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Correctors for Parabolic Equations in a Heterogeneous Fibered Medium.

MOURAD SFAXI - ALI SILI

Sunto. – *Studiamo un problema di correttori nell'ambito dell'omogeneizzazione di equazioni lineari paraboliche in un mezzo eterogeneo Ω formato da due materiali. Il primo materiale è localizzato in un insieme F_ε di fibre cilindriche parallele, periodicamente distribuite con un periodo di taglia ε, il secondo materiale è localizzato nella "matrice" $M_\varepsilon = \Omega - F_\varepsilon$. Il rapporto tra i coefficienti di conduttività dei due materiali è dell'ordine di $1/\varepsilon^2$.*

Dopo aver scritto il problema omogeneizzato, diamo un risultato di correttore e proviamo che la soluzione u_ε del problema iniziale è della forma $u_\varepsilon = \tilde{u}_\varepsilon + \hat{u}_\varepsilon$, dove $\tilde{u}_\varepsilon$ è un correttore per u_ε e $\hat{u}_\varepsilon$ è uno strato limite nel tempo. Contrariamente ai risultati noti per le equazioni paraboliche, tale strato limite non è concentrato vicino all'origine $t = 0$, ma perdura almeno per ogni $t \in (0, m)$ per un certo $m > 0$. La dimostrazione di tale risultato è basata sul fatto che $\hat{u}_\varepsilon$ non converge, in generale, nella topologia forte di $L^2(\Omega \times (0, T))$.

Summary. – *We study the problem of correctors in the framework of the homogenization of linear parabolic equations posed in a heterogeneous medium Ω made of two materials.*

The first one is located in a set F_ε of cylindrical parallel fibers periodically distributed with a period of size ε, and the second one is located in the "matrix" $M_\varepsilon = \Omega - F_\varepsilon$. The ratio between the conductivity coefficients of the two materials is of order $1/\varepsilon^2$.

After writing the homogenized problem, we give a corrector result and prove that the solution u_ε of the starting problem is of the form $u_\varepsilon = \tilde{u}_\varepsilon + \hat{u}_\varepsilon$, where $\tilde{u}_\varepsilon$ is a corrector for u_ε and $\hat{u}_\varepsilon$ is a time boundary layer. In contrast to the known results for parabolic equations, this boundary layer is not concentrated about the time origin $t = 0$, but it remains at least for all $t \in (0, m)$ with some $m > 0$. The proof of the latter is based on the fact that $\hat{u}_\varepsilon$ does not converge, in general, in $L^2(\Omega \times (0, T))$ for the strong topology.

1. – The problem in the heterogeneous medium.

This paper is devoted to the study of homogenization and of problem of correctors for the two following initial boundary value problems:

$$
(1.1)\qquad \begin{cases} \rho_\varepsilon \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div} C_\varepsilon(x)\nabla u_\varepsilon & = & f(x,t) & \text{in} & \Omega \times (0,T), \\ u_\varepsilon & = & 0 & \text{on} & \Gamma \times (0,T), \\ C_\varepsilon(x)\nabla u_\varepsilon . n & = & 0 & \text{on} & (\partial\Omega - \Gamma) \times (0,T), \\ u_\varepsilon(x,0) & = & u_0^\varepsilon(x) & \text{in} & \Omega, \end{cases}
$$

where $\rho_\varepsilon = 1$ or $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$ and where n is the outer normal to the boundary $\partial\Omega$ of the cube Ω. We assume that Ω is the reference configuration of a heterogeneous medium made up from two materials. The first one lying in a set F_ε of parallel cylindrical fibers, is assumed to have a conductivity coefficient of order 1, while the second material occupies the region $M_\varepsilon = \Omega - F_\varepsilon$ and its conductivity coefficient is of order ε^2, see Figure 1.

In the case $\rho_\varepsilon = 1$, we prove that the homogenized problem consists on a parabolic system which couples the temperature v through the fibers and the temperature u outside them. The diffusion, in this case, is instationary along the fibers as well as outside them since the parabolic part of the limiting operator has the form $\partial u/\partial t + \partial v/\partial t$.

In the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$, the homogenized problem which involves once again the two temperatures v and u loses the parabolic part $\partial u/\partial t$, because "the limit" of $\varepsilon^2 \chi_{M_\varepsilon}$ is equal to zero. Hence, in the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$, we obtain at the limit a parabolic equation for the diffusion inside the fibers together with a stationary equation outside the fibers in which the time t plays the role of a parameter.

The case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$ is very close to the elliptic setting addressed in [9].

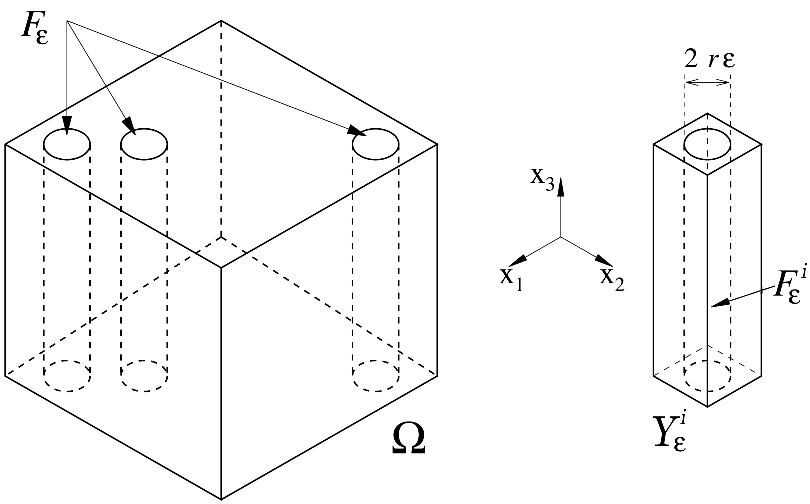

Fig. 1. – The domain Ω, the cell Y_ε^i and the fiber F_ε^i

In particular, one can emphasize as in [9] the nonlocal nature of the homogenized problem through another formula which gives the temperature v and the average z of the temperature outside the fibers, see Theorem 1.2 below.

The key feature of this work is the study of the effect of the initial data on the asymptotic behaviour of the solution u_ε of problem (1.1) as ε goes to zero, or the so-called problem of correctors, see [2], [3] and [4].

In the classical setting where the matrix C_ε is uniformly bounded (*with respect to* ε) from below and from above, such problem for the heat and the wave equations was considered in [3], in the framework of the nonperiodic homogenization. The main tool used in the latter is the H-convergence method due to Murat and Tartar, see [6] and [11].

As far as the parabolic equation is concerned, it was shown in [3] that the solution u_ε of the starting problem may be decomposed as a sum $u_\varepsilon = \tilde{u}_\varepsilon + \hat{u}_\varepsilon$, where $\tilde{u}_\varepsilon$ is a corrector for u_ε, in the sense that the two sequences E_ε and $\widetilde{E}_\varepsilon$ of energies associated respectively to u_ε and $\tilde{u}_\varepsilon$ converge to the same limit.

In the case of a sequence u_0^ε of initial data converging weakly but not strongly in $L^2(\Omega)$, it was proved in [3] that the term $\hat{u}_\varepsilon$ represents a time boundary layer which is concentrated near the time origin $t = 0$. This means in particular that $\hat{u}_\varepsilon$ converges weakly to zero in $L^2(0, T; H^1(\Omega))$, but such convergence becomes strong in $L^2(\delta, T; H^1(\Omega))$ for all δ, $0 < \delta < T$.

In contrast to the previous result, we prove here that if one assume merely the weak convergence in $L^2(\Omega)$ of the sequence u_0^ε, then the time boundary layer $\hat{u}_\varepsilon$ associated to the solution u_ε of (1.1) remains at least until a time $t = m$, with some m satisfying $0 < m \le T$. This result holds in both cases $\rho_\varepsilon = 1$ and $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$.

Let us point out that if one takes $\rho_\varepsilon = \chi_A + \varepsilon^2 \chi_{\Omega - A}$, where A is a fixed subdomain of Ω, then the boundary layer is concentrated at $t = 0$, as shown in [7]. Hence the propagation of the perturbation we observe here with $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$ clearly comes from the homogenization process.

Before stating our results, let us make more precise our notations and hypotheses. Let C be the square $C = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)^2$ and let $\Omega = C \times \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) = C \times I$ be the corresponding cube of $\mathbb{R}^3$. The union of the upper and lower faces of Ω is denoted by Γ. Let F_ε be a set of parallel fibers periodically distributed in Ω with a period εY where $Y = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)^2$ and let $D = D(0, r)$ be the disc of radius r, $0 < r < \frac{1}{2}$, contained in Y. Finally, we set $M_\varepsilon = \Omega - F_\varepsilon$ for the matrix surrounding the fibers F_ε, see Figure 1.

Troughout this paper, we denote a generic point x of $\mathbb{R}^3$ by $x = (x', x_3)$, with $x' = (x_1, x_2)$. Similarly, we denote a vector-field g in $\mathbb{R}^3$ by $g = \begin{pmatrix} g' \\ g_3 \end{pmatrix}$, with $g' = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$.

We use the notation ∇'_y to express the gradient with respect to the variable $y = (y_1, y_2)$. The characteristic function of a Borel set A will be denoted by χ_A.

We assume that C is a partition of squares C^i_ε of size ε, each of them containing the disc D^i_ε of center $x^i_\varepsilon = \varepsilon(i_1, i_2)$ and of radius $r\varepsilon$, with $0 < r < \dfrac{1}{2}$.

