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Remarks on $S$-Closedness in Topological Spaces


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Remarks on $S$-Closedness in Topological Spaces.

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Sunto. – Relativamente al [27], sono provate alcune proprietà dei sottospazi $S$-chiusi e dei sottoinsiemi $S$-chiusi di uno spazio topologico. Sono studiate delle condizioni mediante le quali le applicazioni conservano alcuni sottospazi $S$-chiusi.

Summary. – Corresponding to [27], some properties of $S$-closed subspaces and subsets $S$-closed relative to a topological space are proved. Conditions under which mappings preserve certain $S$-closed subspaces are investigated.

1. – Preliminaries.

Topological spaces are denoted by $(X, \tau)$. Let $S$ be a subset of a space $(X, \tau)$. We denote the interior and the closure of $S$ in this space by $\text{int}(S)$ (or $\text{int}_X(S)$) and $\text{cl}(S)$ (or $\text{cl}_X(S)$), respectively. The set $S$ is said to be regular open (resp. regular closed) in $(X, \tau)$, if $S = \text{int}(\text{cl}(S))$ (resp. $S = \text{cl}(\text{int}(S))$). The $S$ is said to be $a$-open [23] (resp. semi-open [19]; preopen [20]; semi-preopen [1, 2]), if $S \subset \text{int}(\text{cl}(\text{int}(S)))$ (resp. $S \subset \text{cl}(\text{int}(S))$; $S \subset \text{int}(\text{cl}(S))$). The complement to $X$ of an $a$-open (resp. semi-open; preopen; semi-preopen) set is said to be an $a$-closed (resp. semi-closed; preclosed; semi-preclosed) set. The intersection of all $a$-closed (resp. semi-closed; preclosed; semi-preclosed) sets (in $(X, \tau)$) containing $S$ is called the $a$-closure (resp. semi-closure; preclosure; semi-preclosure) of $S$ in $(X, \tau)$, and it is denoted respectively by $a\text{-cl}(S)$ (or $a\text{-cl}_X(S)$), $\text{sc}(S)$ (or $\text{sc}_X(S)$), $\text{pcl}(S)$ (or $\text{pcl}_X(S)$), $\text{spcl}(S)$ (or $\text{spcl}_X(S)$). The set $S$ is $a$-closed (resp. semi-closed; preclosed; semi-preclosed) if and only if $a\text{-cl}(S) = S$ (resp. $\text{sc}(S) = S$; $\text{pcl}(S) = S$; $\text{spcl}(S) = S$). Each closed subset of a space $(X, \tau)$ is $a$-closed, semi-closed, preclosed, and semi-preclosed. The collection of all $a$-open (resp. semi-open; preopen; semi-preopen) subsets of a space $(X, \tau)$ is denoted by $a\text{-O}(X, \tau)$ or $a^c$ (resp. $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$; $\text{SPO}(X, \tau)$). The family of all regular open (resp. regular closed; semi-closed) subsets of $(X, \tau)$ is denoted by $\text{RO}(X, \tau)$ (resp. $\text{RC}(X, \tau)$, $\text{SC}(X, \tau)$). Members of the intersection $\text{SR}(X, \tau) = \text{SO}(X, \tau) \cap \text{SC}(X, \tau)$ are called semi-regular sets [9]. A space $(X, \tau)$ is extremally disconnected (briefly e.d.) if $\text{cl}(V) \in \tau$ for each $V \in \tau$. 
T. Thompson [40] has defined an $(X, \tau)$ to be \textit{S-closed}, if for every cover \(\{V_a : a \in A\} \subset \text{SO}(X, \tau)\) of \(X\) there exists a finite subfamily \(A_1 \subset A\) such that \(X = \bigcup_{a \in A_1} \text{cl}_X(V_a)\). T. Noiri [27] has defined a subset \(S\) of \((X, \tau)\) to be \textit{S-closed relative} to \((X, \tau)\), if for every cover \(\{V_a : a \in A\} \subset \text{SO}(X, \tau)\) of \(S\) there exists a finite subfamily \(A_1 \subset A\) such that \(S \subset \bigcup_{a \in A_1} \text{cl}_X(V_a)\).

2. – \textit{S-closed subspaces}.

From [27, Corollary 3.4] we obtain, as a particular case, the following

\textsc{Corollary 2.1.} – If \(A\) and \(B\) are both \textit{S-closed regular open} subspaces of a space \((X, \tau)\), then \(A \cap B\) is an \textit{S-closed} subspace of \((X, \tau)\).

Utilizing [27, Theorem 3.1], we can reexpress Corollary 2.1 as follows:

\textsc{Corollary 2.1‘.} – If sets \(A\) and \(B\) are both regular open and are \textit{S-closed relative} to \((X, \tau)\), then \(A \cap B\) is an \textit{S-closed} subspace of \((X, \tau)\).

This result we generalize in the following way.

\textsc{Theorem 2.2.} – Let \(A, B \in \text{SC}(X, \tau)\) and \(A \cap B \in \tau\). If \(A\) and \(B\) are both \textit{S-closed relative} to \((X, \tau)\), then \(A \cap B\) is an \textit{S-closed} subspace of \((X, \tau)\).

\textsc{Proof.} – Since \(A \cap B \in \text{SC}(X, \tau) \cap \tau\), clearly we have \(A \cap B \in \text{RO}(X, \tau)\) (see also for instance [11, Lemma 2.2 (2)]). So, it follows from [27, Theorems 3.3 and 3.1] that \(A \cap B\) is an \textit{S-closed} subspace.

