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Dedicated to my father on his 60th

**Sunto.** – Rivestimenti di $\mathbb{P}^1$ con gruppo di monodromia un gruppo di Weyl di tipo $W(D_d)$ sono stati studiati da Biggers e Fried che hanno provato l’irriducibilità dei corrispondenti spazi di Hurwitz. In questo articolo viene dimostrata l’irriducibilità degli spazi di Hurwitz che parametrizzano rivestimenti di una curva proiettiva complessa, connessa, non singolare di genere $\geq 0$, il cui gruppo di monodromia è un gruppo di Weyl di tipo $W(B_d)$.

**Summary.** – Let $Y$ be a smooth, connected, projective complex curve of genus $\geq 0$. Biggers and Fried proved the irreducibility of the Hurwitz spaces which parametrize coverings of $\mathbb{P}^1$ whose monodromy group is a Weyl of type $W(D_d)$. Here we prove the irreducibility of Hurwitz spaces that parametrize coverings of $Y$ with monodromy group a Weyl group of type $W(B_d)$.

**Introduction.**

Let $X$, $X'$ and $Y$ be smooth, connected, projective complex curves of genus $\geq 0$. Let $H_{d,n}(Y)$ be the Hurwitz space which parametrizes degree $d$ coverings of $Y$ simply branched in $n$ points. The irreducibility of the spaces $H_{d,n}(\mathbb{P}^1)$ is proved by Hurwitz in [9]. Coverings of $\mathbb{P}^1$ simply branched in all but one point of the discriminant were studied by Natanzon [13], Kluitmann [11] and Mochizuki [12], who proved the irreducibility of the corresponding Hurwitz spaces. More generally one can consider sequences of coverings of type $X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1$. This is what Biggers and Fried did in [2]. They proved the irreducibility of Hurwitz spaces which parametrize equivalence classes of sequences of coverings $f \circ \pi$, where $\pi$ is an unramified cyclic covering of degree $l$ and $f$ is a covering simply branched of $\mathbb{P}^1$ of degree $m$. Only recently Graber, Harris and Starr considered the Hurwitz spaces $H_{d,n}^0(Y)$ parametrizing coverings with full monodromy group $S_d$ of curves of genus $\geq 1$. They proved in [8] the irreducibility of these spaces for $n \geq 2d$. Kanev sharpened this result proving in [10] the irreducibility of $H_{d,n}^0(Y)$ in the case $n \geq \max\{2, 2d - 4\}$ if $g \geq 1$ and $n \geq \max\{2, 2d - 6\}$ if $g = 1$. Moreover Kanev extended the result to coverings which are simply branched in all but one point of the discriminant. Fixing the branching data of special point,
i.e., a partition \( \mathcal{g} = (e_1, \ldots, e_r) \) of \( d \) where \( e_1 \geq \cdots \geq e_r \), he obtained the Hurwitz spaces \( H_{d,n,\mathcal{g}}^0(Y) \) parametrizing coverings with monodromy group \( S_d \), simply branched in \( n \) points and ramified with multiplicities \( e_1, \ldots, e_r \) over one additional point. Kanev proved they are irreducible if \( n \geq 2d - 2 \). We sharpened the latter result proving in [15] the irreducibility of \( H_{d,n,\mathcal{g}}^0(Y) \) for \( n + |\mathcal{g}| \geq 2d \), where \( |\mathcal{g}| = \sum_{i=1}^r (e_i - 1) \).