The cell Y^i_ε is defined by $Y^i_\varepsilon = C^i_\varepsilon \times I$, in such a way that

$$\Omega = \bigcup_{i \in I_\varepsilon} Y^i_\varepsilon, \quad \text{where} \quad I_\varepsilon = \big\{\, i = (i_1, i_2) \in \mathbb{Z}^2,\, C^i_\varepsilon \subset\subset C \,\big\}. \tag{1.2}$$

The fiber F^i_ε, the set F_ε of all fibers and the matrix M_ε surrounding them are defined by

$$F^i_\varepsilon = D^i_\varepsilon \times I, \qquad F_\varepsilon = \bigcup_{i \in I_\varepsilon} F^i_\varepsilon, \qquad M_\varepsilon = \Omega - F_\varepsilon. \tag{1.3}$$

We assume that the matrix C_ε arising in (1.1) is defined by

$$C_\varepsilon(x) = A\left(x, \frac{x'}{\varepsilon}\right)\chi_{F_\varepsilon}(x) + \varepsilon^2 B\left(x, \frac{x'}{\varepsilon}\right)\chi_{M_\varepsilon}, \quad \text{a.e. } x \in \Omega, \tag{1.4}$$

where $A(x, y)$ and $B(x, y)$ are two positive definite symmetric matrices defined on $\Omega \times Y$, Y-periodic in y with coefficients in $L^\infty(\Omega \times Y)$, i.e., there exist two positive constants $0 < \delta \leq \gamma$, such that

$$\delta \sum_{i=1}^{3} \xi_i^2 \leq \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}(x, y)\xi_j.\xi_i \leq \gamma \sum_{i=1}^{3} \xi_i^2, \quad \forall\, \xi \in \mathbb{R}^3, \text{ a.e. } (x, y) \in \Omega \times Y, \tag{1.5}$$

Furthermore, A and B are assumed to be admissible test functions in the sense of the two-scale convergence (see [1]): this means that the coefficients $A_{ij}\left(x, \dfrac{x'}{\varepsilon}\right)$ are measurable and satisfy

$$\lim_{\varepsilon \to 0} \int_\Omega \left| A_{ij}\left(x, \frac{x'}{\varepsilon}\right) \right|^2 dx = \int_\Omega \int_Y |A_{ij}(x, y)|^2 \, dxdy. \tag{1.6}$$

We also need to introduce the following spaces

$$V = \big\{\, u \in H^1(\Omega),\, u = 0 \quad \text{on} \quad \Gamma \,\big\}, \tag{1.7}$$

where $H^1(\Omega)$ is the usual Sobolev space,

$$H^1_{\#(Y-D)}(Y) = \Big\{\, u \in H^1_{\text{loc}}(\mathbb{R}^2),\, u \text{ is } Y\text{-periodic, } \operatorname{supp} u_{|Y} \subset Y - D \,\Big\}, \tag{1.8}$$

$$H^1_m(D) = \left\{\, u \in H^1(D), \int_D u \, dy = 0 \,\right\}, \tag{1.9}$$

$$H_0^1(I) = \left\{ u \in H^1(I),\ u\left(-\frac{1}{2}\right) = u\left(\frac{1}{2}\right) = 0 \right\}, \tag{1.10}$$

$$\mathcal{U}_1 = \left\{ \begin{array}{l} u \in L^2\Big(\Omega \times (0,T); H^1_{\#(Y-D)}(Y)\Big) \cap C\big([0,T]; L^2(\Omega \times Y)\big), \\ \dfrac{\partial u}{\partial t} \in L^2\big(0,T; L^2(\Omega \times Y)\big) \end{array} \right\}, \tag{1.11}$$

$$\mathcal{U}_2 = L^2\Big(\Omega \times (0,T); H^1_{\#(Y-D)}(Y)\Big), \tag{1.12}$$

$$\mathcal{V} = \left\{ v \in L^2\big(C \times (0,T); H_0^1(I)\big) \cap C\big([0,T]; L^2(\Omega)\big)\ ;\ \frac{\partial v}{\partial t} \in L^2\big(0,T; L^2(\Omega)\big) \right\}, \tag{1.13}$$

$$\mathcal{W} = L^2\big(\Omega \times (0,T); H^1_m(D)\big). \tag{1.14}$$

We also assume

$$f \in H^1\big(0,T; L^2(\Omega)\big), \quad \text{if} \quad \rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}, \tag{1.15}$$

but only

$$f \in L^2\big(0,T; L^2(\Omega)\big), \quad \text{if } \rho_\varepsilon = 1, \tag{1.15'}$$

and

$$\begin{cases} u_0^\varepsilon \in H^1(\Omega) \quad \forall\, \varepsilon, \\ \displaystyle\int_\Omega |u_0^\varepsilon|^2\, dx \le c, \\ \displaystyle\int_\Omega \Big\{ |\nabla u_0^\varepsilon|^2 \chi_{F_\varepsilon} + \varepsilon^2 |\nabla u_0^\varepsilon|^2 \chi_{M_\varepsilon} \Big\}\, dx \le c, \\ \text{there exists } u_0 \in L^2(\Omega \times Y) \text{ such that } u_0^\varepsilon \rightharpoonup\!\!\rightharpoonup u_0(x,y), \end{cases} \tag{1.16}$$

where the symbol "$\rightharpoonup\!\!\rightharpoonup$" stands for the two-scale convergence of the sequence u_0^ε to u_0.

REMARK 1.1. – Hypotheses (1.15) and (1.16) are not optimal since other weaker hypotheses are possible in order to get existence and uniqueness of the solution u_ε of problem (1.18) below in a larger class.

However, we assume (1.15) and (1.16) in order to get solutions u_ε with uniformly (with respect to ε) bounded derivatives in time in the space $L^2\big(0,T; L^2(\Omega)\big)$ (see inequality (4.11) in the Appendix). Note also (see the Appendix) that the hypothesis $\dfrac{\partial f}{\partial t} \in L^2\big(0,T; L^2(\Omega)\big)$ is useful only in the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$.

An example of a sequence satisfying (1.16) can be obtained for instance by considering the solution u_0^ε of the elliptic problem:

$$\begin{cases} -\operatorname{div} C_\varepsilon(x)\nabla u_0^\varepsilon &= g \quad \text{in} \quad \Omega, \\ u_0^\varepsilon &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where g is a given function of $L^2(\Omega)$ and the matrix C_ε is defined in (1.4).

From the results obtained in [9], it is clear that the sequence u_0^ε satisfies (1.16). Furthermore, it was shown in [9], that

$$u_0^\varepsilon \rightharpoonup\rightharpoonup u_0(x,y) + v_0(x),$$

$$(\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon})u_0^\varepsilon \rightharpoonup\rightharpoonup \chi_D(y)\, v_0(x),$$

where

$$u_0 \in L^2\Big(\Omega; H^1_{\#(Y-D)}(Y)\Big) \quad \text{and} \quad v_0 \in L^2\big(C; H^1_0(I)\big).$$

One can easily verify that the sequences u_0^ε and $(\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon})u_0^\varepsilon$ cannot strongly converge in $L^2(\Omega)$.

To describe respectively the average of the temperature in the fibers and the effect of the anisotropy of the fibers, we introduce the following sequences:

$$\begin{cases} v_\varepsilon = \dfrac{1}{\pi r^2}\, u_\varepsilon \chi_{F_\varepsilon}, \\ w_\varepsilon = \displaystyle\sum_{i\in I_\varepsilon} \left(\frac{u_\varepsilon}{\varepsilon} - \frac{1}{|D^i_\varepsilon|} \int_{D^i_\varepsilon} \frac{u_\varepsilon}{\varepsilon}\, dx' \right) \chi_{D^i_\varepsilon}. \end{cases} \tag{1.17}$$

Let $T > 0$ be a given positive number and consider the following variational formulation of problems (1.1).

$$\begin{cases} u_\varepsilon \in L^2(0,T;V) \cap C\big([0,T]; L^2(\Omega)\big), \quad \rho_\varepsilon \dfrac{\partial u_\varepsilon}{\partial t} \in L^2(0,T;L^2(\Omega)), \\ \displaystyle\int_0^T\!\!\int_\Omega \rho_\varepsilon(x') \frac{\partial u_\varepsilon}{\partial t}(x,t)\phi(x,t)\, dxdt \\ \displaystyle + \int_0^T\!\!\int_\Omega \left(A\Big(x, \frac{x'}{\varepsilon}\Big) \nabla u_\varepsilon \chi_{F_\varepsilon}(x') + \varepsilon^2 B\Big(x, \frac{x'}{\varepsilon}\Big) \nabla u_\varepsilon \chi_{M_\varepsilon}(x') \right) \nabla\phi(x,t)\, dxdt \\ \displaystyle = \int_0^T\!\!\int_\Omega f(x,t)\phi(x,t)\, dxdt, \quad \forall\, \phi \in L^2(0,T;V), \\ u_\varepsilon(x,0) = u_0^\varepsilon(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1.18}$$

Under hypotheses (1.15) and (1.16), the proof of the existence and the uniqueness of the solution u_ε of (1.18) is given at the end of the paper, in the Appendix, see also [5].

The homogenized problems.

Our result concerning the homogenized problems may be stated as follows.

THEOREM 1.1. – *Let u_ε be the sequence of solutions of* (1.18). *Then:*

i) If $\rho_\varepsilon(x') = 1$, there exists

$$(u, v, w) \in \mathcal{U}_1 \times \mathcal{V} \times \mathcal{W}, \tag{1.19}$$

such that:

$$u_\varepsilon \rightharpoonup\rightharpoonup u(x, y, t) + v(x, t), \tag{1.20}$$

$$u_\varepsilon \rightharpoonup z := \int_{Y-D} u(x, y, t)\, dy + v(x, t) \quad \textit{weakly in} \quad L^2\big(0, T, L^2(\Omega)\big), \tag{1.21}$$

$$v_\varepsilon \rightharpoonup v \quad \textit{weakly in} \quad L^2\big(0, T; L^2(C; H_0^1(I))\big), \tag{1.22}$$

$$\nabla u_\varepsilon \chi_{F_\varepsilon} \rightharpoonup\rightharpoonup \chi_D(y) \begin{pmatrix} \nabla'_y w \\ \dfrac{\partial v}{\partial x_3} \end{pmatrix}, \tag{1.23}$$

$$\varepsilon \nabla u_\varepsilon \chi_{M_\varepsilon} \rightharpoonup\rightharpoonup \chi_{(Y-D)}(y) \begin{pmatrix} \nabla'_y u \\ 0 \end{pmatrix}, \tag{1.24}$$

$$\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup\rightharpoonup \frac{\partial u}{\partial t}(x, y, t) + \frac{\partial v}{\partial t}(x, t). \tag{1.25}$$

Furthermore, (u, v, w) *is the unique solution of the homogenized problem:*

$$\begin{cases} (u, v, w) \in \mathcal{U}_1 \times \mathcal{V} \times \mathcal{W}, \\ \displaystyle\int_0^T \int_\Omega \int_Y \left(\frac{\partial u}{\partial t}(x, y, t) \ + \ \frac{\partial v}{\partial t}(x, t) \right) \big(\bar{u}(x, y, t) \ + \ \bar{v}(x, t)\big)\, dx dy dt \\ \displaystyle + \int_0^T \int_\Omega \int_Y \chi_D(y) A(x, y) \begin{pmatrix} \nabla'_y w \\ \dfrac{\partial v}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} \nabla'_y \bar{w} \\ \dfrac{\partial \bar{v}}{\partial x_3} \end{pmatrix} dx dy dt \\ \displaystyle + \int_0^T \int_\Omega \int_Y \chi_{(Y-D)}(y) B(x, y) \begin{pmatrix} \nabla'_y u \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla'_y \bar{u} \\ 0 \end{pmatrix} dx dy dt \\ \displaystyle = \int_0^T \int_\Omega \int_Y f(x, t) \big(\bar{u}(x, y, t) + \bar{v}(x, t)\big)\, dx dy dt, \\ \forall (\bar{u}, \bar{v}, \bar{w}) \in \mathcal{U}_1 \times \mathcal{V} \times \mathcal{W}, \\ \displaystyle u(x, y, 0) = u_0(x, y) - \frac{1}{\pi r^2} \int_D u_0(x, y)\, dy, \\ \displaystyle v(x, 0) = \frac{1}{\pi r^2} \int_D u_0(x, y)\, dy. \end{cases} \tag{1.26}$$

ii) If $\rho_\varepsilon(x') = \chi_{F_\varepsilon}(x') + \varepsilon^2\chi_{M_\varepsilon}(x')$*, there exists*

$$(u, v, w) \in \mathcal{U}_2 \times \mathcal{V} \times \mathcal{W}, \tag{1.27}$$

such that convergences (1.20)-(1.24) *remain true, while convergence* (1.25) *must be replaced by*

$$\rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup\!\!\rightharpoonup \chi_D(y) \frac{\partial v}{\partial t}(x,t). \tag{1.28}$$