\textsc{Lemma 2.3.} – Let \(A\) be a subset of \((X, \tau)\). Then, the following holds:

\textbf{(a)} [18, Proposition 2.7]. \(A \in \text{PO}(X, \tau)\) iff \(\text{scl}(A) = \text{int}(\text{cl}(A))\).

\textbf{(b)} [2, Theorem 2.20(a)]. \(A \in a-O(X, \tau)\) iff \(\text{spcl}(A) = \text{int}(\text{cl}(\text{int}(A)))\).

\textbf{(c)} [2, Theorem 2.20(c)]. \(A \in \text{SPO}(X, \tau)\) iff \(a-\text{cl}(A) = \text{cl}(\text{int}(\text{cl}(A)))\).

\textbf{(d)} \(A \in \text{SO}(X, \tau)\) iff \(\text{pcl}(A) = \text{cl}(\text{int}(A))\).

\textsc{Proof.} – To prove the case (d) we use [2, Theorem 1.5 (e)].

\textsc{Theorem 2.4.} – Let \(A\) be \textit{S-closed relative} to \((X, \tau)\). Then,

\textbf{(a)} \(a-\text{cl}_X(A), \text{scl}_X(A), \text{pcl}_X(A), \text{spcl}_X(A)\) are \textit{S-closed relative} to \((X, \tau)\);

\textbf{(b)} \(A \in \text{PO}(X, \tau)\), then \(\text{int}_X(\text{scl}_X(A))\) is \textit{S-closed relative} to \((X, \tau)\);

\textbf{(b)} \(A \in a-O(X, \tau)\), then \(\text{int}_X(\text{spcl}_X(A))\) is \textit{S-closed relative} to \((X, \tau)\);

\textbf{(b)} \(A \in \text{SPO}(X, \tau)\), then \(\text{int}_X(a-\text{cl}_X(A))\) is \textit{S-closed relative} to \((X, \tau)\);

\textbf{(b)} \(A \in \text{SO}(X, \tau)\), then \(\text{int}_X(\text{pcl}_X(A))\) is \textit{S-closed relative} to \((X, \tau)\).
PROOF. – (a). Proofs for all kinds of closures are quite similar to that of [27, Theorem 3.4] (for \( \text{cl}(A) \)).

(b). We apply: respective parts of the case (a), Lemma 2.3, and [27, Theorem 3.3]. \( \square \)

**Corollary 2.5.** – If \( A \in \text{PO}(X, \tau) \) (resp. \( A \in a-\text{O}(X, \tau) \); \( A \in \text{SPO}(X, \tau) \); \( A \in \text{SO}(X, \tau) \)) is \( S \)-closed relative to \((X, \tau)\), then the set \( \text{int}_X(\text{scl}_X(A)) \) (respectively \( \text{int}_X(\text{spcl}_X(A)); \text{int}_X(a-\text{cl}_X(A)); \text{int}_X(\text{pcl}_X(A)) \)) is an \( S \)-closed subspace of \((X, \tau)\).

**Proof.** – Follows from Theorem 2.4(b), Lemma 2.3, and [27, Theorem 3.1]. \( \square \)

**Remark 2.6.** – Without difficulties one checks that [27, Theorem 3.5] is also true if we replace “\( A \) is \( S \)-open ...” by “\( A \) is an \( S \)-closed \( a \)-open ...”. We obtain below that similar results hold also for weaker kinds of closure of \( a \)-open sets.

**Theorem 2.7.** – Let \( A \) be an \( S \)-closed \( a \)-open subspace of \((X, \tau)\). Then, \( \text{scl}_X(A), \text{spcl}_X(A), a-\text{cl}_X(A), \) and \( \text{pcl}_X(A) \) are \( S \)-closed subspaces of \((X, \tau)\).

**Proof.** – We apply respective parts of Lemma 2.3 for each considered case of weak closures, which are of the form \( \text{int}(\text{cl}(.) \) or \( \text{cl}(\text{int}(.) \). So, \( \text{scl}_X(A), \text{spcl}_X(A), a-\text{cl}_X(A), \) and \( \text{pcl}_X(A) \) are semi-open in \((X, \tau)\). The proof for each case is quite similar to that of [27, Theorem 3.5] and hence we can leave details to the reader. \( \square \)

**Theorem 2.8.** – Let \((X, \tau)\) be an \( S \)-closed space.

(a) Let \( A \) be a semi-closed (resp. a semi-preclosed) subset of \((X, \tau)\). If \( A \in \text{PO}(X, \tau) \) (resp. \( A \in a-\text{O}(X, \tau) \)) then \( A \) is an \( S \)-closed subspace of \((X, \tau)\).

(b) Let \( A \) be an \( a \)-closed (resp. a preclosed) subset of \((X, \tau)\). If \( A \in \text{SPO}(X, \tau) \) (resp. \( A \in \text{SO}(X, \tau) \)) and \( \text{Fr}(A) \) is \( S \)-closed relative to \((X, \tau)\), then \( A \) is \( S \)-closed relative to \((X, \tau)\).

**Proof.** – (A). This follows from Lemma 2.3 and [27, Corollary 3.2].

(B) follows from Lemma 2.3, [27, Theorem 3.3], and [27, Theorem 3.6] (see the proof of [27, Theorem 3.7]). \( \square \)

Recall that a space \((X, \tau)\) is called **locally \( S \)-closed** [27, Definition 4.1], if each point of \( X \) has an open neighbourhood which is an \( S \)-closed subspace of \((X, \tau)\).