Here we consider sequences of coverings \( X \xrightarrow{\pi} X' \xrightarrow{f} Y \) where \( \pi \) is a branched covering of degree 2 and \( f \) of degree \( d \geq 3 \). Denote by \( D_\pi \), \( D_f \), \( D \) respectively the branch locus of \( \pi \), \( f \) and \( f \circ \pi \). Let \( b_0 \in Y - D \) and let \( \phi : (f \circ \pi)^{-1}(b_0) = \{-d, \ldots, -1, 1, \ldots, d\} \) be a bijection. In this paper we are interested in Hurwitz spaces that parametrize equivalence classes of pairs \( [f \circ \pi, \phi] \) of sequences of coverings \( f \circ \pi \) and bijections \( \phi \) satisfying the following: \( \pi \) is a covering as above and either \( f \) is a covering simply branched of degree \( d \) of \( Y \) with \( n_2 > 0 \) branch points and monodromy group \( S_d \), or \( f \) is a covering of degree \( d \) of \( Y \) with \( n_2 > 0 \) points of simple branching and one special point \( c \), whose local monodromy has cyclic type given by the partition \( \mathcal{g} \) of \( d \) and furthermore \( f \) has full monodromy group \( S_d \). Denote these spaces respectively by \( H_{W(B_d),(n_1,n_2)}(Y, b_0) \) and \( H_{W(B_d),(n_1,n_2,\mathcal{g})}(Y, b_0), H_{W(B_d),\{n_1,n_2,\{j_1,\ldots,j_\ell\}\}}(Y, b_0) \). We prove their irreducibility when \( Y \cong \mathbb{P}^1 \) and then we extend the result to smooth, connected, projective complex curves of genus \( \geq 1 \). Here we follow the standard approach. Let \( g \) be the genus of \( Y \), associate to \( [f \circ \pi, \phi] \) an ordered \( (n + 2g) \)-tuple of elements of \( S_{2d} \), \((t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)\), satisfying the following: for each \( i = 1, \ldots, n \), \( t_i \not\equiv id \), \( t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g] \), \( n_1 \) among the \( t_i \) are transpositions of type \( (j - j) \) and \( n_2 \) are permutations of type either \((j h) \cdots (j h)\) or \((j - h) \cdots (j h)\). Moreover the group generated by \( t_i, \lambda_k, \mu_k \) is the Weyl group of type \( B_d \). Observe that when \( g = 0 \) the \((n + 2g)\)-tuples \((t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)\) are of the form \((t_1, \ldots, t_n)\) and \( t_1 \cdots t_n = id \). So we prove the irreducibility of our Hurwitz spaces by proving the transitivity of the action of the braid group \( \pi_1((Y - b_0)^{\mathbb{A}} - \Delta, D) \) on the set of ordered \((n + 2g)\)-tuple as above. In order to prove the transitivity of the action of \( \pi_1((\mathbb{P}^1 - b_0)^{\mathbb{A}} - \Delta, D) \), we prove that applying elementary transformations of the Artin’s braid group it is possible to bring each \((t_1, \ldots, t_n)\) as above to a given normal form. Once this is proved, to extend the result to curve of genus \( \geq 1 \), it is sufficient to prove that applying braid moves it is possible to replace every \((t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)\) by \((t_1, \ldots, t_n; id, id, \ldots, id, id)\). In order to prove this we use the results obtained by Kanev in [10] and by the author in [15]. So we prove the irreducibility of the Hurwitz space \( H_{W(B_d),\{n_1,n_2}\}}(Y, b_0) \) for \( n_2 \geq 2d - 2 \) and the irreducibility of the spaces \( H_{W(B_d),\{n_1,n_2,\mathcal{g}\}}(Y, b_0) \) and \( H_{W(B_d),\{n_1,n_2,\{j_1,\ldots,j_\ell\}\}}(Y, b_0) \) for \( n_2 + |\mathcal{g}| \geq 2d \).

Moreover here we prove the irreducibility of Hurwitz spaces that parametrize equivalence classes \( [f \circ \pi] \) where \( f \) is a branched covering of degree \( d \geq 3 \) of \( \mathbb{P}^1 \), with \( n_2 \) points of simple branching and one special point \( c \) and \( \pi \) is a branched covering of degree 2 such that \( D_\pi \subset f^{-1}(c) \).
Conventions. – In this paper we work with sequences of branched coverings
$X \xrightarrow{\pi'} X' \xrightarrow{\pi} Y$. Two degree $d$ branched coverings $h_1 : X_1 \to Y$ and $h_2 : X_2 \to Y$
are called equivalent if there exists a biholomorphic map $p : X_1 \to X_2$ such that
$h_2 \circ p = h_1$. Two sequences of coverings $X_1 \xrightarrow{\pi} X'_1 \xrightarrow{\pi} Y$ and $X_2 \xrightarrow{\pi} X'_2 \xrightarrow{\pi} Y$ are
called equivalent if there exist two biholomorphic maps $p : X_1 \to X_2$ and
$p' : X'_1 \to X'_2$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. The equivalence class
containing the covering $f \circ \pi$ is denoted by $[f \circ \pi]$. We denote by $D$ the branch
locus of $f \circ \pi$ and by $b_0$ a point in $Y - D$. We number the fiber $(f \circ \pi)^{-1}(b_0)$
through a bijection $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \ldots, -1, 1, \ldots, d\}$ and denote by
$m : \pi_1(Y - D, b_0) \to S_{2d}$ the monodromy homomorphism associated to $[f \circ \pi, \phi]$.
Moreover here the natural action of $S_d$ on $\{1, \ldots, d\}$ is on the right and multi-
plification of permutation is by $\sigma \cdot \tau = \tau \circ \sigma$, e.g., $(12)(13) = (123)$.