Furthermore, (u, v, w) *is the unique solution of the problem:*

$$\begin{cases} (u, v, w) \in \mathcal{U}_2 \times \mathcal{V} \times \mathcal{W}, \\ \pi r^2 \int_0^T \int_\Omega \frac{\partial v}{\partial t}(x,t)\, \bar{v}(x,t)\, dxdt \\ + \int_0^T \int_\Omega \int_Y \chi_D(y) A(x,y) \begin{pmatrix} \nabla'_y w \\ \dfrac{\partial v}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} \nabla'_y \bar{w} \\ \dfrac{\partial \bar{v}}{\partial x_3} \end{pmatrix} dxdydt \\ + \int_0^T \int_\Omega \int_Y \chi_{(Y-D)}(y) B(x,y) \begin{pmatrix} \nabla'_y u \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla'_y \bar{u} \\ 0 \end{pmatrix} dxdydt \\ = \int_0^T \int_\Omega \int_Y f(x,t) \big(\bar{u}(x,y,t) + \bar{v}(x,t) \big)\, dxdydt, \\ \forall (\bar{u}, \bar{v}, \bar{w}) \in \mathcal{U}_2 \times \mathcal{V} \times \mathcal{W}, \\ v(x,0) = \dfrac{1}{\pi r^2} \displaystyle\int_D u_0(x,y)\, dy. \end{cases} \tag{1.29}$$

In the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}$, the equation wearing on u in (1.29) may be separated from the equation on v in such a way that the homogenized problem takes the following form:

THEOREM 1.2. – *The solution* (u, v, w) *of problem* (1.29) *satisfies the following equality:*

$$u(x,y,t) = f(x,t)\, \hat{u}(x,y,t), \tag{1.30}$$

$$w(x,y,t) = \frac{\partial v}{\partial x_3}(x,t)\, \hat{w}(x,y,t), \tag{1.31}$$

where $\hat{u}$ *and* $\hat{w}$ *are the solutions of the problems:*

$$
\left\{
\begin{aligned}
&\hat{u} \in L^\infty\big(\Omega \times (0,T); H^1_{\#(Y-D)}(Y)\big),\\
&\int\limits_{Y-D} \left[B(x,y)\begin{pmatrix}\nabla'_y \hat{u}(x,y,t)\\ 0\end{pmatrix}\right]'.\nabla'_y \bar{u}(y)\,dy = \int\limits_{Y-D} \bar{u}(y)\,dy,\\
&\forall\, \bar{u} \in H^1_{\#(Y-D)}(Y), \quad \text{a.e} \quad (x,t) \in \Omega \times (0,T),
\end{aligned}
\right.
\tag{1.32}
$$

$$
\left\{
\begin{aligned}
&\hat{w} \in L^\infty\big(\Omega \times (0,T); H^1_m(D)\big),\\
&\int\limits_D \left[A(x,y)\begin{pmatrix}\nabla'_y \hat{w}(x,y,t)\\ 1\end{pmatrix}\right]'.\nabla'_y \bar{w}(y)dy = 0,\\
&\forall\, \bar{w} \in H^1_m(D), \quad \text{a.e} \quad (x,t) \in \Omega \times (0,T).
\end{aligned}
\right.
\tag{1.33}
$$

Defining $m(x,t)$ *and* $A_0(x,t)$ *as:*

$$
m(x,t) = \frac{1}{\displaystyle\int\limits_{Y-D} \hat{u}(x,y,t)\;dy}, \tag{1.34}
$$

$$
A_0(x,t) = \int\limits_D \left[A(x,y)\begin{pmatrix}\nabla'_y \hat{w}(x,y,t)\\ 1\end{pmatrix}\right]_3 dy, \tag{1.35}
$$

the homogenized problem (1.29) *may be written as:*

$$
\left\{
\begin{aligned}
&m(x,t)\,\Big(z(x,t) - v(x,t)\Big) = f(x,t) \quad in \quad \Omega \times (0,T),\\
&\pi r^2 \int\limits_0^T\int\limits_\Omega \frac{\partial v}{\partial t}(x,t)\,\bar{v}(x,t)\,dxdt + \int\limits_0^T\int\limits_\Omega A_0(x,t)\frac{\partial v}{\partial x_3}(x,t)\,\frac{\partial \bar{v}}{\partial x_3}(x,t)\,dxdt\\
&= \int\limits_0^T\int\limits_\Omega f(x,t)\bar{v}(x,t)\,dxdt, \quad \forall\, \bar{v} \in L^2\big(0,T; L^2\big(C; H^1_0(I)\big)\big),\\
&v(x,0) = \frac{1}{\pi r^2}\int\limits_D u_0(x,y)\,dy,
\end{aligned}
\right.
\tag{1.36}
$$

where z is defined by (1.21).

The proof of (1.36) is very close to the one given in [9] in the elliptic setting. It will not be reproduced here.

The corrector results.

We now state our corrector results. For this purpose, we introduce the following equations.

We define $\tilde{u}_\varepsilon$ and $\hat{u}_\varepsilon$ as the solutions of

$$
(1.37)\qquad \begin{cases} \rho_\varepsilon \dfrac{\partial \tilde{u}_\varepsilon}{\partial t} - \operatorname{div} C_\varepsilon(x)\nabla \tilde{u}_\varepsilon = f(x,t) & \text{in} \quad \Omega \times (0,T), \\ \tilde{u}_\varepsilon = 0 & \text{on} \quad \Gamma \times (0,T), \\ C_\varepsilon(x)\nabla \tilde{u}_\varepsilon . n = 0 & \text{on} \quad (\partial\Omega - \Gamma) \times (0,T), \\ \tilde{u}_\varepsilon(x,0) = a_\varepsilon(x) & \text{in} \quad \Omega, \end{cases}
$$

$$
(1.38)\qquad \begin{cases} \rho_\varepsilon \dfrac{\partial \hat{u}_\varepsilon}{\partial t} - \operatorname{div} C_\varepsilon(x)\nabla \hat{u}_\varepsilon = 0 & \text{in} \quad \Omega \times (0,T), \\ \hat{u}_\varepsilon = 0 & \text{on} \quad \Gamma \times (0,T), \\ C_\varepsilon(x)\nabla \hat{u}_\varepsilon . n = 0 & \text{on} \quad (\partial\Omega - \Gamma) \times (0,T), \\ \hat{u}_\varepsilon(x,0) = u_0^\varepsilon(x) - a_\varepsilon(x) & \text{in} \quad \Omega, \end{cases}
$$

where a_ε is defined as follows:

$$
(1.39)\qquad \begin{cases} a_\varepsilon(x) = u_0\Big(x, \dfrac{x'}{\varepsilon}\Big) \quad \text{if} \quad \rho_\varepsilon = 1, \\ a_\varepsilon(x) = \dfrac{1}{\pi r^2} \displaystyle\int_D u_0(x,y)\,dy \quad \text{if} \quad \rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}. \end{cases}
$$

Then, we have the following proposition.

PROPOSITION 1.1. -- *If* a_ε *satisfies the hypothesis* (1.16), *then we have:*

$$
(1.40)\qquad u_\varepsilon = \tilde{u}_\varepsilon + \hat{u}_\varepsilon .
$$

Indeed, assuming the uniform estimate arising in (1.16) on a_ε, $\tilde{u}_\varepsilon$ and $\hat{u}_\varepsilon$ belong to the same class (*defined in* (1.18)) as u_ε. Hence uniqueness of u_ε leads to identity (1.40).

Remark that if one choose u_0^ε to be the sequence given after Remark 1.1, then one can verify that the sequence a_ε defined by (1.39) satisfies hypothesis (1.16) at least when some regularity is assumed on u_0 and v_0.

From now on, we assume that (1.16) is satisfied by a_ε.

THEOREM 1.3. – *Assume that the limit* (u,v,w) *and* u_0 *satisfy the following regularity hypotheses:*

$$
(1.41)\qquad \begin{cases} u \in L^2\big(\Omega \times (0,T); C_\#(Y)\big), \ \nabla'_y u\,, \ \nabla'_y w \in \big(L^2\big(\Omega \times (0,T); C_\#(Y)\big)\big)^2, \\ \lim\limits_{\varepsilon \to 0} \displaystyle\int_\Omega \Big| u_0\Big(x, \frac{x'}{\varepsilon}\Big)\Big|^2 dx = \int_\Omega\int_Y |u_0(x,y)|^2\, dx dy. \end{cases}
$$

Then the following convergence holds true:

$$\lim_{\varepsilon\to 0}\left\{\int_0^T\int_{F_\varepsilon}\left|\nabla\tilde{u}_\varepsilon-\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\right|^2 dxdt+\int_0^T\int_{M_\varepsilon}|\varepsilon\nabla\tilde{u}_\varepsilon-\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}|^2\,dxdt\right\}=0 \tag{1.42}$$

Convergence (1.42) means that the oscillations of the sequences $\nabla\tilde{u}_\varepsilon\chi_{F_\varepsilon}$ and $\varepsilon\nabla\tilde{u}_\varepsilon\chi_{M_\varepsilon}$ are all contained in their respective two-scale limits which are the same as those of $\nabla u_\varepsilon\chi_{F_\varepsilon}$ and $\varepsilon\nabla u_\varepsilon\chi_{M_\varepsilon}$.