**Theorem 2.9.** – (see [27, Theorem 4.1]). For a space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) is locally \( S \)-closed.
(2) Each point of $X$ has an open neighbourhood which is $S$-closed relative to $(X, \tau)$.

(3) Each point of $X$ has an open neighbourhood $V$ such that $a$-$\text{cl}_X(V)$ (resp. $\text{sc}_X(V); \text{pel}_X(V); \text{sc}_{\text{pel}}_X(V)$) is $S$-closed relative to $(X, \tau)$.

(4) Each point of $X$ has an open neighbourhood $V$ such that $\text{int}_X(a$-$\text{cl}_X(V))$ (resp. $\text{int}_X(\text{sc}_X(V)); \text{int}_X(\text{pel}_X(V); \text{int}_X(\text{sc}_{\text{pel}}_X(V)$) is $S$-closed relative to $(X, \tau)$.

(5) Each point of $X$ has an open neighbourhood $V$ such that $\text{int}_X(a$-$\text{cl}_X(V)$ (resp. $\text{int}_X(\text{sc}_X(V); \text{int}_X(\text{pel}_X(V); \text{int}_X(\text{sc}_{\text{pel}}_X(V)$) is an $S$-closed subspace of $(X, \tau)$.

Proof. – (1)$\Rightarrow$(2) and (4)$\Rightarrow$(5) follow from [27, Theorem 3.1]. (2)$\Rightarrow$(3): the case (a) of Theorem 2.4. (3)$\Rightarrow$(4): Theorem 2.4. (5)$\Rightarrow$(1) is obvious. 

In every space $(X, \tau)$
$$V \cap \text{sc}(S) \subset \text{cl}(\text{sc}(V \cap S))$$
for each $S \subset X$ and $V \in \text{SO}(X, \tau)$.

Remark 2.11. – Recall that
$$\text{RO}(X, \tau) \cup \text{RC}(X, \tau) \subset \text{SR}(X, \tau),$$
[39, Lemma 2.3]. This inclusion is proper, in general.

The following theorem is a slight improvement of [27, Theorem 3.3] for the case of spaces that are not e.d.

Theorem 2.12. – Assume that a space $(X, \tau)$ is not e.d. Let an $A \subset X$ be $S$-closed relative to $(X, \tau)$ and a set $B \in \text{RO}(X, \tau)$ or $B \in \text{SR}(X, \tau) \setminus \text{RO}(X, \tau)$ with $\text{cl}(B) = \text{sc}(B)$. Then $A \cap B$ is $S$-closed relative to $(X, \tau)$.

Proof. – Suppose $A \cap B \subset \bigcup_{a \in A} V_a$, where $V_a \in \text{SO}(X, \tau)$ for each $a \in A$. Since $B \in \text{SR}(X, \tau)$, thus $X \setminus B \in \text{SO}(X, \tau)$ [39, Lemma 2.2 (ii)] and
$$A \subset (X \setminus B) \bigcup \bigcup_{a \in A} V_a.$$  
But $A$ is an $S$-closed relative to $(X, \tau)$, thus there exists a finite subfamily $A_1 \subset A$ such that
$$A \subset \text{cl}(X \setminus B) \bigcup \bigcup_{a \in A_1} \text{cl}(V_a).$$
Utilizing Lemma 2.10 we obtain
\[ A \cap B \subseteq (B \cap \text{scl}(X \setminus B)) \cup \bigcup_{a \in A_1} \text{cl}(V_a) = \bigcup_{a \in A_1} \text{cl}(V_a). \]

This shows that \( A \cap B \) is \( S \)-closed relative to \((X, \tau)\).

\( \Box \)

**Remark 2.13.** – (a). The author proved in [14] that a space \((X, \tau)\) is e.d. if and only if for each \( S \in \text{SO}(X, \tau) \), \( \text{sel}(S) = \text{int}(\text{cl}(S)) = \text{cl}(\text{int}(S)) \). Thus, by [24, Lemma 2] we obtain that in e.d. spaces \( \text{cl}(S) = \text{scl}(S) \) for each \( S \in \text{SO}(X, \tau) \). The reversed implication is also true. This equivalence was proved in [9, Proposition 2.4].

(b). The author proved in [14] that a space \((X, \tau)\) is e.d. if and only if \( \text{RO}(X, \tau) = \text{RC}(X, \tau) \). On the other hand, by [10, Proposition 2(i)] \( \text{SR}(X, \tau) = \text{RO}(X, \tau) \cap \text{RC}(X, \tau) \) and [39, Lemma 2.3] (see Remark 2.11), we have \( \text{RO}(X, \tau) \cup \text{RC}(X, \tau) = \text{SR}(X, \tau) \) in any e.d. space. Consequently, in these spaces we have \( \text{RO}(X, \tau) = \text{SR}(X, \tau) \). To give an example of a set \( B \in \text{SR}(X, \tau) \setminus \text{RO}(X, \tau) \) for which \( \text{cl}(B) = \text{scl}(B) \), it is enough to consider the space of reals with usual topology and \( B = [0, 1] \).

