &= x_{31}z_{31} \\
t_3 &= t_{31}
\end{align*}
\] The total transform of $B_{z_3}^\ast (\Phi_m)$ with respect to $B_{x_3}$ is given by \[
B_{x_3}^\ast (B_{z_3}^\ast (\Phi_m)) : \psi''_m (1, x_{31}^2 z_{31}, x_{31} z_{31}) = 0.
\]

Next, we write the form $\psi''_m (X_0, X_1, X_3)$ as follows \[
\psi''_m (X_0, X_1, X_3) = \sum_{i+j+k=m} c_{ijk} X_0^i X_1^j X_3^k.
\]
where $c_{ijk} \in k$. We thus obtain the total transform \[
B_{x_3}^\ast (B_{z_3}^\ast (\Phi_m)) : \sum_{i+j+k=m} c_{ijk} x_{31}^{2j+k} z_{31}^{j+k} = 0.
\]

If we want $\Phi_m$ to be an $m$-canonical adjoint, from the expression of $D_m$ in (*), section 4, we must put $x_{31}^m$ in evidence, modulo $V : f_6 = 0$, in the latter total transform. But, considering $V_{U_0}$ given by \[
f_{6U_0} (x, y, z, t) = a_{31200} xy^2 + \phi''_4 (x, y, z, t) + \phi''_0 (x, y, z, t) + \phi''_6 (x, y, z, t) = 0,
\]
where $\phi''_i (x, y, z, t)$ is a form of degree $i$ in $x, y, z, t$, we obtain that the strict trasform of $f_{6U_0}$ with respect to $B_{z_3}$ and $B_{x_3}$ is \[
a_{31200} y_{31}^2 + \phi''_4 (\cdots) x_{31} + \phi''_0 (\cdots) x_{31}^2 + \phi''_6 (\cdots) x_{31}^3.
\]

Given the presence of $a_{31200} y_{31}^2$, we deduce that putting $x_{31}^m$ in evidence, modulo $V : f_6 = 0$, in $B_{z_3}^\ast (B_{z_3}^\ast (\Phi_m)) : \sum_{i+j+k=m} c_{ijk} x_{31}^{2j+k} z_{31}^{j+k} = 0$, is equivalent to put $x_{31}^m$ in evidence without “modulo $V : f_6 = 0$”. Here, as in the proof of Lemma 1, we use the fact that $B_{x_3} \circ B_{z_3}$ coincides with the desingularization $\sigma_{x_3}$ on the affine open set $V_{x_3}$.

So, it remains for us to establish when we can put $x_{31}^m$ in evidence in \[
\sum_{i+j+k=m} c_{ijk} x_{31}^{2j+k} z_{31}^{j+k} = 0,
\]
without “modulo $V : f_6 = 0$” and this can be done immediately.

**Conclusion.** In the above polynomial, we can put $x_{31}^m$ in evidence if and only if $2j + k \geq m$.

Finally, if we consider the triple points $A_1$ and $A_3$ as both having an infinitely near triple line, then for the singularities given by $A_1$ and $A_3$, we also get much the same results as in the above Conclusion, that we obtained for the singularity given by $A_0$. That is to say, we obtain $2k + i \geq m$ in the case of $A_1$, and $2i + j \geq m$.
in the case of $A_3$. Adding the three inequalities, we obtain $3(i + j + h) \geq 3m$. Since $i + j + h = m$, we deduce that all the inequalities are equalities. Therefore $i = j = h$, i.e.

$$
\psi''_m(X_0, X_1, X_3) = \sum_{3i=m} c_{iii} X_0^i X_1^i X_3^i = c_{iii} X_0^i X_1^i X_3^i.
$$

This shows that the $m$-canonical adjoints to $V$ are of the type

$$
\phi_m : c_{iii}(X_0 X_1 X_3)^i = 0,
$$

for $i > 0$, $3i = m$ and Proposition 1 is proved.

\[\square\]

6. – Computing the irregularities of $Z_3$.

There remains for us to prove that $q_i = \dim_k H^i(Z_3, O_{Z_3}) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(Z_3, O_{Z_3}) = q(S_r) = \dim_k H^1(S_r, O_{S_r})$, where $S_r \subset Z_3$ is the strict transform of a generic hyperplane section $S$ of $V$ (cf. [S1], section 4, for instance). $S$ has several isolated (actual or infinitely near) double points and no other singularities. This follows from the fact that the hypersurface $V$, outside the points $A_0, A_1, A_2, A_3$ and $A_4$, only has actual or infinitely near double curves. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36), section 4 in [S1], which states that:

$$
q_2 = p_g(X) + p_g(S_r) - \dim_k(W_2),
$$

where $W_2$ is the vector space of the degree 2 forms defining global adjoints $\phi_2$ to $V$, i.e. defining hyperquadrics $\phi_2$ such that

$$
\pi_r^* \ldots \pi_2^*[\pi_1^*(\phi_2)] = E_1 - E_2 - E_4 - E_6 - E_8 \geq 0,
$$

(cf. the expression of $D_m$ in (*), section 4). So the above hyperquadrics $\phi_2$ are those passing through the points $A_0, A_1, A_2, A_3$ and $A_4$. Thus, we have: $\dim_k(W_2) = 15 - 5 = 10$. It follows from $p_g(S_r) = 10$ and $p_g(X) = 0$ (cf. Proposition 1 in section 5), that $q_2 = 0$.

7. – A net of elliptic curves on $Z_3$.

Let us consider the two 4-ple points $A_2$ and $A_4$ and the double line $A_2 A_4$ on $V \subset \mathbb{P}^4$. Clearly, there is a net of planes passing through the two points (and the double line). The generic plane of the net cuts out a degree six plane curve on $V$, which is split into the line $A_2 A_4$ counted twice, and into an irreducible quartic $C_4$ having exactly two nodes (ordinary double points) (according to Bertini). If
\( \widetilde{C}_4 \to C_4 \) is a desingularization of \( C_4 \), then \( \widetilde{C}_4 \) is an elliptic curve and we can assume \( \widetilde{C}_4 \in \mathbb{Z}_3 \). This shows that on \( \mathbb{Z}_3 \) there is a net of elliptic curves.

This completes the construction and the description of the first threefold \( \mathbb{Z}_3 \).

Construction of the threefold \( \mathbb{Z}_4 \).

8. – Imposing four triple points with an infinitely near double surface and a 4-ple point on a degree six hypersurface \( V' \) in \( \mathbb{P}^4 \).

Let \( (x_0, x_1, x_2, x_3, x_4) \) be homogeneous coordinates in \( \mathbb{P}^4 \) and let us indicate as \( f_0'(X_0, X_1, X_2, X_3, X_4) \) a form (homogeneous polynomial) of degree 6, in the variables \( X_0, X_1, X_2, X_3, X_4 \), defining a hypersurface of degree six \( V' \subset \mathbb{P}^4 \). We impose a triple point with an infinitely near double (singular) surface on \( V' \) at each of the four vertices \( A_0 = (1, 0, 0, 0, 0), A_1 = (0, 1, 0, 0, 0), A_3 = (0, 0, 0, 1, 0) \) and \( A_4 = (0, 0, 0, 0, 1) \), with an ordinary 4-ple (quadruple) point at the remaining vertex \( A_2 = (0, 0, 1, 0, 0) \) of the fundamental tetrahedron \( X_0X_1X_2X_3X_4 = 0 \). We have already considered the singularity given by a triple point with an infinitely near double surface in [\( S_2 \)].