REMARK 1.2. – If one assume regularity hypotheses (1.41), then we have the following convergences as a consequence of (1.42):

if $\rho_\varepsilon=1$, then

$$\lim_{\varepsilon\to 0}\left\|\tilde{u}_\varepsilon(x,t)-u\left(x,\frac{x'}{\varepsilon},t\right)-v(x,t)\right\|_{L^2(\Omega\times(0,T))}=0.$$

If $\rho_\varepsilon=\chi_{F_\varepsilon}+\varepsilon^2\chi_{M_\varepsilon}$, then

$$\lim_{\varepsilon\to 0}\left\|\rho_\varepsilon\tilde{u}_\varepsilon(x,t)-\sqrt{\pi r^2}\,v(x,t)\right\|_{L^2(\Omega\times(0,T))}=0.$$

Indeed, in the case $\rho_\varepsilon=1$, multiplying equation (1.37) by $\tilde{u}_\varepsilon$ and integrating over $(0,t)\times\Omega$, we obtain

$$\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\tilde{u}_\varepsilon|^2\,dx = \int_0^t\int_\Omega f\,\tilde{u}_\varepsilon\,dxds-\int_0^t\int_\Omega A\left(x,\frac{x'}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon.\nabla\tilde{u}_\varepsilon\chi_{F_\varepsilon}\,dxds\\ \qquad -\displaystyle\int_0^t\int_\Omega\varepsilon^2 B(x,\frac{x'}{\varepsilon})\nabla\tilde{u}_\varepsilon.\nabla\tilde{u}_\varepsilon\chi_{M_\varepsilon}\,dxds+\frac{1}{2}\int_\Omega|a_\varepsilon|^2\,dx.\end{cases}$$

Once again, we integrate over $(0,T)$ to get

$$\begin{cases}\dfrac{1}{2}\displaystyle\int_0^T\int_\Omega|\tilde{u}_\varepsilon|^2\,dxdt = \int_0^T\int_0^t\int_\Omega f\,\tilde{u}_\varepsilon\,dxdsdt-\int_0^T\int_0^t\int_\Omega A(x,\frac{x'}{\varepsilon})\nabla\tilde{u}_\varepsilon.\nabla\tilde{u}_\varepsilon\chi_{F_\varepsilon}\,dxdsdt\\ \qquad -\displaystyle\int_0^T\int_0^t\int_\Omega\varepsilon^2 B(x,\frac{x'}{\varepsilon})\nabla\tilde{u}_\varepsilon.\nabla\tilde{u}_\varepsilon\chi_{M_\varepsilon}\,dxdsdt+\frac{T}{2}\int_\Omega|a_\varepsilon|^2\,dx.\end{cases}$$

Note that, due to hypothesis (1.6) and strong convergence (1.42), the functions $A\left(x,\dfrac{x'}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon\chi_{F_\varepsilon}$ and $B\left(x,\dfrac{x'}{\varepsilon}\right)\varepsilon\nabla\tilde{u}_\varepsilon\chi_{M_\varepsilon}$ are admissible in the sense of the two-scale convergence (see [1]). Then, passing to the limit with respect to ε and taking into account the equality arising in (1.41), we get:

$$\left\{\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2}\int_0^T\int_\Omega |\tilde{u}_\varepsilon(x,t)|^2\,dxdt \;=\; & \int_0^T\int_0^t\int_{\Omega\times Y} f(x,t)\,(u(x,y,t)+v(x,t))\,dxdydsdt\\ & -\int_0^T\int_0^t\int_{\Omega\times Y}\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}.A(x,y)\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}dxdydsdt\\ & -\int_0^T\int_0^t\int_\Omega\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}.B(x,y)\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}dxdydsdt\\ & +\frac{T}{2}\int_{\Omega\times Y}|u_0(x,y)|^2\,dxdy.\end{aligned}\right.$$

Now, taking $(\bar{u},\bar{v},\bar{w})=\chi_{(0,t)}(u,v,w)$ as test function in (1.26) and integrating by parts, we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega |\tilde{u}_\varepsilon(x,t)|^2\,dxdt=\int_0^T\int_{\Omega\times Y}|u(x,y,t)+v(x,t)|^2\,dxdydt.$$

which is the desired result.

In the case $\rho_\varepsilon=\chi_{F_\varepsilon}+\varepsilon^2\chi_{M_\varepsilon}$, a similar proof can be performed by the use of equation (1.29).

As far as $\hat{u}_\varepsilon$ is concerned, we establish the following theorem:

THEOREM 1.4. – *If one replaces all the limits by zero, then convergences* (1.20)-(1.25) *and* (1.28) *take place in the sense of the weak topology of* $L^2(\Omega\times(0,T))$. *These convergences are strong in* $L^2(\Omega\times(0,T))$ *if and only if,*

$$\sqrt{\rho_\varepsilon}\,(u_0^\varepsilon-a_\varepsilon)\longrightarrow 0\quad\textit{strongly in } L^2(\Omega). \tag{1.43}$$

If convergence (1.43) *fails, then*

$$\left\{\begin{array}{l}\limsup\limits_{\varepsilon}\displaystyle\int_0^t\int_\Omega \rho_\varepsilon\,|\hat{u}_\varepsilon(x,s)|^2\,dxds\;>\;0,\quad\forall\,t\in(0,T),\\ \qquad\textit{and there exists }\; m,\,0<m\le T\quad\textit{such that}\\ \limsup\limits_{\varepsilon}\displaystyle\int_a^b\int_\Omega \rho_\varepsilon\,|\hat{u}_\varepsilon(x,s)|^2\,dxds\;>\;0,\quad\forall\,a,\,b\;:\;0\le a<b\le m.\end{array}\right. \tag{1.44}$$

REMARK 1.3. – One can easily verify that, without extra assumption on the regularity of the matrix C_ε, the sequence defined in Remark 1.1 does not satisfy (1.43) in the both cases $\rho_\varepsilon=1$ and $\rho_\varepsilon=\chi_{F_\varepsilon}+\varepsilon^2\chi_{M_\varepsilon}$.

The first inequality in (1.44) means that there is a subsequence of ε such that the L^2-norm of $\sqrt{\rho_\varepsilon}\,\hat{u}_\varepsilon$ tends to a strictly positive constant, while the second inequality in (1.44) shows that the time boundary layer $\hat{u}_\varepsilon$ remains at least until the time $t = m$.

In the next section, we shall prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.3. and of Theorem 1.4.

2. – Proof of Theorem 1.1.

Throughout the paper, we denote by c various constants which do not depend on ε.

As usual, we first establish apriori estimates.

LEMMA 2.1. – *The sequence* u_ε *of solutions of* (1.18) *and the sequences* v_ε, w_ε *defined by* (1.17) *fulfil the following apriori estimates:*

$$\| u_\varepsilon \|_{L^2(\Omega\times(0,T))} \leq c, \tag{2.1}$$

$$\| \sqrt{\rho_\varepsilon}\, u_\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{2.2}$$

$$\| v_\varepsilon \|_{L^2(C\times(0,T);H^1_0(I))} \leq c, \tag{2.3}$$

$$\| w_\varepsilon \|_{L^2(\Omega\times(0,T))} \leq c, \tag{2.4}$$

$$\| \nabla u_\varepsilon \chi_{F_\varepsilon} \|_{(L^2(\Omega\times(0,T)))^3} \leq c, \tag{2.5}$$

$$\| \varepsilon \nabla u_\varepsilon \chi_{M_\varepsilon} \|_{(L^2(\Omega\times(0,T)))^3} \leq c, \tag{2.6}$$

$$\| \sqrt{\rho_\varepsilon}\, \frac{\partial u_\varepsilon}{\partial t} \|_{L^2(\Omega\times(0,T))} \leq c. \tag{2.7}$$

PROOF OF LEMMA 2.1. – The following inequality

$$\int_\Omega |Z(x)|^2\, dx \leq c \int_\Omega (\chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}) |\nabla Z(x)|^2\, dx \quad \forall Z \in H^1(\Omega),\ Z = 0 \text{ on } \Gamma_D,$$

was proved in [9], inequality (3.16). The proof is based on the particular geometry of Ω together with the boundary condition satisfied by Z, and it does not depend on the equation solved by Z.

Hence, taking $Z = u_\varepsilon(.,t)$, $t \in (0,T)$, we get:

$$\int_\Omega |u_\varepsilon(x,t)|^2\, dx \leq c \int_\Omega (\chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}) |\nabla u_\varepsilon(x,t)|^2\, dx. \tag{2.8}$$

Taking $\phi(x,s) = u_\varepsilon(x,s)\chi_{(0,t)}$ in equation (1.18) and using the coerciveness of the matrices A and B, we get

$$\tag{2.9} \begin{cases} \displaystyle\frac{1}{2}\int_\Omega \rho_\varepsilon(x')\,|u_\varepsilon(x,t)|^2\,dx + \delta\int_0^t\int_\Omega (\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon})\,|\nabla u_\varepsilon(x,s)|^2\,dxds \\ \displaystyle\qquad\qquad \le \frac{1}{2}\int_\Omega \rho_\varepsilon(x')\,|u_0^\varepsilon(x)|^2\,dx + \int_0^t\int_\Omega f(x,s)\,u_\varepsilon(x,s)\,dxds. \end{cases}$$

Applying Young's inequality in the last integral, we get thanks to (1.15') (which is of course valid in the case $\rho_\varepsilon = 1$), (1.16) and (2.8)

$$\tag{2.10} \int_0^t\int_\Omega (\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon})\,|\nabla u_\varepsilon(x,s)|^2\,dxds \le c.$$

Turning back to (2.8), we see that (2.10) implies (2.1) in both cases $\rho_\varepsilon(x') = 1$ and $\rho_\varepsilon(x') = \chi_{F_\varepsilon}(x') + \varepsilon^2\chi_{M_\varepsilon}(x')$.

Estimate (2.2) is a consequence of (2.9), (2.1) and hypotheses (1.15)-(1.16).

Estimate (2.5) and (2.6) follow from (2.10).

To get (2.4), we use the following inequality

$$\int_\Omega \left(Z - \sum_{i\in I_\varepsilon} \frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} Z\,dx' \right)^2 \chi_{F_\varepsilon}\,dx \le c\int_\Omega \varepsilon^2|\nabla Z|^2\chi_{F_\varepsilon}\,dx, \quad \forall Z \in H^1(\Omega),$$

which was also proved in [9] (inequality (3.3)) with the help of Poincaré-Wirtinguer's inequality.

Since $\dfrac{\partial v_\varepsilon}{\partial x_3}(x,t) = \dfrac{1}{\pi r^2}\dfrac{\partial u_\varepsilon}{\partial x_3}(x,t)\,\chi_{F_\varepsilon}(x')$, estimate (2.3) follows from (2.10) and the boundary condition $u_\varepsilon = 0$ on $\Gamma \times (0,T)$.