**Corollary 2.14.** – (see [27, Corollary 3.2]). If \((X, \tau)\) is an \( S \)-closed not e.d. space and an \( A \in \text{RO}(X, \tau) \) or \( A \in \text{SR}(X, \tau) \setminus \text{RO}(X, \tau) \) with \( \text{cl}(A) = \text{scl}(A) \), then \( A \) is \( S \)-closed relative to \((X, \tau)\).

**Proof.** – To see \( S \)-closedness of \( A \) relative to \((X, \tau)\) we apply [27, Theorem 3.1] and Theorem 2.12. Notice that if \( A \in \text{SR}(X, \tau) \) and \( \text{cl}(A) = \text{scl}(A) \), then \( A \in \text{RC}(X, \tau) \).

\( \Box \)

**Corollary 2.15.** – (see [27, Corollary 3.3]). Let \((X, \tau)\) be not an e.d. space. If an \( A \) is \( S \)-closed relative to \((X, \tau)\) and a set \( B \in \text{RO}(X, \tau) \) or \( B \in \text{SR}(X, \tau) \setminus \text{RO}(X, \tau) \) with \( \text{cl}(B) = \text{scl}(B) \), then

1. \( A \cap B \) is \( S \)-closed relative to \( B \).
2. \( B \) is \( S \)-closed relative to \((X, \tau)\), if \( B \subseteq A \).

**Proof.** – (1) follows from Theorem 2.12 and [27, Theorem 3.2] (strong sufficiency). (2): Theorem 2.12.

\( \Box \)

The following corollary is an improvement of [27, Corollary 3.1].

**Corollary 2.16.** – Let \( A \) and \( X_0 \) be \( a \)-open subsets of a space \((X, \tau)\) such that \( A \subset X_0 \). Then, \( A \) is an \( S \)-closed subspace of \((X_0, \tau_{X_0})\) if and only if \( A \) is an \( S \)-closed subspace of \((X, \tau)\).
Proof. – We use [33, Lemma 2], [27, Theorem 3.1], and [27, Theorem 3.2].

The next corollary is an immediate consequence of [27, Theorem 3.2].

Corollary 2.17. – Let $A \subset X_0 \subset X_1 \subset X$ and $X_0, X_1$ be $a$-open subsets of $(X, \tau)$. Then, $A$ is an $S$-closed relative to $(X_0, \tau_{X_0})$ if and only if $A$ is an $S$-closed relative to $(X_1, \tau_{X_1})$.

Using [33, Lemma 2] and Corollary 2.16 we infer what follows.

Corollary 2.18. – Let $A \subset X_0 \subset X_1 \subset X$ and $A, X_0, X_1$ be $a$-open subsets of $(X, \tau)$. Then, $A$ is $S$-closed subspace of $(X_0, \tau_{X_0})$ if and only if $A$ is $S$-closed subspace of $(X_1, \tau_{X_1})$.

Theorem 2.19. – Let $A$ be an $S$-closed $a$-open subspace of $(X, \tau)$. Then, $scl_X(A)$ is $S$-closed relative to $(cl_X(A), \tau_{cl_X(A)})$.

Proof. – Let $\{V_a : a \in A\} \subset SO(cl_X(A), \tau_{cl_X(A)})$ be a cover of $scl_X(A)$. Obviously, $\{V_a : a \in A\}$ is a cover of $A$. Since $A \in \tau^a$, $cl_X(A) \in SO(X, \tau)$. Hence $V_a \in SO(X, \tau)$ for each $a \in A$ [24, Theorem 1]. By [27, Theorem 3.1] the set $A$ is $S$-closed relative to $(X, \tau)$. Thus, there exists a finite subset $A_1 \subset A$ with $A \subset \bigcup_{a \in A_1} cl_X(V_a)$. This inclusion implies that $scl_X(A) \subset \bigcup_{a \in A_1} cl_X(V_a)$. So, we obtain $scl_X(A) \subset \bigcup_{a \in A_1} (cl_X(V_a) \cap cl_X(A)) = \bigcup_{a \in A_1} cl_{cl_X(A)}(V_a)$ and the proof is complete. □

Remark 2.20. – Let an $A \in RO(X, \tau)$ be such that $cl_X(A) \in \tau^a$. The set $scl_X(A)$ is $S$-closed relative to $(cl_X(A), \tau_{cl_X(A)})$ if and only if $A$ is an $S$-closed subspace of $(X, \tau)$.

Proof. – It is enough to show that $A$ is closed in $(X, \tau)$. Namely, we have

\[
cl_X(A) \subset int_X(cl_X(int_X(cl_X(A)))) \subset cl_X(A).
\]

So, $cl_X(A) = int_X(cl_X(A))$ and by hypothesis $cl_X(A) = A$. □

Corollary 2.21. – Let an $A \in RO(X, \tau)$ be such that $cl_X(A) \in \tau^a$. Then, $A$ is an $S$-closed subspace of $(X, \tau)$ if and only if $scl_X(A)$ is $S$-closed relative to $(cl_X(A), \tau_{cl_X(A)})$.

Recall that a topological space $(X, \tau)$ is said to be semi-connected [32], if $X$ cannot be written as a union of two nonempty disjoint semi-open sets in $(X, \tau)$. In the opposite case a space is called semi-disconnected.
Theorem 2.22. Let $A \neq \emptyset$ be $S$-closed relative to $(X, \tau)$ and $\text{cl}(A) \subseteq X_0 \subset X$. If there exists a subfamily $\{V_a : a \in A\} \subset \text{SO}(X, \tau)$ such that $(a_1) A \supset \bigcup_{a \in A} V_a$ and $(a_2) A \subset \bigcup_{a \in A} \text{cl}(V_a)$, then $(X_0, \tau_{X_0})$ is semi-disconnected.