The equation of \( V' \) with the above imposed singularities contains 27 coefficients, but the essential coefficients for our purposes are fewer (as in the case of \( V \), section 1); in fact, we only need 12 coefficients. We write the equation of \( V' \) directly, with the imposed singularities with the 12 essential coefficients.

\[
V' : f_0'(X_0, X_1, X_2, X_3, X_4) =
\]
\[
a_{31002}X_0^3X_1X_2^3 + \\
a_{13020}X_0X_1^3X_2^3 + \\
a_{20031}X_0^2X_1^3X_2X_3 + \\
a_{02013}X_0^3X_1X_3X_4 + \\
a_{21201}X_0^2X_1X_2^2X_4 + a_{20211}X_0^3X_2^3X_3X_4 + a_{12210}X_0X_1^3X_2^2X_3 + a_{11220}X_0X_1X_2X^2X_4 + \\
a_{11202}X_0X_1X_2^2X_4^2 + a_{10221}X_0X_1^2X_3^3X_4 + a_{02211}X_1X_2^2X_3X_4 + a_{01221}X_1X_2^2X_3X_4 = 0.
\]

From here on, \( V' \) denotes this last hypersurface defined by the above last form \( f_0'(X_0, X_1, X_2, X_3, X_4) \) for a generic choice of the parameters \( a_{ijkl} \).

9. – The unimposed actual singularities on \( V' \).

We consider the hypersurface \( V' \) at the end of section 8.

Close to the singularities imposed on \( V' \), new singularities appear on the generic \( V' \); they are actual or infinitely near singularities. The actual unimposed singularities are given by six actual double (straight) lines on \( V' \) given by \( A_0A_1 \),
$A_0A_2, A_1A_2, A_2A_3, A_2A_4$ and $A_3A_4$, according to the following picture, where the double lines are drawn in bold type.

The generic $V'$ has no other actual singularities, so the generic $V'$ is reduced, irreducible and normal.

10. – The infinitely near singularities of $V'$.

Resolution of singularities of $V'$

Here again, the desingularization of $V'$ is very long but very easy. New unimposed infinitely near singularities appear on the generic $V'$, close to the imposed infinitely near singularities. They are double singular curves. Here, there are none of the infinitely near isolated double points seen in the case of $V$.

So, none of the unimposed singularities affect the birational invariants of a desingularization $Z_4$ of $V'$, such as the irregularities and the plurigenera of $Z_4$, i.e. in calculating these invariants, we can assume that there are only the imposed singularities on $V'$.

The desingularization of $V'$ is, more or less, a repetition of the one in $[S_1], [S_2]$ and in section 3, so only the tree of the first blow-ups of $V'_{U_0} = V' \cap U_0$, where $U_0 = \{X_0 \neq 0\}$, is reproduced here.
11. – The m-canonical adjoints to $V' \subset \mathbb{P}^4$. 

Following the notations in section 4, an order can be established in the sequence of blow-ups in the example $V'$: let us assume that $\pi_1$ is the blow-up at the triple point $A_0$ and $\pi_2$ is the blow-up along the double surface infinitely near $A_0$, $\pi_3$ is the blow-up at the triple point $A_1$, and $\pi_4$ is the blow-up along the double surface infinitely near $A_1$, $\pi_5$ is the blow-up at the 4-ple point $A_2$, $\pi_6$ is the blow-up at the triple point $A_3$, $\pi_7$ is the blow-up along the double surface infinitely near $A_3$, $\pi_8$ is the blow-up at the triple point $A_4$, and $\pi_9$ is the blow-up along the double surface infinitely near $A_4$. Let $\sigma = \pi_r \circ \cdots \circ \pi_1$ be a sequence of blow-ups solving the singularities of $V'$.

The equivalent formula of (*), section 4, is given here by:

\[
D'_m = \pi_r \cdots \{\pi_2[\Phi_m] - mE_2\} - mE_4 - mE_5 - mE_7 - mE_9,
\]

where $E_i$ is the exceptional divisor of the blow-up $\pi_i$, i.e. $E_2$ is the exceptional divisor of the blow-up $\pi_2$ along the double surface infinitely near $A_0$, $E_5$ is the exceptional divisor of the blow-up at the 4-ple point $A_2$, ... and $E_9$ is the exceptional divisor of the blow-up $\pi_9$ along the double surface infinitely near $A_4$.

No other exceptional divisors appear in $D'_m$ because the unimposed singularities are either actual or infinitely near double singular curves on our (generic) $V'$.

12. – The plurigenera of a desingularization $Z_4$ of $V'$.

Let $\sigma_{|Z_4} : Z_4 \rightarrow V'$ be a desingularization of the hypersurface $V' \subset \mathbb{P}^4$, where $\sigma = \pi_r \circ \cdots \circ \pi_1$ (section 11).

**Proposition 2.** – The plurigenera of $Z_4$ are given by $P_{4i} = 1$, $\forall i \geq 1$, and $P_m = 0$ if $m \neq 4i$.

**Proof.** – Let us consider the equation of $V'$: $f'_6(X_0, X_1, X_2, X_3, X_4) = 0$, and we arrange the form $f'_6$ to the powers of $X_2$.

\[
f'_6 = \varphi_4(X_0, X_1, X_3, X_4)X_2^3 + \varphi_5(X_0, X_1, X_3, X_4)X_2 + \varphi_6(X_0, X_1, X_3, X_4) = 0,
\]

where $\varphi_i(X_0, X_1, X_3, X_4)$ is a form of degree $i$ in $X_0, X_1, X_3, X_4$.

Next, let us consider the hypersurface $\Phi_m$ appearing in (*) section 11, assuming that its equation is $F_m(X_0, X_1, X_2, X_3, X_4) = 0$, of degree $m$. Arranging the form $F_m$ to the powers of $X_2$, we can write

\[
F_m(X_0, X_1, X_2, X_3, X_4)
= \psi_s(X_0, X_1, X_3, X_4)X_2^{m-s} + \psi_{s+1}(X_0, X_1, X_3, X_4)X_2^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_3, X_4),
\]
where \( \psi_j(X_0, X_1, X_3, X_4) \) is a form of degree \( j \) in \( X_0, X_1, X_3, X_4 \) and \( s \) is an integer satisfying \( 0 \leq s \leq m \). So, \( \Phi_m \) has an \( s \)-ple point at \( A_2 \), with \( 0 \leq s \leq m \).

Let us assume that \( \Phi_m \) is an \( m \)-canonical adjoint to \( V' \), i.e. \( D'_m|_{z_4} \geq 0 \). From Lemma 1, section 5, we have the following result: modulo \( f'_6 = 0 \), we can assume \( s \geq m - 1 \); i.e. that \( \Phi_m \), modulo \( f'_6 \), is defined by the form

\[
F_m = \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).
\]

From the (*) in the proof of Proposition 1, section 5, we have the following equality:

\[
\psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\phi_4(X_0, X_1, X_3, X_4),
\]

where \( B'''(X_0, X_1, X_3, X_4) \) is a form of degree \( m - 5 \) in \( X_0, X_1, X_3, X_4 \) or zero-form.

**Lemma 2.** – The \( m \)-canonical adjoint to \( V' \) given by

\[
\Phi_m : \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0,
\]

where \( \psi_{m-1}(X_0, X_1, X_3, X_4) = B'''(X_0, X_1, X_3, X_4)\phi_4(X_0, X_1, X_3, X_4) \), has the following property

\[
D'_m|_{z_4} \geq 0 \iff D'_m + E_5 \geq 0,
\]

where \( D'_m = \pi_1^* \{ \pi_2^*[\pi_1^*(\Phi_m)] - mE_2 \} - mE_4 - mE_5 - mE_7 - mE_9 \), is defined in (*), section 11.