Finally, (2.7) is a consequence of the lower semi-continuity of the L^2-norm and estimate (4.11) below. □

PROOF OF THEOREM 1.1. – Since from the point of view of two-scale convergence, the time variable t plays the role of a parameter, the proof of (1.19)-(1.24) is very close to the corresponding proof given in [9] in the elliptic setting, so that we only recall the main steps.

From (2.5) and (2.6), we deduce the following estimate:

$$\tag{2.11} \|\varepsilon\,\nabla u_\varepsilon\|_{L^2(\Omega\times(0,T))} \le c.$$

Using estimate (2.1) and (2.11), one can extract a subsequence (*still denoted* ε)

and find $U(x, y, t) \in L^2\big(\Omega \times (0, T); H^1_\#(Y)\big)$ such that:

$$u_\varepsilon \rightharpoonup\rightharpoonup U(x, y, t), \tag{2.12}$$

$$\varepsilon \nabla u_\varepsilon \rightharpoonup\rightharpoonup \begin{pmatrix} \nabla'_y U(x, y, t) \\ 0 \end{pmatrix}. \tag{2.13}$$

Note that the two-scale convergence $\varepsilon \dfrac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup\rightharpoonup 0$ in (2.13) holds true because there is no fast variable in the x_3-direction.

On the other hand, we get from (2.3) and (2.4) the existence of $v \in L^2\big(C \times (0, T); H^1_0(I)\big)$ and $\zeta \in L^2\big(0, T; L^2(\Omega \times Y)\big)$ such that:

$$v_\varepsilon \rightharpoonup v \quad \text{weakly in} \quad L^2\big(C \times (0, T); H^1_0(I)\big), \tag{2.14}$$

$$w_\varepsilon \rightharpoonup\rightharpoonup \zeta(x, y, t). \tag{2.15}$$

Note that χ_{F_ε} and χ_{M_ε} two-scale converge respectively to χ_D and $\chi_{(Y-D)}$, so that the use of (2.12) and (2.14) and the fact that

$$u_\varepsilon \chi_{M_\varepsilon} = u_\varepsilon - u_\varepsilon \chi_{F_\varepsilon} = u_\varepsilon - \pi r^2\, v_\varepsilon, \tag{2.16}$$

give us

$$u_\varepsilon \chi_{M_\varepsilon} \rightharpoonup \int_Y U(x, y, t)\, dy - \pi r^2\, v(x, t) \quad \text{weakly in} \quad L^2(\Omega \times (0, T)). \tag{2.17}$$

On the other hand,

$$u_\varepsilon \chi_{M_\varepsilon} \rightharpoonup \int_Y U(x, y, t)\, \chi_{(Y-D)}(y)\, dy = \int_{Y-D} U(x, y, t)\, dy \ \text{ weakly in } \ L^2(\Omega \times (0, T)). \tag{2.18}$$

It results from (2.17) and (2.18) that

$$v(x, t) = \frac{1}{\pi r^2} \int_D U(x, y, t)\, dy. \tag{2.19}$$

Using estimate (2.5) and test function $\varphi(x, t)\, \psi(y)$ with $\psi \in \mathcal{C}^\infty_o(D)$ extended by periodicity to the whole of $\mathbb{R}^2$, we show as in [9] that

$$\nabla'_y U(x, y, t) = 0, \quad \text{a.e. in} \quad \Omega \times D \times (0, T), \tag{2.20}$$

so that (2.19) leads to

$$U(x, y, t) = v(x, t), \quad \text{a.e. in} \quad \Omega \times D \times (0, T). \tag{2.21}$$

Since $U(x, ., t) \in H^1(Y)$ for almost all $(x, t) \in \Omega \times (0, T)$, it is easily seen that equality (2.21) remains true in $\Omega \times \bar{D} \times (0, T)$.

Putting

$$u(x,y,t) = U(x,y,t) - v(x,t) \quad \text{in} \quad \Omega \times Y \times (0,T), \tag{2.22}$$

we get

$$u \in L^2\Big(\Omega \times (0,T); H^1_{\#(Y-D)}(Y)\Big), \tag{2.23}$$

and that u obviously satisfies (1.20), (1.21) and (1.24).

Choosing test functions of the form $\varphi(x,t)\psi(y)$ with $\varphi \in L^2(\Omega \times (0,T))$ and $\psi \in C_0^\infty(D)$, and using estimates (2.4) and (2.5), one obtains as in [9] that

$$\nabla' u_\varepsilon \chi_{F_\varepsilon} \rightharpoonup\!\!\rightharpoonup \nabla'_y w(x,y,t), \tag{2.24}$$

with some $w \in L^2\big(\Omega \times (0,T); H^1_m(D)\big)$, while (2.14) yields

$$\chi_{F_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \pi r^2 \frac{\partial v}{\partial x_3} \quad \text{weakly in} \quad L^2\big(\Omega \times (0,T)\big), \tag{2.25}$$

so that

$$\chi_{F_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup\!\!\rightharpoonup \chi_D(y) \frac{\partial v}{\partial x_3}(x,t). \tag{2.26}$$

Hence, (2.24) and (2.26) imply (1.23).

In the case $\rho_\varepsilon(x') = 1$, (1.25) follows immediately from the two-scale convergence (1.20) and estimate (2.7).

In the case $\rho_\varepsilon(x') = \chi_{F_\varepsilon}(x') + \varepsilon^2 \chi_{M_\varepsilon}(x')$, estimate (2.7) and the following two-scale convergence

$$\rho_\varepsilon u_\varepsilon \rightharpoonup\!\!\rightharpoonup \chi_D(y)\, U(x,y,t) = \chi_D(y)\, v(x,t), \tag{2.27}$$

lead to

$$\rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup\!\!\rightharpoonup \chi_D(y) \frac{\partial v}{\partial t}(x,t), \tag{2.28}$$

which is nothing but (1.28).

To identify the limit problem, we need the following lemma, proved in [9] (emma 3.2), in the more general setting of nonlinear monotone operators.

LEMMA 2.2. – *The following equality holds true*

$$\int_D \left[A(x,y) \begin{pmatrix} \nabla'_y w(x,y,t) \\ \frac{\partial v}{\partial x_3}(x,t) \end{pmatrix} \right]' dy = 0 \quad a.e. \quad (x,t) \in \Omega \times (0,T), \tag{2.29}$$

where v and w are those defined in (2.14) *and* (2.24).

We are now in a position to identify the homogenized problems.

We choose test functions of the form

$$\Phi(x,t) \;=\; \bar{u}\left(x,\frac{x'}{\varepsilon},t\right)+\bar{v}(x,t)+\varepsilon\bar{w}\left(x,\frac{x'}{\varepsilon},t\right), \tag{2.30}$$

where

$$(\bar{u},\bar{v},\bar{w})\in\mathcal{D}\Big(\Omega\times(0,T);C^{\infty}_{\#(Y-D)}(Y)\Big)\times\mathcal{D}(\Omega\times(0,T))\times\mathcal{D}\big(\Omega;C^{\infty}_{o}(\bar{D})\big).$$

($\bar{w}$ being extended by periodicity to the whole of $\mathbb{R}^2$) and then we can pass to the limit in (1.18) with the help of equality (2.29) and hypothesis (1.6). Finally, a density argument leads to the desired equations valid for all $(\bar{u},\,\bar{v},\,\bar{w})\in\mathcal{U}_1\times\mathcal{V}\times\mathcal{W}$, (respectively $(\bar{u},\,\bar{v},\,\bar{w})\in\mathcal{U}_2\times\mathcal{V}\times\mathcal{W}$).

We now proceed to find the initial conditions arising in (1.26) and (1.29).

We first remark, see [5], that

$$u+v\in C\big([0,T];L^2(\Omega\times Y)\big), \tag{2.31}$$

because of $u+v\in H^1\big(0,T;L^2(\Omega\times Y)\big)$ by construction. We have also

$$v\in C\big([0,T];L^2(\Omega)\big). \tag{2.32}$$

Hence, from (2.31) and (2.32), we get

$$u\in C\big([0,T];L^2(\Omega\times Y)\big). \tag{2.33}$$

Taking $\varphi(t)\,\psi(x,y)\in C([0,T])\otimes\mathcal{D}\big(\Omega;C^{\infty}_{\#}(Y)\big)$, with $\varphi(T)=0$, as a test function and integrating by parts, we obtain

$$\left\{\begin{aligned}\int_0^T\!\!\int_\Omega \frac{\partial u_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi\left(x,\frac{x'}{\varepsilon}\right)dxdt \;&=\; -\,\varphi(0)\int_\Omega u_0^\varepsilon(x)\,\psi\left(x,\frac{x'}{\varepsilon}\right)dx\\ &\quad-\int_0^T\!\!\int_\Omega u_\varepsilon(x,t)\,\varphi'(t)\,\psi\left(x,\frac{x'}{\varepsilon}\right)dxdt.\end{aligned}\right. \tag{2.34}$$

Passing to the limit $\varepsilon\to0$ in (2.34), we get by the use of convergences (1.16) and (1.20)

$$\left\{\begin{aligned}\lim_{\varepsilon\to0}\int_0^T\!\!\int_\Omega \frac{\partial u_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi\left(x,\frac{x'}{\varepsilon}\right)dxdt &= -\,\varphi(0)\int_{\Omega\times Y}u_0(x,y)\,\psi(x,y)\,dxdy\\ &\quad-\int_0^T\!\!\int_{\Omega\times Y}\big(u(x,y,t)+v(x,t)\big)\,\varphi'(t)\,\psi(x,y)\,dxdydt.\end{aligned}\right. \tag{2.35}$$

Integrating once again the last integral, we get

$$(2.36)\quad \begin{cases} \displaystyle\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi\left(x,\frac{x'}{\varepsilon}\right)dxdt = -\,\varphi(0)\int_{\Omega\times Y} u_0(x,y)\,\psi(x,y)\,dxdy \\ \displaystyle\qquad +\int_0^T\int_{\Omega\times Y}\left(\frac{\partial u}{\partial t}(x,y,t)+\frac{\partial v}{\partial t}(x,t)\right)\varphi(t)\,\psi(x,y)\,dxdydt \\ \displaystyle\qquad +\,\varphi(0)\int_{\Omega\times Y}\big(u(x,y,0)+v(x,0)\big)\,\psi(x,y)\,dxdy. \end{cases}$$

Using convergence (1.25) in the left hand side of (2.36), we deduce

$$(2.37)\qquad u(x,y,0)+v(x,0)=u_0(x,y),\quad \text{a.e}\quad (x,y)\in\Omega\times Y.$$

On the other hand, from (1.17), we have

$$(2.38)\qquad v_\varepsilon(x,0)=\frac{1}{\pi r^2}\,u_0^\varepsilon(x)\chi_{F_\varepsilon},$$

so that

$$(2.39)\qquad v_\varepsilon(x,0)\rightharpoonup\frac{1}{\pi r^2}\int_D u_0(x,y)\,dy,\quad \text{weakly in}\quad L^2(\Omega).$$