Proof. We have $\text{cl}(V_a) \subset \text{SO}(X, \tau)$ for each $a \in A$. Since $A$ is an $S$-closed relative to $(X, \tau)$, thus by $(a_2)$ there exists a finite subset $A_1 \subset A$ such that $A \subset \bigcup_{a \in A_1} \text{cl}(V_a)$. Hence $\text{cl}(A) \subset \bigcup_{a \in A_1} \text{cl}(V_a)$. On the other hand, by $(a_1)$ we have $\text{cl}(A) \supset \bigcup_{a \in A_1} \text{cl}(V_a) \supset \bigcup_{a \in A_1} \text{cl}(V_a)$. Thus, $\text{cl}(A) = \bigcup_{a \in A_1} \text{cl}(V_a)$ and hence

$$
\text{cl}_{X_0}(A) = \bigcup_{a \in A_1} \text{cl}_{X_0}(V_a).
$$

By $(a_1)$, $V_a \subset X_0$ for each $a \in A$, thus by [19, Theorem 6] every set $V_a \subset \text{SO}(X_0, \tau_{X_0})$. So, $(1)$ implies that $\text{cl}_{X_0}(A) \subset \text{SO}(X_0, \tau_{X_0})$ [19, Theorem 2]. To finish the proof it is enough to observe that $\emptyset \neq X_0 \setminus \text{cl}_{X_0}(A) \in \tau_{X_0}$. \hfill \Box

It is known that the family $\tau^a$ induced by $\tau$ forms a topology on $X$ [23], which is different than $\tau$, in general. Recall that for any $S \in \text{SO}(X, \tau)$ we have $a\text{-cl}(S) = \text{cl}(S)$ [18, Proposition 2.22].

Theorem 2.23. Let $(X, \tau)$ be a space, $A \in \text{SO}(X, \tau)$, $B \in \tau^a$, and $A \cap B = \emptyset$. If the set $A \cup B$ is $S$-closed relative to $(X, \tau)$, then the set $B$ is $S$-closed relative to $(X, \tau)$.

Proof. Let $\mathcal{F} = \{U_a : a \in \nabla\} \subset \text{SO}(X, \tau)$ be a cover of $B$. Then, the family $\mathcal{F} \cup \{A\}$ covers $A \cup B$. By hypothesis there exists a finite subfamily $\mathcal{F}' = \{U_{a_i} : i = 1, \ldots, n\} \subset \mathcal{F}$ such that

$$
A \cup B \subset \bigcup_{i=1}^n \text{cl}(U_{a_i}) \cup \text{cl}(A).
$$

So, we obtain

$$
B \subset \bigcup_{i=1}^n \text{cl}(U_{a_i}) \cup a\text{-cl}(A \cap B) = \bigcup_{i=1}^n \text{cl}(U_{a_i}).
$$

This shows that $B$ is $S$-closed relative to $(X, \tau)$. \hfill \Box

In [15] the author has proved that a space $(X, \tau)$ is semi-disconnected if and only if there exist nonempty sets $U_1 \in \text{SO}(X, \tau)$, $U_2 \in \tau^a$ such that $X = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Directly from this result and Theorem 2.23 one obtains the following corollary.
COROLLARY 2.24. – Let $(X, \tau)$ be a semi-disconnected and $S$-closed space. Then there exists a nonempty a-open proper subset of $X$, $S$-closed relative to $(X, \tau)$.

REMARK 2.25. – Let $(X, \tau)$ be $S$-closed and $A$ be clopen. Then $X \setminus A$ is $S$-closed relative to $(X, \tau)$ (hence $S$-closed subspace of $(X, \tau)$ [27, Theorem 3.1]).

PROOF. – Obvious since $X \setminus A$ is clopen in $(X, \tau)$. □

THEOREM 2.26. – Let a subset $A$ of a space $(X, \tau)$ be clopen and be an $S$-closed subspace of $(X, \tau)$. Then $(X, \tau)$ is $S$-closed if and only if $X \setminus A$ is an $S$-closed subspace of $(X, \tau)$.

PROOF. – It follows from Remark 2.25, [27, Theorem 3.1], and [27, Theorem 3.6]. □

In [5] Cameron introduced the concept of $I$-compactness of a space. It was established [5, Corollary 3] that $I$-compact spaces are precisely the $S$-closed spaces which are e.d. Recall that a subset $S$ of a space $(X, \tau)$ is $I$-compact relative to $(X, \tau)$ if every cover of $S$ with semi-open sets has a finite subfamily interiors of closures of whose members cover $S$ [37]. A subset $A$ of a space $(X, \tau)$ is $N$-closed if every cover with regular open sets has a finite subcover [6]. A space is said to be weakly Hausdorff if for each point $x \in X$, $\{x\}$ is the intersection of all regular closed sets containing $x$ [38].

THEOREM 2.27. – Let $(X, \tau)$ be weakly Hausdorff. If $A \in \text{PO}(X, \tau)$ is $I$-closed relative to $(X, \tau)$ and $X \setminus A$ is $N$-closed, then there exists a finite partition of $X$ by regular open subsets of $(X, \tau)$.

PROOF. – This is an immediate consequence of [37, Lemma 4.13] and [37, Theorem 4.16] (we use [11, Lemma 2.2(4)] and the well known fact that the intersection of two regular open sets is regular open too [12, p. 92, 22g]). □

3. – Mappings and $S$-closedness.