**Proof of Lemma 2.** Let us consider the affine open set \( U_0 = \{ X_0 \neq 0 \} \) as in section 1. Locally, the blow-up \( \pi_1 \) of \( \mathbb{P}^3 \) at \( A_0 \) is given by \( B_{x_1}, B_{y_2}, B_{z_2}, B_t \) (cfr. section 1).

The total transform of \( \Phi_m \cap U_0 \) with respect to \( B_{x_1} \) is given by

\[
B_{x_1}^*(\Phi_m \cap U_0) : \psi_{m-1}(1, x_1, x_1z_1, x_1t_1)x_1y_1 + \psi_m(1, x_1, x_1z_1, x_1t_1) = 0.
\]

The double surface \( S_0 \) infinitely near \( A_0 \) in affine coordinates \((x_1, y_1, z_1, t_1)\) is given by \( \begin{cases} x_1 = 0 \\ t_1 = 0 \end{cases} \) and the blow-up \( \pi_2 \) along \( S_0 \) is locally given by the formulas:

\[
B_{x_1} : \begin{cases} x_1 = x_{11} \\ y_1 = y_{11} \\ z_1 = z_{11} \\ t_1 = t_{11} \end{cases}; \quad B_{y_1} : \begin{cases} x_1 = x_{12}t_{12} \\ y_1 = y_{12} \\ z_1 = z_{12} \\ t_1 = t_{12} \end{cases}.
\]

The total transform of \( B_{x_1}^*(\Phi_m \cap U_0) \) with respect to \( B_{t_1} \) is given by

\[
B_{t_1}^*[B_{x_1}^*(\Phi_m \cap U_0)] : \psi_{m-1}(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2)x_{12}y_{12}t_{12}
+ \psi_m(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}t_{12}^2) = 0.
\]
Since $\Phi_m$ is an $m$-canonical adjoint to $V'$, the latter equation, modulo
\[
\frac{1}{t_{12}^6} f_6'(1, x_{12} t_{12}, x_{12} y_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12}^2 t_{12}) = 0,
\]
must be of the type $t_{12}^m(...)=0$.

The latter statement follows, as in the proof of Lemma 1, from the fact that $B_{t_{12}} \circ B_{x_1}$ coincides with the desingularization $\sigma_{|z_4}$ on the affine open set $V'_{t_{12}}$ (see the tree of blow-ups).

In other words, the following equality of polynomials must hold
\[
(\ast) \quad \psi_{m-1}(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12} t_{12}) + \psi_m(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12}^2 t_{12}) + A(x_{12}, y_{12}, z_{12}, t_{12}) \cdots \quad a_{21201} x_{12} y_{12}^2 + a_{20211} x_{12} y_{12}^2 z_{12} t_{12} \cdots = \frac{1}{t_{12}^6} f_6'.
\]

Note that the variable $y_{12}$ in the equality $(\ast)$ appears in
\[
\psi_{m-1}(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12} t_{12}) + \psi_m(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12}^2 t_{12})
\]
with power one and in $[\cdots + a_{21201} x_{12} y_{12}^2 + a_{20211} x_{12} y_{12}^2 z_{12} t_{12} \cdots]$ with power two.

It follows that the equality $(\ast)$ holds if and only if
\[
(\ast\ast) \quad \psi_{m-1}(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12} t_{12}) + \psi_m(1, x_{12} t_{12}, x_{12} z_{12} t_{12}, x_{12} y_{12}^2 t_{12}) = t_{12}^m(...)
\]
and moreover $A(x_{12}, y_{12}, z_{12}, t_{12}) = t_{12}^m(...)$ or $A(x_{12}, y_{12}, z_{12}, t_{12})$ is zero.

We obtain the result given by the equality $(\ast\ast)$, obtained in the affine open set $U_0$, in the affine open sets $U_1$, $U_3$ and $U_4$ ($U_i = \{ X_i \neq 0 \}$) too. So the equality $(\ast\ast)$, and the analogous equalities in $U_1$, $U_3$ and $U_4$, tell us that
\[
\Phi_m : \psi_{m-1}(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0,
\]
where $\psi_{m-1}(X_0, X_1, X_3, X_4) = B''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)$ satisfies
\[
D_{m|z_4}' \geq 0 \iff D_m' + mE_5 \geq 0.
\]

Finally, if we consider $U_2 = \{ X_2 \neq 0 \}$, then we obtain
\[
D_{m|z_4}' \geq 0 \iff D_m' + E_5 \geq 0.
\]

This proves Lemma 2.

**Proof of Proposition 2 (continuation).** From the result of Lemma 2, to compute the plurigenera of $Z_4$, we consider
\[
F_m = B''(X_0, X_1, X_3, X_4)\varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4).
\]
where
\[
\varphi_4(X_0, X_1, X_3, X_4) = a_{21201}X_0^2X_1X_4 + a_{20211}X_0^2X_3X_4 + a_{12210}X_0X_1^2X_3 \\
+ a_{11220}X_0X_1X_3^2 + a_{11202}X_0X_1X_4^2 + a_{10221}X_0X_3X_4 + a_{02211}X_1^2X_3X_4 + a_{01212}X_1X_3X_4^2.
\]

Let us write
\[
B'''(X_0, X_1, X_3, X_4)X_2 = \left( \sum_{i+j+h+l=m-5} b_{ijkl}X_0^iX_1^jX_3^hX_4^l \right)X_2,
\]
\[
\psi_m(X_0, X_1, X_3, X_4) = \sum_{i'+j'+h'+l'=m} c_{ijkl}X_0^{i'}X_1^{j'}X_3^{h'}X_4^{l'},
\]
where \( b_{ijkl}, c_{ijkl} \in k \).

With these notations, and considering \( \Phi_m \cap U_0 \), the equality in \((\cdot \cdot \cdot)\) becomes
\[
\left( \sum_{i+j+h+l=m-5} b_{ijkl}x_{12}^{j+h+l}x_{12}^{i+j+2l} \right)\varphi_4(1, x_{12}t_{12}, x_{12}z_{12}t_{12}, x_{12}z_{12}^2)(x_{12}y_{12}t_{12}) \\
+ \sum_{i'+j'+h'+l'=m} c_{ijkl}x_{12}^{j+h'+l'}x_{12}^{i'+j'+h'+2l'} = t_{12}^m(...).
\]
The latter equality is equivalent to the inequalities \[
\begin{cases} 
  j + h + 2l + 4 \geq m \\
  j' + h' + 2l' \geq m 
\end{cases}
\]

Similarly, if we consider \( \Phi_m \cap U_1 \), the analogous equality of \((\cdot \cdot \cdot)\) provides the inequalities \[
\begin{cases} 
  i + 2h + l + 4 \geq m \\
  i' + 2h' + l' \geq m 
\end{cases}
\]

Again, \( \Phi_m \cap U_3 \) and \( \Phi_m \cap U_4 \) provide the inequalities
\[
\begin{cases} 
  2i + j + l + 4 \geq m \\
  2i' + j' + l' \geq m 
\end{cases} \quad \text{and} \quad \begin{cases} 
  i + 2j + h + 4 \geq m \\
  i' + 2j' + h' \geq m 
\end{cases}
\]

Combining all the inequalities gives us
\[
\begin{cases} 
  l \geq i + 1 \geq h + 2 \geq j + 3 \geq l + 4 \\
  l' \geq i' \geq h' \geq j' \geq l'
\end{cases}
\]

The first line tells us that \( B'''(X_0, X_1, X_3, X_4) \) is the zero-form and the second line shows that \( i' = j' = h' = l' \), i.e. the form defining \( \Phi_m \) is
\[
c_{rrrr}X_0^rX_1^rX_3^rX_4^r, \ \forall r \geq 1.
\]

This proves Proposition 2. \( \square \)

13. – The irregularities of \( Z_4 \).