Let $\varphi(t)\,\psi(x)$ be a function in $C([0,T])\otimes\mathcal{D}(\Omega)$, with $\varphi(T)=0$. We have

$$(2.40)\quad \begin{cases} \displaystyle\int_0^T\int_\Omega \frac{\partial v_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi(x)\,dxdt = -\,\varphi(0)\int_\Omega v_\varepsilon(x,0)\,dx \\ \displaystyle\qquad -\int_0^T\int_\Omega v_\varepsilon(x,t)\,\varphi'(t)\,\psi(x)\,dxdt, \end{cases}$$

so that letting $\varepsilon\to 0$ in (2.40), we get with the help of (2.39) and (1.22)

$$(2.41)\quad \begin{cases} \displaystyle\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial v_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi(x)\,dxdt = -\,\varphi(0)\int_\Omega\left(\frac{1}{\pi r^2}\int_D u_0(x,y)\,dy\right)\psi(x)\,dx \\ \displaystyle\qquad -\int_0^T\int_\Omega v(x,t)\,\varphi'(t)\,\psi(x)\,dxdt, \end{cases}$$

which can be rewritten as

$$
(2.42)\quad \begin{cases} \lim\limits_{\varepsilon\to 0} \int\limits_0^T\int\limits_\Omega \dfrac{\partial v_\varepsilon}{\partial t}(x,t)\,\varphi(t)\,\psi(x)\,dxdt = -\ \varphi(0)\int\limits_\Omega \left(\dfrac{1}{\pi r^2}\int\limits_D u_0(x,y)\,dy\right)\psi(x)\,dx \\ \qquad\qquad + \ \varphi(0)\int\limits_\Omega v(x,0)\,\psi(x)\,dx \\ \qquad\qquad + \int\limits_0^T\int\limits_\Omega \dfrac{\partial v}{\partial t}(x,t)\,\varphi(t)\,\psi(x)\,dxdt. \end{cases}
$$

On the other hand, since

$$
(2.43)\qquad \frac{\partial v_\varepsilon}{\partial t} = \frac{1}{\pi r^2}\frac{\partial u_\varepsilon}{\partial t}\chi_{F_\varepsilon} \rightharpoonup \frac{1}{\pi r^2}\left(\frac{\partial u}{\partial t}(x,y,t) + \frac{\partial v}{\partial t}(x,t)\right)\chi_D(y),
$$

we get

$$
(2.44)\qquad \frac{\partial v_\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{weakly in} \quad L^2(\Omega\times(0,T)).
$$

Using (2.44) in the left hand side of (2.42), we get

$$
(2.45)\qquad v(x,0) = \frac{1}{\pi r^2}\int\limits_D u_0(x,y)\,dy.
$$

Equalities (2.37) and (2.45) give us

$$
(2.46)\qquad u(x,y,0) = u_0(x,y) - \frac{1}{\pi r^2}\int\limits_D u_0(x,y)\,dy \quad \text{a.e} \quad (x,y)\in\Omega\times Y.
$$

The uniqueness of the solution of (1.26) as well as of that (1.29) does not lead to any special difficulty. $\square$

3. – Proof of the corrector results.

PROOF OF THEOREM 1.3. – Note first, that all results proved at this stage remain valid for the sequence $\tilde u_\varepsilon$.

Thanks to the coercivity hypothesis (1.5), one can write

$$
(3.1)\quad \begin{cases} \delta\int\limits_0^T\int\limits_\Omega \left|\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\right|^2 \chi_{F_\varepsilon}\,dxdt + \delta\int\limits_0^T\int\limits_\Omega \left|\varepsilon\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}\right|^2\chi_{M_\varepsilon}\,dxdt \\ \le \int\limits_0^T\int\limits_\Omega A\left(x,\dfrac{x'}{\varepsilon}\right)\left(\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\right)\cdot\left(\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\right)\chi_{F_\varepsilon}\,dxdt \\ +\int\limits_0^T\int\limits_\Omega B\left(x,\dfrac{x'}{\varepsilon}\right)\left(\varepsilon\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}\right)\cdot\left(\varepsilon\nabla\tilde u_\varepsilon - \begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}\right)\chi_{M_\varepsilon}\,dxdt := J_\varepsilon \end{cases}
$$

Mutiplying (1.37) by $\tilde{u}_\varepsilon$ and integrating over $\Omega \times (0,T)$, we obtain

$$
\begin{cases}
J_\varepsilon = \int_0^T \int_\Omega f(x,t)\,\tilde{u}_\varepsilon(x,t)\,dxdt - \frac{1}{2}\int_\Omega \rho_\varepsilon |\tilde{u}_\varepsilon(x,T)|^2\,dx + \frac{1}{2}\int_\Omega \rho_\varepsilon |a_\varepsilon(x)|^2\,dx \\
-\int_0^T \int_\Omega \Bigg\{ A\left(x,\frac{x'}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon.\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\chi_{F_\varepsilon} - A\left(x,\frac{x'}{\varepsilon}\right)\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}.\left(\nabla\tilde{u}_\varepsilon - \begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\right)\chi_{F_\varepsilon} \\
-\varepsilon B\left(x,\frac{x'}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon.\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}\chi_{M_\varepsilon} - B\left(x,\frac{x'}{\varepsilon}\right)\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}.\left(\varepsilon\nabla\tilde{u}_\varepsilon - \begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}\right)\chi_{M_\varepsilon}\Bigg\}\,dxdt
\end{cases}
\tag{3.2}
$$

Before passing to the limit, note that (remark that u and v are continuous with respect to t)

$$
\tilde{u}_\varepsilon(x,T) \rightharpoonup\rightharpoonup u(x,y,T) + v(x,T). \tag{3.3}
$$

A classical result of two-scale convergence, see [1], implies

$$
\liminf_\varepsilon \int_\Omega |\tilde{u}_\varepsilon(x,T)|^2\,dx \geq \int_\Omega\int_Y |u(x,y,T)+v(x,T)|^2\,dxdy. \tag{3.4}
$$

Then, using two-scale convergences arising in Theorem 1.1, taking into account regularity hypotheses (1.41) and assumptions (1.5)-(1.6) on the coefficients of A and B, one can pass to the limit in (3.2) to obtain:

In the case $\rho_\varepsilon(x') = 1$, taking into account the equality $u_0(x,y) = u(x,y,0) + v(x,0)$,

$$
\begin{cases}
\limsup_{\varepsilon\to 0} J_\varepsilon \leq \int_0^T \int_{\Omega\times Y} f(x,t)\big(u(x,y,t)+v(x,t)\big)\,dxdydt \\
\quad -\frac{1}{2}\int_{\Omega\times Y} |u(x,y,T)+v(x,T)|^2\,dxdy + \frac{1}{2}\int_{\Omega\times Y} |u(x,y,0)+v(x,0)|^2\,dxdy \\
\quad -\int_0^T \int_{\Omega\times Y} \chi_D(y) A(x,y)\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}.\begin{pmatrix}\nabla'_y w\\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}dxdydt \\
\quad -\int_0^T \int_{\Omega\times Y} \chi_{(Y-D)}(y) B(x,y)\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}.\begin{pmatrix}\nabla'_y u\\ 0\end{pmatrix}dxdydt.
\end{cases}
\tag{3.5}
$$

In the case $\rho_\varepsilon(x') = \chi_{F_\varepsilon}(x') + \varepsilon^2\chi_{M_\varepsilon}(x')$, recalling that $v(x,0) = \dfrac{1}{\pi r^2}\int_D u_0(x,y)\,dy$,

$$
(3.6)\quad \begin{cases}
\limsup_{\varepsilon\to 0} J_\varepsilon \le \int_0^T \int_{\Omega\times Y} f(x,t)\big(u(x,y,t)+v(x,t)\big)\,dxdydt \\
\qquad -\dfrac{\pi r^2}{2}\int_\Omega |v(x,T)|^2\,dx + \dfrac{\pi r^2}{2}\int_\Omega |v(x,0)|^2\,dx \\
\qquad -\int_0^T \int_{\Omega\times Y} \chi_D(y)\,A(x,y)\begin{pmatrix}\nabla'_y w \\ \dfrac{\partial v}{\partial x_3}\end{pmatrix}\cdot\begin{pmatrix}\nabla'_y w \\ \dfrac{\partial v}{\partial x_3}\end{pmatrix} dxdydt \\
\qquad -\int_0^T \int_{\Omega\times Y} \chi_{(Y-D)}(y)\,B(x,y)\begin{pmatrix}\nabla'_y u \\ 0\end{pmatrix}.\begin{pmatrix}\nabla'_y u \\ 0\end{pmatrix} dxdydt.
\end{cases}
$$

Taking $(\bar{u},\bar{v},\bar{w})=(u,v,w)$ in (1.26) (respectively (1.29)), we deduce that the right hand side of (3.5) (respectively (3.6)) is equal to zero, which proves the corrector result on $\tilde{u}_\varepsilon$. $\square$

PROOF OF THEOREM 1.4. – First, one can easily prove that the limits arising in (1.20)-(1.25) and (1.28) are equal to zero when we replace u_ε by $\hat{u}_\varepsilon$. Let us prove that these convergences are strong if and only if (1.43) holds true.

To do this, we just need to prove the following equivalence:

$$
(3.7)\quad \lim_{\varepsilon\to 0} \int_0^t\int_\Omega \Big\{ |\nabla \hat{u}_\varepsilon|^2 \chi_{F_\varepsilon} + |\varepsilon \nabla \hat{u}_\varepsilon|^2 \chi_{M_\varepsilon} \Big\}\, dxds = 0 \iff \lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon |\hat{u}_\varepsilon(x,0)|^2\,dx = 0.
$$

Multiplying (1.38) by $\hat{u}_\varepsilon \chi_{(0,t)}$ and integrating by parts the parabolic term, we get, for all $t\in(0,T)$:

$$
(3.8)\quad \begin{cases}
\dfrac{1}{2}\int_\Omega \rho_\varepsilon |\hat{u}_\varepsilon(x,t)|^2\,dxds \\
+\int_0^t\int_\Omega \Big\{ A(x,\dfrac{x'}{\varepsilon})\,\nabla\hat{u}_\varepsilon.\nabla\hat{u}_\varepsilon\,\chi_{F_\varepsilon} + \varepsilon^2 B(x,\dfrac{x'}{\varepsilon})\,\nabla\hat{u}_\varepsilon.\nabla\hat{u}_\varepsilon\,\chi_{M_\varepsilon}\Big\}\,dxds \\
\qquad\qquad = \dfrac{1}{2}\int_\Omega \rho_\varepsilon |\hat{u}_\varepsilon(x,0)|^2\,dx.
\end{cases}
$$

Suppose that $\lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon |\hat{u}_\varepsilon(x,0)|^2\,dx = 0$. Then, the coerciveness of A and B implies that each term in the left hand side of (3.8) tends to zero.