DEFINITION 3.1. – [17]. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be almost continuous (in the sense of Husain), if for each $x \in X$ and each neighbourhood $V$ of $f(x)$, $\text{cl} \ (f^{-1}(V))$ is a neighbourhood of $x$.

Mashhour et al. observed [20] that almost continuity in the sense of Husain coincides with precontinuity (i.e., $f^{-1}(V) \subset \text{int} \ (\text{cl} \ (f^{-1}(V)))$ for each $V \in \sigma$).
DEFINITION 3.2. – [19]. A mapping $f : (X, \tau) \to (Y, \sigma)$ is **semi-continuous** if $f^{-1}(V) \in \text{SO}(X, \tau)$ for every set $V \in \sigma$.

A. Neubrunnová showed [22] that precontinuity and semi-continuity are independent of each other.

DEFINITION 3.3. – [29]. A mapping $f : (X, \tau) \to (Y, \sigma)$ is **strongly semi-continuous** (Mashhour et al. [21] call these mappings a-continuous) if $f^{-1}(V) \in \tau'$ for each $V \in \sigma$.

DEFINITION 3.4. – [8]. A mapping $f : (X, \tau) \to (Y, \sigma)$ is **irresolute** if $f^{-1}(V) \in \text{SO}(X, \tau)$ for each $V \in \text{SO}(Y, \sigma)$.

Each irresolute mapping is semi-continuous ($\tau \subset \text{SO}(X, \tau)$). Each a-continuous mapping is semi-continuous and precontinuous ($\tau \subset \text{SO}(X, \tau) \cap \text{PO}(X, \tau) = \tau'$ [30, Lemma 3.1]).

Janković showed the following.

THEOREM 3.5. – [18, Corollary 4.14]. Let $f : (X, \tau) \to (Y, \sigma)$ be a precontinuous and irresolute mapping. If $G$ is a subset $S$-closed relative to $(X, \tau)$, then $f(G)$ is $S$-closed relative to $(Y, \sigma)$.

Without difficulties it may be observed that this theorem can be obtained with the use of [18, Proposition 3.1(c)].

Notions of precontinuity and irresoluteness are independent of each other as the following examples show.

**Example 3.6.** – We apply [30, Example 3.11]. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$, and let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then $f$ is irresolute and it is not precontinuous because $f^{-1}(\{b, c\}) \notin \text{PO}(X, \tau)$.

**Example 3.7.** – (a). [30, Theorem 3.12] shows that there exists an a-continuous mapping which is not irresolute. We shall give an example of such a mapping. (b). Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, and let $f : (X, \tau) \to (Y, \sigma)$ be defined as follows: $f(a) = f(b) = a$, $f(c) = c$. Then $f$ is continuous but it is not irresolute since $f^{-1}(\{b, c\}) = \{c\} \notin \text{SO}(X, \tau)$.

The above examples show that a-continuity and irresoluteness are independent of each other, as it was observed in [30]. We recall now definitions of some weak forms of openness of mappings.

**Definition 3.8.** – [36]. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be **almost open in the sense of Singal** (briefly a.o.S.), if $f(U) \in \sigma$ for each $U \in \text{RO}(X, \tau)$.
Definition 3.9. – [41]. A mapping \( f : (X, \tau) \to (Y, \sigma) \) is said to be **almost open in the sense of Wilansky** (briefly a.o.W.), if \( f^{-1}(\text{cl}(V)) \subseteq \text{cl}(f^{-1}(V)) \) for each \( V \in \sigma \).

Rose has proved [35, Theorem 11], that a mapping \( f : (X, \tau) \to (Y, \sigma) \) is a.o.W. if and only if \( f(U) \in \text{PO}(Y, \sigma) \) for each subset \( U \in \tau \).

Definition 3.10. – [3]. A mapping \( f : (X, \tau) \to (Y, \sigma) \) is **semi-open** if \( f(U) \in \text{SO}(Y, \sigma) \) for each \( U \in \tau \).

Notions of a.o.S., a.o.W., and of semi-openness (as given above), are independent of each other (see respective examples in [28]).

Definition 3.11. – [34]. A mapping \( f : (X, \tau) \to (Y, \sigma) \) is **weakly open** if \( f(U) \subseteq \text{int}(f(\text{cl}(U))) \) for each set \( U \in \tau \).

Each a.o.S. mapping is weakly open [28, Lemma 1.4], but the converse is not true, in general [28, Example 1.5]. Notions of weak openness and a.o.W. are independent of each other (respective examples in [28]).

Lemma 3.12. – [25, Theorem 1]. Every a.o.W. and semi-continuous mapping is irresolute.

Combining Theorem 3.5 and Lemma 3.12 we obtain the following generalization of [26, Theorem 2.1].

Theorem 3.13. – If a mapping \( f : (X, \tau) \to (Y, \sigma) \) is a-continuous and a.o.W., and if \( G \) is S-closed relative to \( (X, \tau) \), then \( f(G) \) is S-closed relative to \( (Y, \sigma) \).

Remark 3.14. – Notions of a.o.W. and a-continuity are independent of each other. The mapping \( f \) from [28, Example 1.6] is a.o.W., while it is not a-continuous. The mapping \( f \) from Example 3.7(b) is a-continuous and it is not a.o.W., since \( f^{-1}(\text{cl}([b])) \not\subseteq \text{cl}(f^{-1}([b])) = \emptyset \).