With the same proof as in the case of \( Z_3 \), cf. section 6, the irregularities of \( Z_4 \) are \( q_1 = q_2 = 0 \).
This completes the construction of the second threefold $Z_4$. 

Construction of the threefold $Z_5$. 

14. – Imposing five triple points with an infinitely near double surface on a degree six hypersurface $V''$ in $\mathbb{P}^4$. 

The simplest equation of a degree six hypersurface $V''$ having five triple points with an infinitely near double surface is given by

$$V'' : f''(X_0, X_1, X_2, X_3, X_4) =$$

$$a_{30201}X_0^3X_1^2X_4 +$$

$$a_{13020}X_0X_1^3X_2^2 +$$

$$a_{10302}X_1X_2^3X_3^2 +$$

$$a_{20130}X_0^2X_2X_3^3 +$$

$$a_{02013}X_1^2X_3X_4^3 = 0.$$ 

The rotations of indices and variables passing from $A_i$ to $A_{i+1}$ and returning to $A_0$ are as follows. 

Rotations of indices (and variables)

$$A_0 \mapsto A_1 \mapsto A_2 \mapsto A_3 \mapsto A_4 \mapsto A_0$$

$$ijkhl \mapsto lijkhl \mapsto hlijk \mapsto khlij \mapsto jkhli \mapsto ijkhl$$

Here, the equation $f''(X_0, X_1, X_2, X_3, X_4) = 0$ is invariant with respect to the five rotations. So, a statement on the equation that is true for the point $A_i$, for the affine open set $U_i$, ... holds true for any other point $A_j$, for any other affine open set $U_j$, ....

From now on, $V''$ denotes the degree six hypersurface defined by the $f''(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of parameters $a_{ijkhl}$.

15. – The unimposed actual singularities on $V''$. 

There are five unimposed actual double (straight) lines on $V''$: $A_0A_1$, $A_0A_4$, $A_1A_2$, $A_2A_3$, $A_3A_4$; see below, where the double lines are drawn in bold type.
The (generic) $V''$ has no other actual singularities, so $V''$ is reduced, irreducible and normal.

16. – The infinitely near singularities of $V''$.

Here, since the equation of $V''$ is invariant with respect to the rotations of indices and variables, all we have to do is resolve the singularities in just one of the affine open sets $U_i = \{X_i \neq 0\}$.

New unimposed infinitely near singularities appear on the generic $V''$ close to the imposed infinitely near singularities; they are only double singular curves. So, here again, none of the unimposed singularities affect the birational invariants of a desingularization $Z_5$ of $V''$.

The desingularization of $V''$ is, more or less, the same as in $[S_1], [S_2]$ and section 3, so only a part of the tree of the blow-ups of $V''_{U_0} = V'' \cap U_0$ is given here.

The affine threefolds $V''_1$, $V''_5$ and $V''_6$ are singular along (locally) double straight lines. Here again, they have local double lines infinitely near and, after some blow-ups, we can resolve all the singularities.

17. – The m-canonical adjoints to $V'' \subset \mathbb{P}^4$.

As in the previous constructions, an order can be established in the sequence of blow-ups, e.g. let us assume that $\pi_1$ is the blow-up at the triple point $A_0$, $\pi_2$ is the blow-up along the double surface infinitely near $A_0$, $\pi_3$ is the blow-up at $A_1$, $\pi_4$ is the blow-up along the double surface infinitely near $A_1$, ..., and $\pi_{10}$ is the blow-up along the double surface infinitely near $A_4$. 
The equivalent formula of (*), section 4, is given by:

\[ D''_m = \pi'_2 \cdots \pi'_2(\Phi_m) - mE_2 - mE_4 - mE_6 - mE_8 - mE_{10}, \]

where \( E_i \) is the exceptional divisor of the blow-up \( \pi_i \), i.e. \( E_2 \) is the exceptional divisor of the blow-up \( \pi_2 \) along the double surface infinitely near \( A_0 \), and so on.

No other exceptional divisors appear in \( D''_m \) because the unimposed singularities are either actual or infinitely near double singular curves on our (generic) \( V'' \).

18. – The plurigenera of a desingularization \( Z_5 \) of \( V'' \).

Let \( \sigma_{|Z_5} : Z_5 \rightarrow V'' \) be a desingularization of the hypersurface \( V'' \subset \mathbb{P}^4 \), where \( \sigma = \pi_r \circ \cdots \circ \pi_1 \) (section 17).

**Proposition 3.** – The plurigenera of \( Z_5 \) are given by \( P_{5i} = 1 \), \( \forall i \geq 1 \), and \( P_m = 0 \) if \( m \neq 5i \).

The statement of Proposition 3 is much the same as those of Propositions 1 (section 5) and 2 (section 12), but the proof is completely different, essentially because the hypersurface \( V'' \) has no 4-ple points in this case.

To prove Proposition 3, we need some preliminary results on global and non-global \( m \)-canonical adjoints to \( V'' \) (cf. section 4, or [S1] for the definitions).

**Lemma 3.** – The global \( m \)-canonical adjoints to \( V'' \) are given by

\[ \Phi_m : c_i X_0^i X_1^i X_2^i X_3^i X_4^i = 0, \ \forall i \geq 0, \]

where \( c_i \in k \).

In particular, the global \( m \)-canonical adjoints exist if and only if \( m = 5i \), \( \forall i \) and there is only one of them for \( m = 5i \).

**Proof of Lemma 3.** Let us consider a global \( m \)-canonical adjoint to \( V'' \)

\[ \Phi_m : \sum_{i+j+k+l+m} b_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^m = 0, \]

Locally, the blow-up \( \pi_1 \) of \( \mathbb{P}^4 \) at \( A_0 \) is given by \( B_{r_1}, B_{r_2}, B_{z_3}, B_{t_3} \) (section 1).

The total transform \( \Phi^* \) of \( \Phi_m \cap U_0 \) with respect to \( B_{t_3} \) is given by

\[ \Phi^* = B_{z_3}^* (\Phi_m \cap U_0) : \sum_{i+j+k+l+m} b_{ijkl}(x_3 z_3)^i (y_3 z_3)^j z_3^k (z_3 t_3)^l = 0, \]

The double surface \( S_0 \) infinitely near \( A_0 \) in affine coordinates \( (x_3, y_3, z_3, t_3) \) is
given by \( \begin{cases} y_3 = 0 \\ z_3 = 0 \end{cases} \) and the blow-up \( \pi_2 \) along \( S_0 \) is given locally by the formulas:

\[
B_{y_3} : \begin{cases} 
  x_3 = x_{31} \\
  y_3 = y_{31} \\
  z_3 = z_{31} \\
  t_3 = t_{31} 
\end{cases} \quad ; \quad 
B_{z_{32}} : \begin{cases} 
  x_3 = x_{32} \\
  y_3 = y_{32} z_{32} \\
  z_3 = z_{32} \\
  t_3 = t_{32} 
\end{cases} .
\]

The total transform \( \Phi^{**} \) of \( \Phi^* = B_{z_{32}}^* (\Phi_m \cap U_0) \) with respect to \( B_{z_{32}} \) is given by

\[
\Phi^{**} = B_{z_{32}}^* (\Phi^*) : \sum_{i+j+k+h+l=m} b_{ijklh}(x_{32} y_{32} z_{32})^j (y_{32} z_{32})^k z_{32}^l (z_{32} t_{32})^l = 0,
\]

Since \( \Phi_m \) is a global \( m \)-canonical adjoint to \( V'' \), by definition in (\( ^\land \)), section 17, we have \( D_m \geq 0 \). This implies that

\[
\frac{\Phi^{**}}{(z_{32}^m)} = \left( \sum_{i+j+k+h+l=m} b_{ijklh}(x_{32} y_{32} z_{32})^j (y_{32} z_{32})^k z_{32}^l (z_{32} t_{32})^l = 0 \right) \geq 0.
\]

Here, as well as in the proof of Lemma 1, we use the fact that \( B_{z_{32}} \circ B_{z_{31}} \) coincides with the desingularization \( \sigma_{z_{31}} \) on the affine open set \( V''_{z_{32}} \), in fact \( V''_{z_{32}} \) is nonsingular (see the tree of blow-ups, section 16).