Conversely, suppose that

$$
(3.9)\quad \lim_{\varepsilon\to 0} \int_0^T\int_\Omega \Big\{ |\nabla \hat{u}_\varepsilon|^2 \chi_{F_\varepsilon} + |\varepsilon \nabla \hat{u}_\varepsilon|^2 \chi_{M_\varepsilon} \Big\}\, dxdt = 0.
$$

Then, one can assume, up to substracting a new subsequence, that

$$\lim_{\varepsilon}\int_{\Omega}\left\{|\nabla\hat{u}_\varepsilon|^2\chi_{F_\varepsilon}+\varepsilon^2|\nabla\hat{u}_\varepsilon|^2\chi_{M_\varepsilon}\right\}dx=0,\quad \text{a.e. } t\in(0,T),$$

in such a way that (2.8) (applied to $\hat{u}_\varepsilon$) gives us:

$$\lim_{\varepsilon\to 0}\int_{\Omega}|\hat{u}_\varepsilon(x,t)|^2\,dx=0,\quad \text{a.e. } t\in(0,T). \tag{3.10}$$

As a consequence of (3.10), we obviously obtain

$$\lim_{\varepsilon}\int_{\Omega}\rho_\varepsilon|\hat{u}_\varepsilon(x,t)|^2\,dx=0,\quad \text{a.e. } t\in(0,T). \tag{3.11}$$

Remark now that the second integral in the left hand side of (3.8) is bounded from above by the left hand side of (3.9) due to the L^∞-boundedness of the coefficients of A and B. Hence it tends to zero.

As a consequence, we get the right hand side of (3.7).

We now assume that (1.43) is failed.

We begin by proving the first inequality of (1.44). We argue by contradiction.

Suppose that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega}\rho_\varepsilon|\hat{u}_\varepsilon(x,t)|^2\,dxdt=0. \tag{3.12}$$

But

$$\int_0^T\int_0^t\int_{\Omega}\rho_\varepsilon\frac{\partial\hat{u}_\varepsilon}{\partial t}(x,s)\hat{u}_\varepsilon(x,s)\,dxdsdt=\frac{1}{2}\int_0^T\int_{\Omega}\rho_\varepsilon\left|\hat{u}_\varepsilon(x,t)\right|^2dx-\frac{T}{2}\int_{\Omega}\rho_\varepsilon\left|\hat{u}_\varepsilon(x,0)\right|^2dx. \tag{3.13}$$

Since $\sqrt{\rho_\varepsilon}\,\dfrac{\partial\hat{u}_\varepsilon}{\partial t}$ is bounded in $L^2\big(\Omega\times(0,T)\big)$, we deduce from (3.12) that the left hand side of (3.13) tends to zero. And then $\lim\limits_{\varepsilon\to 0}\int_{\Omega}\rho_\varepsilon\left|\hat{u}_\varepsilon(x,0)\right|^2dx=0$, which is a contradiction.

We now prove the second inequality of (1.44).

Define the function Z by:

$$Z(t)=\limsup_{\varepsilon}\int_{\Omega}\rho_\varepsilon\left|\hat{u}_\varepsilon(x,t)\right|^2dx,\quad\forall\, t\in(0,T). \tag{3.14}$$

If there exists $t_0\in(0,T)$ such that

$$Z(t_0)=0, \tag{3.15}$$

then the number m defined as

$$m = \inf \{t \in]0,T] : Z(t) = 0\},$$

satisfies

$$m > 0. \tag{3.15'}$$

Indeed, if (3.15') is not true, i.e., if $m = 0$, then for all $a > 0$, there exists $t_a \in]0,T]$ such that

$$Z(t_a) = 0 \quad \text{and} \quad a > t_a. \tag{3.16}$$

Since the positive function Z decreases over $(0,T)$ as it can be seen from the following equality

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \rho_\varepsilon \left|\hat{u}_\varepsilon(x,t)\right|^2 dx \\ \displaystyle\int_0^t\int_\Omega \left\{ A\left(x,\frac{x'}{\varepsilon}\right)\nabla\hat{u}_\varepsilon.\nabla\hat{u}_\varepsilon\,\chi_{F_\varepsilon} + \varepsilon^2 B\left(x,\frac{x'}{\varepsilon}\right)\nabla\hat{u}_\varepsilon.\nabla\hat{u}_\varepsilon\chi_{M_\varepsilon}\right\} dxds \\ = \dfrac{1}{2}\displaystyle\int_\Omega \rho_\varepsilon \left|\hat{u}_\varepsilon(x,0)\right|^2 dx, \end{cases}$$

we deduce from (3.16) that

$$Z(a) = 0 \qquad \forall\, a \in\ ]0,T]. \tag{3.17}$$

Moreover, due to estimate (2.2), the sequence $\int_\Omega \rho_\varepsilon \left|\hat{u}_\varepsilon(x,t)\right|^2 dx$ is bounded in $L^\infty(0,T)$, so that, one can apply Lebesgue's Theorem and get

$$\lim_{\varepsilon\to 0} \int_0^T\int_\Omega \rho_\varepsilon \left|\hat{u}_\varepsilon(x,t)\right|^2 dxdt = 0, \tag{3.18}$$

which is in contradiction with the first inequality of (1.44).

Let a be a real such $0 \le a < m$.

We have $Z(a) > 0$ by definition of m.

Assume that, for a real b such that $a < b \le m$, we have

$$\limsup_\varepsilon \int_a^b\int_\Omega \rho_\varepsilon|\hat{u}_\varepsilon(x,t)|^2 dxdt = 0. \tag{3.19}$$

On the other hand, for all $t \in (a,b)$, we have

$$\int_a^t\int_\Omega \rho_\varepsilon \frac{\partial\hat{u}_\varepsilon}{\partial t}(x,s)\,\hat{u}_\varepsilon(x,s)\,dxds = \frac{1}{2}\int_\Omega \rho_\varepsilon\left|\hat{u}_\varepsilon(x,t)\right|^2 dx - \frac{1}{2}\int_\Omega \rho_\varepsilon\left|\hat{u}_\varepsilon(x,a)\right|^2 dx.$$

And then

$$\int_a^b\int_a^t\int_\Omega \rho_\varepsilon \frac{\partial \hat{u}_\varepsilon}{\partial t}(x,s)\,\hat{u}_\varepsilon(x,s)\,dxdsdt = \frac{1}{2}\int_a^b\int_\Omega \rho_\varepsilon \left|\hat{u}_\varepsilon(x,t)\right|^2 dxdt - \frac{b-a}{2}\int_\Omega \rho_\varepsilon\left|\hat{u}_\varepsilon(x,a)\right|^2 dx.$$

Taking into account estimate (2.7) and using convergence (3.19) $\Big($which implies $\lim\limits_{\varepsilon\to 0}\int_a^b\int_\Omega \rho_\varepsilon\,|\hat{u}_\varepsilon(x,t)|^2 dxdt = 0\Big)$, we deduce that, for all the sequence ε,

$$\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\left|\hat{u}_\varepsilon(x,a)\right|^2 dx \;=\; 0,$$

which is a contradiction with $Z(a) > 0$.

Hence,

$$\limsup_{\varepsilon}\int_a^b\int_\Omega \rho_\varepsilon\,|\hat{u}_\varepsilon(x,t)|^2\,dxdt \;>\; 0 \qquad \forall\; 0\le a<b\le m.$$

If

$$Z(t) \;>\; 0 \quad \forall\, t\in[0,T],$$

then,

$$\limsup_{\varepsilon}\int_a^b\int_\Omega \rho_\varepsilon\,|\hat{u}_\varepsilon(x,t)|^2\,dxdt \;>\; 0, \qquad \forall\; 0\le a<b\le T,$$

i.e., one can take an arbitrary a in $]0,T]$ and argue as above. □

4. – Appendix

In this section, we prove the existence and the uniqueness of solutions to problem (1.1) for each $\varepsilon > 0$ fixed, assuming that the initial data belong to the class given by (1.16) and that the matrices A and B satisfy the hypotheses (1.5). We use the classical Galerkin method (see [5]).

Let $(\omega_\upsilon)_{\upsilon\in\mathbb{N}}$ be a basis in V, and let

$$u_{\varepsilon m}(t) = \sum_{j=1}^{m} g^j_{\varepsilon m}(t)\,\omega_j \tag{4.1}$$

be the solution to the Cauchy problem

$$(4.2)\qquad \begin{cases} \displaystyle\int_\Omega \rho_\varepsilon \frac{\partial u_{\varepsilon m}}{\partial t}(t)\,\omega_j\,dx + \int_\Omega C_\varepsilon \nabla u_{\varepsilon m}(t).\nabla\omega_j\,dx = \int_\Omega f(t)\,\omega_j\,dx \quad 1\le j\le m, \\ u_{\varepsilon m}(0) = u^0_{\varepsilon m},\ u^0_{\varepsilon m} = \displaystyle\sum_{j=1}^m \alpha^j_{\varepsilon m}\,\omega_j \to u^\varepsilon_0 \quad \text{in} \quad V \quad \text{as} \quad m\to\infty. \end{cases}$$

By standard methods in differential equations, we can prove the existence of a solution of (4.2) on some interval $(0, t_{\varepsilon m})$. Then, this solution can be extended to the whole interval $(0, T)$ thanks to the estimate (4.6) below.