Lemma 3.15. – [28, Theorem 1.12]. Every a.o.S. and semi-continuous mapping is irresolute.

Combining Theorem 3.5 and Lemma 3.15 we get the following.

Theorem 3.16. – If a mapping \( f : (X, \tau) \to (Y, \sigma) \) is a-continuous and a.o.S., and if \( G \) is S-closed relative to \( (X, \tau) \), then \( f(G) \) is S-closed relative to \( (Y, \sigma) \).
REMARK 3.17. – Notions of a.o.S. and $a$-continuity are independent of each other. [28, Example 1.7] shows that there exists an a.o.S. mapping which is not $a$-continuous. In Example 3.7(b) the mapping $f$ is not a.o.S. because $f(X)$ is not open in the range.

**Lemma 3.18.** – [28, Theorem 1.14]. If a space $(Y, \sigma)$ is e.d. and a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is semi-open and semi-continuous, then $f$ is irresolute.

Recall that a semi-open semi-continuous (hence $a$-continuous) mapping, must not be irresolute if the range is not e.d. [31, Example 19].

Applying Theorem 3.5 and Lemma 3.18 we obtain what follows.

**Theorem 3.19.** – Let a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ be $a$-continuous and semi-open. If $(Y, \sigma)$ is e.d. and $G \subset X$ is $S$-closed relative to $(X, \tau)$, then $f(G)$ is $S$-closed relative to $(Y, \sigma)$.

**Remark 3.20.** – Semi-openness and $a$-continuity of an $f$ are independent notions, even if the range of $f$ is e.d. (a). [28, Example 1.8] shows that the $f$ (from this example) is $a$-continuous and not semi-open. (b). [28, Example 1.9] shows that a mapping may be semi-open and not $a$-continuous, but the range in this example is not e.d. (c). Let $X = \{a, b, c\}$, $\tau = \emptyset, X, \{a\}, \{b\}, \{a, b\}$, $Y = \{a, b\}$, and $\sigma = \emptyset, Y, \{a\}, \{b\}$. The mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as follows: $f(a) = f(b) = a, f(c) = b$, is semi-open and not $a$-continuous.

**Definition 3.21.** – [16]. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat continuous if for each set $V \in \sigma$ with $f^{-1}(V) \neq \emptyset$, there exists a set $U \in \tau$ such that $\emptyset \neq U \subset f^{-1}(V)$.

Each semi-continuous mapping is somewhat continuous [16] (semi-continuity and quasi-continuity are equivalent [22]), but the converse is not true in general [16, Example 1].

**Lemma 3.22.** – [28, Theorem 1.11]. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly open somewhat continuous injection, then it is irresolute.

Using once again Theorem 3.5 and Lemma 3.22 we obtain the following.

**Theorem 3.23.** – Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an $a$-continuous weakly open injection. If $G$ is $S$-closed relative to $(X, \tau)$, then $f(G)$ is $S$-closed relative to $(Y, \sigma)$.

**Remark 3.24.** – Weak openness and $a$-continuity are independent notions. (a). The mapping $f$ from [28, Example 1.5] is weakly open, but it is not $a$-continuous.
(in fact, it is not semi-continuous). (b). Let \( X = \{a, b\} = Y, \tau = \{\emptyset, X, \{a\}, \{b\}\}, \) and \( \sigma = \{\emptyset, Y, \{a\}\}. \) Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity mapping. Then \( f \) is continuous and it is not weakly open, because \( f(\{b\}) \not\subset \text{int} \left( \text{cl} \left( \{b\} \right) \right) = \emptyset. \)

The next theorem is an immediate consequence of Theorems 3.13, 3.16, 3.19, 3.23 (for the respective parts).

**Theorem 3.25.** – Let \( f : (X, \tau) \to (Y, \sigma) \) be a mapping.

1. If \( f \) is a-continuous and a.o.W., and if \( (X, \tau) \) is S-closed, then \( f(X) \) is S-closed relative to \( (Y, \sigma) \).
2. If \( f \) is a-continuous and a.o.S., and if \( (X, \tau) \) is S-closed, then \( f(X) \) is S-closed relative to \( (Y, \sigma) \).
3. If \( f \) is a-continuous and semi-open, \( (Y, \sigma) \) is e.d., and if \( (X, \tau) \) is S-closed, then \( f(X) \) is S-closed relative to \( (Y, \sigma) \).
4. If \( f \) is an a-continuous weakly open injection, and if \( (X, \tau) \) is S-closed, then \( f(X) \) is S-closed relative to \( (Y, \sigma) \).

Using Theorem 3.25 (3) one trivially obtains the following corollary.

**Corollary 3.26.** – Let \( f : (X, \tau) \to (Y, \sigma) \) be a surjection and \( (Y, \sigma) \) be e.d. If \( f \) is a-continuous, semi-open and if \( (X, \tau) \) is S-closed, then \( (Y, \sigma) \) is I-compact.

It is interesting to compare this corollary with [37, Theorem 5.5].

**Definition 3.27.** – A mapping \( f : (X, \tau) \to (Y, \sigma) \) is said to be **contra-semiopen** if \( f(U) \in \text{SC}(Y, \sigma) \) for every \( U \in \tau \).

**Lemma 3.28.** – Let \( f : (X, \tau) \to (Y, \sigma) \) be a.o.W. and contra-semiopen. Then \( f(U) \in \text{RO}(Y, \sigma) \) for each \( U \in \tau \).