Clearly, the latter inequality is equivalent to

\[
j + 2k + h + l - m \geq 0, \text{ i.e. } i \leq k.
\]

Note that we obtained the latter result in the affine open set \( U_0 \). Next, without any further computations, simply using the rotations of indices and variables (section 14), we obtain similar results in the other affine open sets \( U_1, U_2, U_3 \) and \( U_4 \):

- in \( U_1 \), we obtain \( j \leq h \);
- in \( U_2 \), we obtain \( k \leq l \);
- in \( U_3 \), we obtain \( h \leq i \);
- in \( U_4 \), we obtain \( l \leq j \).

Composing all the inequalities, we deduce that \( i = j = k = h = l \).

This proves Lemma 3.

**Lemma 4.** – Let us consider an \( m \)-canonical adjoint to \( V'' \) (not necessarily global)

\[
\Phi_m : F_m (X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+h+l=m} b_{ijklh} X_0^i X_1^j X_2^k X_3^h X_4^l = 0,
\]
where \( b_{ijkl} \in k \). \( \Phi_m \) can be replaced with
\[
\Psi_m : F_m - B_0 f_6'' = \sum_{i_0+j_0+k_0+h_0+l_0=m} c_{i_0j_0k_0h_0l_0} x_0^{i_0} x_1^{j_0} x_2^{k_0} x_3^{h_0} x_4^{l_0} = 0,
\]
such that \( i_0 \leq k_0 \), i.e. \( j_0 + 2k_0 + h_0 + l_0 \geq m \), for all monomials.

Before proving Lemma 4, we must point out that the inequality \( i_0 \leq k_0 \) is equivalent to
\[
\Psi^{**}(z_{32}^m) = \frac{1}{(z_{32}^m)^2} \left( \sum_{i_0+j_0+k_0+h_0+l_0=m} c_{i_0j_0k_0h_0l_0} x_3^{j_0} y_3^{k_0} z_3^{i_0+2k_0+h_0+l_0} t_3^{l_0} = 0 \right) \geq 0.
\]
Here, we used the same notations as in the proof of Lemma 3, where \( c_{i_0j_0k_0h_0l_0} \in k \), \( B_0 \) is a suitable form and \( f_6'' = 0 \) is the equation of \( V'' \).

**Proof of Lemma 4.** Since \( \Phi_m : F_m = 0 \) is an \( m \)-canonical adjoint to \( V'' \), by definition (in (\( \diamond \)), section 17) we have \( D''_{m|_{z_3}} \geq 0 \).

Let us consider the first two blow-ups \( \mathbb{P}_2 \to \mathbb{P}_1 \to \mathbb{P}^4 \) for the resolution of the singularities of \( V'' \). If \( V''_1 \) is the strict transform of \( V'' \) with respect to \( \pi_1 \), and \( V''_2 \) is the strict transform of \( V''_1 \) with respect to \( \pi_2 \), then we obtain the sequence
\[
V''_2 \to V''_1 \to V''
\]
of morphisms, where \( \pi_i \) is the restriction of \( \pi_i \) to \( V''_i \), \( i = 1, 2 \).

Now, let us consider the affine open set \( U_0 \) and \( V'' \cap U_0 \). With the notations in the proof of Lemma 3 and the affine open set of affine coordinates \( (x, y, z, t) \), we have that \( V''_2 \) has the equation given by the polynomial
\[
\frac{1}{z_{32}^m} f_6''(1, x_{32} z_{32}^2, y_{32} z_{32}^2, z_{32}^2, t_{32}^2) = a_{30} y_{32}^2 t_{32} + a_{13} y_{32} t_{32}^3
\]
\[+ a_{01} x_{32} y_{32}^2 t_{32} z_{32}^2 + a_{20} y_{32} + a_{02} x_{32} z_{32}^2 t_{32}^2 \]

Next, in the inequality \( D''_{m|_{z_3}} \geq 0 \), we consider only the first two blow-ups. So, in the affine open set of coordinates \( (x, y, z, t) \), we obtain
\[
\left. \left( \frac{\Phi^{**}}{z_{32}^m} \right) \right|_{V''_2} \geq 0.
\]
Here again, we use the fact that \( B_{z_3} \circ B_{z_4} \) coincides with the desingularization \( \sigma_{z_3} \) on the affine open set \( V''_{z_3} \), in fact \( V''_{z_3} \) is nonsingular (see the tree of blow-ups, section 16).

In the language of polynomials, the latter inequality is equivalent to writing
the equality of polynomials
\[
(\diamondsuit) \quad \sum_{i+j+k+h+l=m} b_{ijkl} x_i^j y_k^l z_{32}^{i+j} + B(x_{32}, y_{32}, z_{32}, t_{32}) (a_{30201} y_{32}^2 t_{32} + \dotsc + a_{13020} z_{32}^2 + a_{20130} y_{32} + a_{020130} z_{32} t_{32}) z_{32}^{r} = z_{32}^{m}(\dotsc),
\]
where \( B(x_{32}, y_{32}, z_{32}, t_{32}) \) is a suitable polynomial.

In the particular case, we have \( j + 2k + h + l \geq m \) for all monomials, the Lemma 4 is true with \( B_0 = B = 0 \). So, we assume \( j + 2k + h + l < m \) for some monomials (now \( B \neq 0 \)) and we distinguish the cases \( r \leq j + 2k + h + l < m \), with \( r \geq 0 \). We can conveniently rewrite
\[
\sum_{i+j+k+h+l=m} b_{ijkl} x_i^j y_k^l z_{32}^{i+j} + B(x_{32}, y_{32}, z_{32}, t_{32}) (a_{30201} y_{32}^2 t_{32} + \dotsc + a_{13020} z_{32}^2 + a_{20130} y_{32} + a_{020130} z_{32} t_{32}) z_{32}^{r} = \sum_{j+k+l=m-r} b_{ijkl} x_j^k y_l z_{32}^{j+k+l} t_{32}^{m-r} + \dotsc + \sum_{j+k+l=m-m} b_{ijkl} x_j^k y_l z_{32}^{j+k+l} t_{32}^{m-1} t_{32} + \dotsc,
\]
Substituting in \( (\diamondsuit) \), we obtain
\[
B(x_{32}, y_{32}, z_{32}, t_{32}) = z_{32}^{r} C(x_{32}, y_{32}, z_{32}, t_{32}).
\]
Again, substituting the latter equality in \( (\diamondsuit) \), then simplifying \( z_{32} \) and putting \( z_{32} = 0 \), we obtain the equality of polynomials
\[
(\diamondsuit) \quad \sum_{j+k+l=r} b_{ijkl} x_j^k y_l z_{32}^{j+k+l} t_{32}^{m-r} = C(x_{32}, y_{32}, 0, t_{32}) (a_{30201} y_{32}^2 t_{32} + a_{13020} z_{32}^2 + a_{20130} y_{32}).
\]