4.1 – *A Priori estimates*

First estimate. Multiplying equation (4.2) by $g^j_{\varepsilon m}(t)$, summing over j and integrating over $(0, t)$, $t \in (0, t_{\varepsilon m})$, we obtain

$$(4.3)\qquad \begin{cases} \displaystyle\frac{1}{2}\int_\Omega \rho_\varepsilon\,|u_{\varepsilon m}(t)|^2\,dx + \int_0^t\int_\Omega C_\varepsilon \nabla u_{\varepsilon m}(s).\nabla u_{\varepsilon m}(s)\,dxds \\ \qquad\qquad = \displaystyle\int_0^t\int_\Omega f\,u_{\varepsilon m}\,dxds + \frac{1}{2}\int_\Omega \rho_\varepsilon\,|u^0_{\varepsilon m}|^2\,dx. \end{cases}$$

From assumptions (1.5), (1.15′), (1.16) and (4.2), together with Young's inequality, we get

$$(4.4)\qquad \begin{cases} \displaystyle\int_\Omega \rho_\varepsilon\,|u_{\varepsilon m}(t)|^2\,dx + \delta\int_0^t\int_\Omega \Big\{|\nabla u_{\varepsilon m}(s)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(s)|^2\chi_{M_\varepsilon}\Big\}\,dxds \\ \qquad\qquad \le c + \eta\displaystyle\int_0^t\int_\Omega |u_{\varepsilon m}(s)|^2\,dxds. \end{cases}$$

Applying estimate (2.8) which remains valid for $u_{\varepsilon m}$, in the right hand side of (4.4) and choosing η sufficiently small, we get

$$(4.5)\qquad \int_0^t\int_\Omega \Big\{|\nabla u_{\varepsilon m}(s)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(s)|^2\chi_{M_\varepsilon}\Big\}\,dxds \le c,$$

so that (2.8) leads to

$$(4.5')\qquad \int_0^t\int_\Omega |u_{\varepsilon m}|^2\,dxds \le c.$$

Then (4.4) gives us

$$\int_\Omega \rho_\varepsilon\,|u_{\varepsilon m}(t)|^2\,dx + \int_0^t\int_\Omega \Big\{|\nabla u_{\varepsilon m}(s)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(s)|^2\chi_{M_\varepsilon}\Big\}\,dxds \le L_1, \tag{4.6}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$, and of $\varepsilon > 0$.

Second estimate. Multiplying (4.2) by $\dfrac{dg^j_{\varepsilon m}}{dt}(t)$, summing over j and integrating over $(0,t)$, $t \in (0,T)$, we get

$$\begin{cases} \displaystyle\int_0^t\int_\Omega \rho_\varepsilon\left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxds + \int_0^t\int_\Omega C_\varepsilon\nabla u_{\varepsilon m}(s).\,\frac{\partial}{\partial t}\nabla u_{\varepsilon m}(s)\,dxds \\ \displaystyle\qquad\qquad = \int_0^t\int_\Omega f(s)\,\frac{\partial u_{\varepsilon m}}{\partial t}(s)\,dxds. \end{cases} \tag{4.7}$$

In the case $\rho_\varepsilon = 1$, using (1.15′), one can apply Young's inequality on the right hand side of (4.7) to get

$$\int_0^t\int_\Omega \rho_\varepsilon\left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxdt + \frac{1}{2}\int_\Omega C_\varepsilon\nabla u_{\varepsilon m}(t).\nabla u_{\varepsilon m}(t)\,dx \le c + \frac{1}{2}\int_\Omega C_\varepsilon\nabla u^0_{\varepsilon m}.\nabla u^0_{\varepsilon m}\,dx.$$

The strong convergence in (4.2), assumption (1.16) and the boundedness of the coefficients of C_ε imply

$$\int_0^t\int_\Omega \rho_\varepsilon\left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxds \le L_2,$$

where L_2 is independent of ε and of m. Hence estimate (2.7) is proved.

In the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon}$, we use hypothesis (1.15) which allows us to integrate the right hand side of (4.7) with respect to t. We obtain

$$\begin{cases} \displaystyle\int_0^t\int_\Omega \rho_\varepsilon\left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxdt + \frac{1}{2}\int_\Omega C_\varepsilon\nabla u_{\varepsilon m}(t).\nabla u_{\varepsilon m}(t)\,dx \\ \displaystyle\qquad = \int_\Omega f(t)\,u_{\varepsilon m}(t)\,dx - \int_\Omega f(0)\,u^0_{\varepsilon m}\,dx \\ \displaystyle\qquad\quad - \int_0^t\int_\Omega \frac{\partial f}{\partial t}(s)\,u_{\varepsilon m}(s)\,dxds + \frac{1}{2}\int_\Omega C_\varepsilon\nabla u^0_{\varepsilon m}.\nabla u^0_{\varepsilon m}\,dx. \end{cases} \tag{4.8}$$

Hence, applying Young's and Cauchy-Schwarz's inequalities in the right hand side of (4.8), together with assumption (1.5) we get

$$
(4.9)\quad \begin{cases} \displaystyle\int_0^t\!\!\int_\Omega \rho_\varepsilon \left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxdt + \frac{\delta}{2}\int_\Omega \Big\{|\nabla u_{\varepsilon m}(t)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(t)|^2\chi_{M_\varepsilon}\Big\}\, dx \\ \displaystyle\qquad \le \frac{1}{4\eta}\int_\Omega |f(t)|^2\, dx + \eta\int_\Omega |u_{\varepsilon m}(t)|^2\, dx \\ \displaystyle\qquad + \left(\int_\Omega |f(0)|^2\, dx\right)^{\frac12}\left(\int_\Omega |u^0_{\varepsilon m}|^2\, dx\right)^{\frac12} \\ \displaystyle\qquad + \left(\int_0^t\!\!\int_\Omega \left|\frac{\partial f}{\partial t}(s)\right|^2 dxds\right)^{\frac12}\left(\int_0^t\!\!\int_\Omega |u_{\varepsilon m}(s)|^2\, dxds\right)^{\frac12} \\ \displaystyle\qquad + \frac{\gamma}{2}\int_\Omega \Big\{|\nabla u^0_{\varepsilon m}|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u^0_{\varepsilon m}|^2\chi_{M_\varepsilon}\Big\}\, dx. \end{cases}
$$

Applying once again (2.8) to $u_{\varepsilon m}$, we get

$$
(4.10)\qquad \int_\Omega |u_{\varepsilon m}(t)|^2\, dx \;\le\; c\int_\Omega \Big\{|\nabla u_{\varepsilon m}(t)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(t)|^2\chi_{M_\varepsilon}\Big\}\, dx.
$$

Then, choosing $\eta > 0$ sufficiently small, hypotheses (1.15), (1.16) and strong convergence arising in (4.2), with estimates (4.6) and (4.10) we deduce

$$
(4.11)\quad \int_0^t\!\!\int_\Omega \rho_\varepsilon \left|\frac{\partial u_{\varepsilon m}}{\partial t}(s)\right|^2 dxdt + \int_\Omega \Big\{|\nabla u_{\varepsilon m}(t)|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla u_{\varepsilon m}(t)|^2\chi_{M_\varepsilon}\Big\}\, dx \;\le\; L_3.
$$

where L_3 is a positive constant independent of $m \in \mathbb{N}$ and of $\varepsilon > 0$. This proves estimate (2.7) in the case $\rho_\varepsilon = \chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon}$.

4.2 – *Passage to the limit*

Note first that the sequence $(u_{\varepsilon m})_m$ is bounded in $L^2(0,T;V)$. Indeed, for a fixed ε, estimate (4.11) immediately implies

$$
(4.12)\qquad \|\nabla u_{\varepsilon m}\|_{L^\infty\left(0,T;(L^2(\Omega))^3\right)} \;\le\; c_\varepsilon.
$$

Then, there exist (see [10]) a subsequence $(u_{\varepsilon\mu})_\mu$ of $(u_{\varepsilon m})_m$ and a function $u_\varepsilon : \Omega \times (0,T) \to \mathbb{R}$ such that

$$
(4.13)\quad u_{\varepsilon\mu} \rightharpoonup u_\varepsilon \quad \text{weakly in} \quad L^\infty(0,T;V) \quad \text{and strongly in} \quad C\big([0,T];L^2(\Omega)\big).
$$

Let us now fix j and $\mu > j$. Then, from (4.2) we have

$$\int_\Omega \rho_\varepsilon \frac{\partial u_{\varepsilon\mu}}{\partial t}(t)\,\omega_j\,dx + \int_\Omega C_\varepsilon \nabla u_{\varepsilon\mu}(t).\nabla\omega_j\,dx = \int_\Omega f(t)\,\omega_j\,dx. \tag{4.14}$$

But convergence (4.13) implies that

$$\int_\Omega C_\varepsilon \nabla u_{\varepsilon\mu}(t).\nabla\omega_j\,dx \rightharpoonup \int_\Omega C_\varepsilon \nabla u_\varepsilon(t).\nabla\omega_j\,dx \quad \text{weakly in } L^2(0,T),$$

$$\int_\Omega u_{\varepsilon\mu}(t).\omega_j\,dx \to \int_\Omega u_\varepsilon(t).\omega_j\,dx \quad \text{strongly in } L^2(0,T).$$

We also have, thanks to estimate (4.11),

$$\int_\Omega \rho_\varepsilon \frac{\partial u_{\varepsilon\mu}}{\partial t}(t)\,\omega_j\,dx \rightharpoonup \int_\Omega \rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t}(t)\,\omega_j\,dx. \quad \text{weakly in } L^2(0,T).$$

Thus, from (4.14) and for a fixed j, we deduce

$$\int_\Omega \rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t}(t)\,\omega_j\,dx + \int_\Omega C_\varepsilon \nabla u_\varepsilon(t).\nabla\omega_j\,dx = \int_\Omega f(t)\,\omega_j\,dx. \tag{4.16}$$

So that the density of the basis $(\omega_\upsilon)_{\upsilon\in\mathbb{N}}$ leads to the desired equation on u_ε:

$$\int_\Omega \rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t}(t)\,v\,dx + \int_\Omega C_\varepsilon \nabla u_\varepsilon(t).\nabla v\,dx = \int_\Omega f(t)\,v\,dx \quad \forall\, v \in V. \tag{4.17}$$

Let us now seek for the initial condition associated to u_ε. This is an immediate consequence of the second convergence arising in (4.13), since we have

$$u_{\varepsilon m}(0) \longrightarrow u_\varepsilon(x,0) \quad \text{strongly in } L^2(\Omega),$$

in such a way that the convergence in (4.2) leads to

$$u_\varepsilon(x,0) = u_0^\varepsilon.$$

4.3 – *Uniqueness*

Let u_ε^1 and u_ε^2 be two solutions of (1.1) and put $\omega_\varepsilon = u_\varepsilon^1 - u_\varepsilon^2$. Taking into account the fact that the matrix C_ε is symmetric, we deduce

$$\frac{d}{dt}\left\{\int_\Omega \rho_\varepsilon\,|\omega_\varepsilon(t)|^2\,dx + \int_\Omega C_\varepsilon \nabla\omega_\varepsilon(t).\nabla\omega_\varepsilon(t)\,dx\right\} = 0. \tag{4.18}$$

Integrating (4.18) over $(0,t)$ and using assumption (1.5), we conclude that

$$\int_\Omega \rho_\varepsilon\, |\omega_\varepsilon(t)|^2\, dx = \int_\Omega |\nabla\omega_\varepsilon(t)\chi_{F_\varepsilon}|^2 dx = \int_\Omega |\varepsilon\nabla\omega_\varepsilon(t)\chi_{M_\varepsilon}|^2 dx = 0.$$

And thus, $u_\varepsilon^1 = u_\varepsilon^2$. □

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