**Proof.** – By [35, Theorem 11] and by Definition 3.27 we have

\[
\text{f(U) \subset \text{int}(\text{cl}(f(U))) \subset f(U).}
\]

**Theorem 3.29.** – Let \( f : (X, \tau) \to (Y, \sigma) \) be a-continuous, a.o.W., and contra-semiopen. If \( (X, \tau) \) is an S-closed space and \( G \in \text{RO}(X, \tau) \), then \( f(G) \) is an S-closed subspace of \( (Y, \sigma) \).

**Proof.** – This follows from Lemma 3.28, Theorem 3.13, [27, Corollary 3.2 and Theorem 3.1].

Recall that a mapping \( f : (X, \tau) \to (Y, \sigma) \) is semi-continuous if and only if \( f(\text{scl}(A)) \subset \text{cl}(f(A)) \) for every subset \( A \subset X \) [7, Theorem 1.6].
Lemma 3.30. Let \( f : (X, \tau) \to (Y, \sigma) \) be a.o.S., contra-semiopen, and semi-continuous. Then \( f(U) \in \text{RO}(Y, \sigma) \) for each \( U \in \tau \).

Proof. By [7, Theorem 1.16] and [18, Proposition 2.7(a)] we have \( f(\text{int}(\text{cl}(U))) \subset \text{cl}(f(U)) \). But \( f \) is contra-semiopen, therefore applying [34, Theorem 4] we obtain what follows
\[
f(U) \subset \text{int}(f(\text{int}(\text{cl}(U)))) \subset \text{int}(\text{cl}(f(U))) \subset f(U).
\]
This shows that \( f(U) \in \text{RO}(Y, \sigma) \) for any \( U \in \tau \). \( \square \)

Theorem 3.31. Let \( f : (X, \tau) \to (Y, \sigma) \) be a-continuous, a.o.S., and contra-semiopen. If \((X, \tau)\) is an S-closed space and \( G \in \text{RO}(X, \tau) \), then \( f(G) \) is an S-closed subspace of \((Y, \sigma)\).

Proof. We apply Lemma 3.30, Theorem 3.16, [27, Corollary 3.2 and Theorem 3.1]. \( \square \)

Lemma 3.32. Let a space \((Y, \sigma)\) be e.d. and a mapping \( f : (X, \tau) \to (Y, \sigma) \) be semi-open and contra-semiopen. Then \( f(U) \in \text{RO}(Y, \sigma) \) for each \( U \in \tau \).

Proof. For any \( U \in \tau \) we have what follows:
\[
\text{int}(\text{cl}(f(U))) \subset f(U) \subset \text{cl}(\text{int}(f(U))) = \text{int}(\text{cl}(\text{int}(f(U)))) \subset \text{int}(\text{cl}(f(U))). \quad \square
\]

Theorem 3.33. Let a space \((Y, \sigma)\) be e.d. and a mapping \( f : (X, \tau) \to (Y, \sigma) \) be a-continuous, semi-open, and contra-semiopen. If \((X, \tau)\) is an S-closed space and \( G \in \text{RO}(X, \tau) \), then \( f(G) \) is an S-closed subspace of \((Y, \sigma)\).

Proof. This follows from Lemma 3.32, Theorem 3.19, [27, Corollary 3.2 and Theorem 3.1]. \( \square \)

Lemma 3.34. Let \( f : (X, \tau) \to (Y, \sigma) \) be weakly open, contra-semiopen, and precontinuous. Then \( f(U) \in \text{RO}(Y, \sigma) \) for each \( U \in \tau \).

Proof. By hypothesis and by [18, Proposition 3.1 (c)] we have
\[
f(U) \subset \text{int}(f(\text{cl}(U))) \subset \text{int}(\text{cl}(f(U))) \subset f(U). \quad \square
\]

Theorem 3.35. Let \( f : (X, \tau) \to (Y, \sigma) \) be an a-continuous, weakly open and contra-semiopen injection. If \((X, \tau)\) is an S-closed space and \( G \in \text{RO}(X, \tau) \), then \( f(G) \) is an S-closed subspace of \((Y, \sigma)\). \( \square \)

Remark 3.36. It is easy to see that the statement “\((X, \tau)\) is an S-closed space and \( G \in \text{RO}(X, \tau)\)” in conclusions of Theorems 3.29, 3.31, 3.33, and 3.35 may be
replaced by “$G$ is regular open $S$-closed subspace of $(X, \tau)$”. Thus, these theorems give conditions under which a mapping preserves regular open $S$-closed subspaces.

By Remark 3.36, respective theorems mentioned in this remark, and by [27, Theorem 4.1(5)], we obtain the following result.

**Theorem 3.37.** — Let $(X, \tau)$ be a locally $S$-closed space and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a mapping.

1. If $f$ is an $a$-continuous, a.o.W. (or a.o.S.), and contra-semiopen surjection, then $(Y, \sigma)$ is locally $S$-closed.

2. If $(Y, \sigma)$ is e.d. and $f$ is an $a$-continuous, semi-open, and contra-semiopen surjection, then $(Y, \sigma)$ is locally $S$-closed.

3. If $f$ is an $a$-continuous, weakly open, and contra-semiopen bijection, then $(Y, \sigma)$ is locally $S$-closed.

The reader is advised to compare Theorem 3.37 with [27, Theorem 4.4].

**References**


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