Multiplying the left- and right-hand sides of \( (\diamondsuit) \) by \( z_{32}^{m} \), and taking \( B_{z_{32}} \) and \( B_{z_{32}} \) into account, we obtain
\[
\sum_{j+k+l=r} b_{ijkl} x_j^k y_l z_{32}^{j+k+l} t_{32}^{m-r} = D(x, y, z, t) \left( x y z^{r-5} \right) (a_{30201} y_{32}^2 t_{32} + a_{13020} z_{32}^2 + a_{20130} y_{32}).
\]
We write \( D(x, y, z, t) \left( x y z^{r-5} \right) = \frac{D(x, y, z, t)}{z^r} \). Thus,
\[
\sum_{j+k+l=r} b_{ijkl} x_j^k y_l z_{32}^{j+k+l} t_{32}^{m-r} = z^{r-5-p} D(x, y, z, t) (a_{30201} y_{32}^2 t_{32} + a_{13020} z_{32}^2 + a_{20130} y_{32}).
\]
Replacing \( D(x, y, z, t) \), if necessary, we can assume \( r - 5 - p \geq 0 \). In fact, if \( r - 5 - p < 0 \), then we deduce \( z^{r-3-p} \left( \sum \dotsc \right) = D(x, y, z, t)(\dotsc) \). Since polynomial rings are factorial, we obtain \( D(x, y, z, t) = z^{r+3+p} E(x, y, z, t) \) and \( \sum \dotsc = E(x, y, z, t)(\dotsc) \), as desired.
Now, we suitably homogenize the latter equality to obtain an addendum of the form $F_m$, so we can write

\[(\phi^v) \sum_{j+2k+h+l=r} b_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l = G(X_0, X_1, X_2, X_3, X_4)(a_{30201} X_0^2 X_2^2 X_4 + a_{13020} X_1^2 X_3^2 + a_{20130} X_0 X_2 X_3^2),\]

where $i + j + k + h + l = m$.

From the hypothesis $r \leq m - 1$, we deduce $k \leq i - 1$ and particularly $i \geq 1$. The latter inequality means that $X_0$ can be put in evidence in

\[\sum_{j+2k+h+l=r} b_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l.\]

So, from $(\phi^v)$, we obtain the equality of polynomials $G(X_0, X_1, X_2, X_3, X_4) = X_0 H(X_0, X_1, X_2, X_3, X_4)$, and we can thus rewrite $(\phi^v)$ as follows.

\[\sum_{j+2k+h+l=r} b_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l = H(X_0, X_1, X_2, X_3, X_4)(a_{30201} X_0^2 X_2^2 X_4 + a_{13020} X_1^2 X_3^2 + a_{20130} X_0 X_2 X_3^2).\]

But now $a_{30201} X_0^2 X_2^2 X_4 + a_{13020} X_1^2 X_3^2 + a_{20130} X_0 X_2 X_3^2$ is an addendum of the form $f''_6(X_0, X_1, X_2, X_3, X_4)$ defining $V''$. This enables us to replace the $m$-canonical adjoint $\phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = 0$ with the new one $\phi'_m$ given by

\[F_m(X_0, X_1, X_2, X_3, X_4) - H(X_0, X_1, X_2, X_3, X_4)f''_6(X_0, X_1, X_2, X_3, X_4) = 0,
\]

where the form $F_m - Hf''_6$ is now given by

\[\sum_{j+2k+h+l=r+1} c_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l + \cdots ,\]

i.e. the form $F_m - Hf''_6$ starts with $j + 2k + h + l = r + 1$ instead of $j + 2k + h + l = r$.

By iterating this process, we can replace the $m$-canonical adjoint $\phi_m : F_m = 0$ with $\Psi_m : F_m - B_0 f''_6$, so that

\[F_m - B_0 f''_6 = \sum_{j+2k+h+l=m} c_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l + \cdots .\]

This proves Lemma 4.

The rotations of the indices and variables concerning the affine open set $U_0$ can be repeated for each of the other affine open sets $U_1, U_2, U_3$ and $U_4$ (in the same way as in Lemma 4), i.e. if we choose one affine open set $U_s$, then a result like the one in Lemma 4 holds in $U_s$ too. That is to say, we have
COROLLARY. – If we consider an $m$-canonical adjoint to $V''$

$$\Phi_m : F_m = \sum_{i+j+k+h+l=m} b_{ijkl} X_i^0 X_j^2 X_k^3 X_l^4 = 0,$$

then we can replace $\Phi_m$ with

$$\Phi_m^{(s)} : F_m - B_s f_6'' = \sum_{i+j+k+h+l+m} c_{ijkl} X_i^0 X_j^2 X_k^3 X_l^4 = 0,$$

such that either for $s = 0$, the inequality $i_0 \leq k_0$ holds for all monomials (cf. Lemma 4), or

for $s = 1$, the inequality $j_1 \leq h_1$ holds for all monomials, or

for $s = 2$, the inequality $k_2 \leq l_2$ holds for all monomials, or

for $s = 3$, the inequality $h_3 \leq i_3$ holds for all monomials, or

for $s = 4$, the inequality $l_4 \leq j_4$ holds for all monomials.

LEMMA 5. – If we consider a non-global $m$-canonical adjoint to $V''$

$$\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+h+l=m} b_{ijkl} X_i^0 X_j^2 X_k^3 X_l^4 = 0,$$

then a form $B = B(X_0, X_1, X_2, X_3, X_4)$ exists such that $\Phi_m^* : F_m - B f_6'' = 0$ is a global $m$-canonical adjoint to $V''$. In other words, the following equality holds

$$\Phi_m|_{V''} = \Phi_m^*|_{V''}.$$

Before considering the proof, note that Lemma 5 holds essentially because of the particular rotations of indices and variables. In other words, if we consider other permutations of indices and variables, and we leave the same five imposed singularities, then Lemma 5 may be false (cf. [S_2]).

PROOF OF LEMMA 5. If a form of the type $(\cdots)X_0X_1^2X_3^2$ appears in $F_m$ as an addendum, then we can replace $X_0X_1^2X_3^2$ with

$$\frac{-f_6''}{a_{13020}} + X_0X_1^3X_3^2 = -\frac{a_{30201}}{a_{13020}} X_0^3X_2^2X_4 - \frac{a_{01302}}{a_{13020}} X_1^3X_2^2X_4 - \frac{a_{20130}}{a_{13020}} X_0^2X_2X_3^3 - \frac{a_{02013}}{a_{13020}} X_1^2X_3X_4^3.$$

This is the same as replacing the form $F_m$ with $F_m - (...)f_6''$ and this can be done for the reasons given in the proof of Lemma 1 (section 5). Clearly, we can repeat this replacement several times, obtaining a new form $F_m' = F_m - (...)f_6''$, which contains no addendum of the type $(\cdots)X_0X_1^2X_3^2$. So, from now on, we can consider $\Phi_m' : F_m' = 0$, instead of $\Phi_m : F_m = 0$, because if we prove the lemma for $\Phi_m'$, then it is also proved for $\Phi_m$. If $F_m' = 0$, then $F_m = (...)f_6''$ and $\Phi_m$ is a global adjoint.
(cutting the zero divisor on $Z_5$), and in this case Lemma 5 holds true. So we can assume that $F_{m}' \neq 0$ and $F_{m}'$ contains no addendum of the type $(...)X_0X_1^2X_2^2$.

Let us write $F_{m}' = \sum_{i'+j'+k'+l'=m} b_{i'j'k'l'}X_0^{i'}X_1^{j'}X_2^{k'}X_3^{l'}$. We claim that $i' \leq k'$.

To prove this claim, we assume by contradiction that $i' > k'$. From Lemma 4, $\Phi_{m}'$ can be replaced with

$$\Psi_{m} : F_{m}' - B_0f_6' = \sum_{i_0+j_0+k_0+l_0=m} c_{i_0j_0k_0l_0}X_0^{i_0}X_1^{j_0}X_2^{k_0}X_3^{l_0} = 0,$$

so that $i_0 \leq k_0$. From the proof of Lemma 4, this can be done if and only if we subtract $B_0(a_{30201}X_0^2X_2^2X_4 + a_{13020}X_0X_1^2X_2^2 + a_{20130}X_0^2X_2X_3^2), B_0 \neq 0$, from $F_{m}'$. But this means that the form $F_{m}'$ has the addendum $B_0a_{13020}X_0X_1^2X_2^2$, and this contradiction proves our claim.

Next, we claim that $j' \leq h'$ also holds. Otherwise, we would have to subtract the form $B_1(a_{13020}X_0X_1^2X_2^2 + a_{01302}X_1X_2^2X_4^2 + a_{02013}X_1^2X_2X_3^2), B_1 \neq 0$, from $F_{m}'$ (cf. Corollary). But this is impossible because $F_{m}'$ does not contain the addendum $(...)X_0X_1^2X_2^2$.

Similarly, $h' \leq i'$ also holds. Otherwise, we would have to subtract the form $B_3(a_{13020}X_0X_1^2X_2^2 + a_{20130}X_0^2X_2X_3^2 + a_{02013}X_1^2X_2X_3^2), B_3 \neq 0$, from $F_{m}'$, which is again impossible.

In short, in $F_{m}'$ we have $i' \leq k', j' \leq h', h' \leq i'$. If we also have $k' \leq l'$ and $l' \leq j'$ in $F_{m}'$, then Lemma 5 is demonstrated, because the five inequalities for the same $F_{m}'$ tell us that $\Phi_{m}' : F_{m}' = 0$ is a global $m$-canonical adjoint to $V_{m}'$.

So, let us assume that the two inequalities $k' \leq l'$ and $l' \leq j'$ do not hold, for some monomials. In this case, we show that there is a contradiction, or we construct $F_{m}''$ such that $F_{m}'' = F_{m}' - B_2f_6''$ and $F_{m}''$ defines a global $m$-canonical adjoint, proving Lemma 5.

First we consider $l' > j'$. In this case, as in the proof of Lemma 4 (using the rotations of indices and variables), we see that $F_{m}''$ has an addendum of the type $(...)X_0X_2^2X_4$. We replace $X_0X_2^2X_4$ with the form $\frac{f_6''}{a_{30201}} + X_0^2X_2^2X_4$ in $F_{m}'$ (several times), which is equivalent to replacing the form $F_{m}'$ with $F_{m}'' = F_{m}' - (...)f_6''$. As before, we can see that in the form

$$F_{m}'' = \sum_{i''+j''+k''+l''=m} b_{i''j''k''l''}X_0^{i''}X_1^{j''}X_2^{k''}X_3^{l''}$$

the inequality $l'' \leq j''$ holds (for all monomials). In addition, the inequality $k'' \leq l''$ also holds.

Likewise (or as in the proof of Lemma 4), $l'' \leq j''$ can be obtained if and only if we subtract the addendum $B_2(a_{30201}X_0^2X_2^2X_4 + a_{01302}X_1X_2^2X_4^2 + a_{20130}X_0^2X_2X_3^2)$ from $F_{m}'$. This has two consequences:
1) in $F''_m$ there is the addendum $B_2(a_{30201}X_0^3X_2^3X_4 + a_{01302}X_1X_2^3X_4^2 + a_{20130}X_0X_2X_3)$;

2) in $F'''_m$ there is the addendum $-B_2(\delta_1a_{13020}X_0X_1^2X_2^3 + \delta_2a_{02013}X_1^2X_3X_4^3)$,

where $\delta_i = \{0, 1\}$, $i = 1, 2$, and $-B_2(\delta_1a_{13020}X_0X_1^2X_2^3 + \delta_2a_{02013}X_1^2X_3X_4^3) \neq 0$.

We note that, for example, if $\delta_2 = 0$, then the addendum $B_2a_{02013}X_1^2X_3X_4^3$ appears in $F''_m$.

Now, let us consider the case where $\delta_2 = 1$.

Since $i'$ is the power of the variable $X_0$ and $k'$ is the power of $X_2$ in $F''_m$, we deduce from $i' \leq k'$ and from 1) that $B_2 = \sum X_0^rX_2^{r+s}(\ldots)$, where $r \geq 0$ and $s \geq 1$.

Similarly, $k''$ is the power of $X_2$ and $l''$ is the power of $X_4$ in $F'''_m$, so from $k'' \leq l''$ and from 2), we deduce that $B_2 = \sum X_0^rX_2^{r+s}X_4^{r+l''}(\ldots)$, where $r \geq 0$.

In addition, the inequality $j' \leq h'$ holds in $F''_m$. Again from 1), we obtain $B_2 = \sum X_0^rX_1^2X_2^3X_3^{r+u}X_4^{r+l''}(\ldots)$, where $t \geq 0$ and $u \geq 1$. From $l'' \leq j''$ in $F'''_m$, we obtain that $B_2$ the inequality $r + s + v < t$ must hold.

Finally, from $h' \leq i'$ in $F''_m$, in $B_2$ the inequality $t + u < r$ must hold too.

From $r + s + v < t$ and $t + u < r$, we deduce that $t - s - v > r > t + u$, but this is a contradiction. The contradiction proves that the case of $\delta_2 = 1$ cannot occur.

Next, let us consider the case where $\delta_2 = 0$.

In this case, all the above inequalities hold except $r + s + v < t$, which must be replaced by $r + s + v \leq t + 3$. Here, we also consider the inequality $l' > j'$ in $F'''_m$, which was assumed at the beginning. Said inequality implies that

$r + s + v > t + 1$, so $r + s + v = \{t + 2\}$.

From the inequalities $t + u < r$ and $r + s + v \leq t + 3$, we deduce that $t + u + s < r + s + v = \{t + 2\}$. The case of $t + u + s < t + 2$ does not occur because $u \geq 1$ and $s \geq 1$. There remains the case of $t + u + s < t + 3$. This inequality implies that $s = 1, u = 1, v = 0$ and $t = t + 2$. Thus, the form $B_2$ is of the type $B_2 = \sum X_0^{t+2}X_1^2X_2^3X_3^{t+1}X_4^3$ and

$-B_2a_{13020}X_0X_1^2X_2^3 = -a_{13020}(\sum X_0^{t+3}X_1^2X_2^3X_3^{t+1}X_4^3)$,

where $t \geq 0$. But this is a global 5($t + 3$)-canonical adjoint to $V''$ and the statement in Lemma 5 is true because $F'''_m = -B_2a_{13020}X_0X_1^2X_2^3 + G_m$ and $F'''_m = B_2(a_{30201}X_0^3X_2^3X_4 + a_{01302}X_1X_2^3X_4^2 + a_{20130}X_0X_2X_3 + a_{02013}X_1^2X_3X_4^3) + G_m$, where $G_m$ defines global $m$-canonical adjoints; in fact, the monomials of $G_m$ as addenda of $F'''_m$ satisfy $i'' \leq k', j'' \leq h', k'' \leq i'$ and, as addenda of $F'''_m$ satisfy $l'' \leq j''$, $k'' \leq l''$, with $j'' = j', k'' = k''$, $l'' = l''$.

So, in the case of the inequality $l' > j'$, Lemma 5 is proved.

We have examined the inequality $l' > j'$ and it remains for us to consider the inequality $k' > l'$. Here, applying the same proof as for $l' > j'$, we prove our thesis. So Lemma 5 is completely proved.
Proof of Proposition 3. The proof is immediate because, from Lemma 5, to compute the plurigenera of \( Z_5 \), we can assume that the \( m \)-canonical adjoints to \( V'' \) are global; the statement in Proposition 3 therefore follows from Lemma 1.

\[
\]

19. – The irregularities of \( Z_5 \).

With the same proof as in the case of \( Z_3 \), cf. section 6, the irregularities of \( Z_5 \) are \( q_1 = q_2 = 0 \).

This completes the construction of the third threefold \( Z_5 \).

Appendix

With a construction similar to that of \( Z_4 \), but imposing only three of the four singularities imposed on \( V' \subset \mathbb{P}^4 \) at the four vertices \( A_0, A_1, A_3, A_4 \), we obtain a new hypersurface \( V'' \) such that a desingularization \( X \) of \( V'' \) is a threefold of general type. This threefold \( X \) has the birational invariants \( q_1 = q_2 = 0, p_g = 1, P_2 = 2 \) and its \( m \)-canonical transformation is birational if and only if \( m \geq 11 \).

For instance, let us choose the vertex \( A_4 \) and put no singularities at \( A_4 \), while imposing on \( V'' \) the same singularities as on \( V' \) in section 8 at the other vertices.

Lemma 1 (section 5) and Lemma 2 (section 12) clearly hold, but the proof of Proposition 2 (also in section 12) has to be modified, removing the conditions given by the singularity at \( A_4 \). So, the remaining conditions are the inequalities

\[
\begin{align*}
&j + h + 2l + 4 \geq m \\
&j' + h' + 2l' \geq m \\
&i + 2h + l + 4 \geq m \\
&i' + 2h' + l' \geq m \\
&2i + j + l + 4 \geq m \\
&2i' + j' + l' \geq m
\end{align*}
\]

regarding the following equation of the \( m \)-canonical adjoints to \( V'' \) (loc. cit.)

\[
\Phi_m : B''(X_0, X_1, X_3, X_4) \varphi_4(X_0, X_1, X_3, X_4)X_2 + \psi_m(X_0, X_1, X_3, X_4) = 0.
\]

The union of these inequalities now gives \( \begin{align*}
l \geq i + 1 \geq h + 2 \geq j + 3, \\
l' \geq i' \geq h' \geq j'
\end{align*} \), where \( i + j + h + l = m - 5 \) and \( i' + j' + h' + l' = m \).

In this case, \( B''(X_0, X_1, X_3, X_4) \) can differ from zero: the first value of \( m \) for which \( B''(X_0, X_1, X_3, X_4) \neq 0 \) is 11, according to the values: \( l = 3, i = 2, h = 1 \) and \( j = 0 \). Moreover, the irregularities of \( X \) are \( q_1 = q_2 = 0 \) and the first plurigenera of \( X \) are given by

\[
p_g = 1, \text{ because } \Phi_1 : \lambda_1X_4 = 0 (l' = 1, i' = h' = j' = 0),
\]
\[ P_2 = 2, \text{ because } \Phi_2 : X_4(\lambda_1 X_4 + \lambda_2 X_0) = 0 \begin{cases} l' = 2, l' = h' = j' = 0 \\ l' = i', h' = j' = 0 \end{cases}, \]
\[ P_3 = 3, \text{ because } \Phi_3 : X_4(\lambda_1 X_4^2 + \lambda_2 X_0 X_4 + \lambda_3 X_0 X_3) = 0, \]
\[ P_4 = 5, \text{ because } \Phi_4 : X_4(\lambda_1 X_4^3 + \lambda_2 X_0 X_4^2 + \lambda_3 X_0^2 X_4 + \lambda_4 X_0 X_3 X_4 + \lambda_5 X_0 X_1 X_3) = 0, \]
where \( \lambda_i \in k. \)
\[ P_5 = 6, P_6 = 9, P_7 = 11, P_8 = 14, P_9 = 17. \]

Now, considering the linear system of 4-canonical adjoints to \( V'' \), it is not difficult to see that the 4-canonical transformation \( \varphi_{[4 \mathcal{K}_X]} \), where \( \mathcal{K}_X \) is a canonical divisor on \( X \), is generically a rational transformation \( 2 : 1 \). Roughly speaking, in an open set of \( X \), \( \varphi_{[4 \mathcal{K}_X]} \) can be identified with \( \varphi_{[V'']} \), where \( \varphi \) is the rational transformation defined by the linear system of 4-canonical adjoints (cf. for instance \([S_1]\), section 16); the equation for \( V'' \) is of the type \( (\ldots) X_2^2 + (\ldots) = 0 \); the equations for the 4-canonical adjoints do not contain the variable \( X_2 \); so two distinct points, that are mapped to one point, are of the type \((a, b, x_2, c, d), (a, b, -x_2, c, d)\). Since \( p_g > 0 \), it follows that \( \varphi_{[m \mathcal{K}_X]} \) is either generically \( 2 : 1 \) or birational for \( m > 4 \). Note that \( \varphi_{[m \mathcal{K}_X]} \) is not generically \( n : 1 \) for \( m < 4 \).

Next, the first value of \( m \) for which \( B''(X_0, X_1, X_3, X_4) \) is \( \neq 0 \) is 11 (see above); the \( m \)-canonical adjoint \( \Phi_m : B''(X_0, X_1, X_3, X_4)X_2 = 0 \) “separates” the two points \((a, b, x_2, c, d), (a, b, -x_2, c, d)\) in the rational transformation \( \varphi_{[11 \mathcal{K}_X]} \), thanks to the presence of the variable \( X_2 \) to the power 1. We thus deduce that \( \varphi_{[11 \mathcal{K}_X]} \) is a birational transformation. Again from \( p_g > 0 \), it follows that \( \varphi_{[m \mathcal{K}_X]} \) is a birational transformation for \( m \geq 11. \)

Therefore, we have proved that

the \( m \)-canonical transformation (improperly called a ‘map’) of the threefold \( X \) is generically \( 2 : 1 \) if and only if \( 10 \geq m \geq 4 \) and it is birational if and only if \( m \geq 11. \)

We note that \( X \) is birationally distinct from the threefolds appearing in the lists of \([Re]\), pp. 358-359 and \([F]\), pp. 151-154, because \( X \) has different plurigenera from those of the threefolds in said lists.

Based on a result given by M. Chen \([C]\), we know that a threefold, with the bigenus \( P_2 \geq 2 \), has the \( m \)-canonical transformation that is birational for \( m \geq 16. \)

As a consequence of this and of the above result, the optimal limitation for the birationality of the \( m \)-canonical transformation for threefolds with \( P_2 = 2 \) is now between 11 and 16.

We also constructed a nonsingular threefold \( Y \) of general type in \([S_2]\) where \( \varphi_{[m \mathcal{K}_X]} \) birational if and only if \( m \geq 11 \), but in that case \( Y \) has \( p_g = 0 \) and \( P_2 = 1. \)
Added in proofs. Meng Chen and Kang Zuo have proved that a nonsingular algebraic threefold of general type with \( p_g = 1 \) and \( P_2 = 2 \) has the \( m \)-canonical map (transformation) \( \varphi_{|mK|} \) which is birational for \( m \geq 11 \) (cf. Theorem 4.4 in M. Chen - K. Zuo, Complex projective threefolds with non-negative canonical Euler-Poincaré characteristic, preprint, arXiv:math/0609545v2 [math.AG] 23 Oct. 2007). Our Example in the Appendix proves that such a limitation is optimal. Another example, having \( p_g = 1 \) and \( P_2 = 2 \) with \( \varphi_{|11K|} \) birational and \( \varphi_{|10K|} \) non-birational, like our example, was also presented by Iano-Fletcher (loc. cit. Example 4.8).

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