1. – Preliminaries.

Let $d \geq 3$ be an integer. In § 1.1 we recall some results on Weyl groups of type
$B_d$. We use the notation of [4].

1.1. Let $\{e_1, \ldots, e_d\}$ be the standard base of $\mathbb{R}^d$ and let $R$ be the root system
$\{\pm e_i, \pm e_i \pm e_j : 1 \leq i, j \leq d\}$. Let us denote by $W(B_d)$ the Weyl group of type
$B_d$. $W(B_d)$ is generated by the reflections $s_{e_i}, i = 1, \ldots, d$, and $s_{e_i \pm e_j}, 1 \leq i < j \leq d$. The reflection $s_{e_i} - e_j$ changes $e_i$ and $e_j$, $-e_i$ and $-e_j$, leaving un-
changed $e_h$ for each $h \neq i, \pm j$. The reflection $s_{e_i}$ changes $e_i$ and $-e_i$ while un-
changing each $e_h$ with $h \neq i$. We identify $\{\pm e_i : i = 1, \ldots, d\}$ with
$\{-d, \ldots, -1, 1, \ldots, d\}$ by the map $\pm e_i \to \pm i$. So the action of $W(B_d)$ over
$\{\pm e_i : i = 1, \ldots, d\}$ allows us to define an injective homomorphism $\tau$ from $W(B_d)$
into $S_{2d}$ such that

$$
\tau(s_{e_i} - e_j) = (i \ j)(-i - j), \quad \tau(s_{e_i}) = (i - i)
$$

and

$$
\tau(s_{e_i + e_j}) = \tau(s_{e_i} s_{e_j} s_{e_i - e_j}) = (i - j)(-i j).
$$

Then, by ignoring the sign-changes, each element $w \in W(B_d)$ determines a
permutation of the indexes $1, \ldots, d$. This permutation can be expressed in the
usual way as a product of disjoint cycles. Let $(i_1 i_2 \ldots i_e)$ be such a cycle. Then $w$
sends $e_{i_1}$ to $\pm e_{i_2}$, $\pm e_{i_1}$ to $\pm e_{i_1 + 1}$, $j = 2, \ldots, e - 1$, and $\pm e_{i_1}$ to $\pm e_{i_1}$. The cycle
$(i_1 \ldots i_e)$ is called positive if $w'(e_{i_1}) = e_{i_1}$, and negative if $w'(e_{i_1}) = -e_{i_1}$. A positive
e-cycle of the form $(i_1 \ldots i_e)$ corresponds in $S_{2d}$ to a product of two disjoint e-
cycles, $ss'$, which move the indexes $\{\pm i_1, \ldots, \pm i_e\}$ and are such that if $s$ sends $i_j$
to $i_{j+1}$ ($i_j \to -i_{j+1}$) then $s'$ sends $-i_j$ to $-i_{j+1}$ (resp. $-i_j$ to $i_{j+1}$), where
$\pm i_{e+1} := \pm i_1$. Instead a negative e-cycle of the form $(i_1 i_2 \ldots i_e)$ corresponds in $S_{2d}$ to a 2e-cycle of type $(i_1 \pm i_2 \ldots \pm i_e - i_1 \mp i_2 \ldots \mp i_e)$. The lengths of the
cycles together with their signs give a set of positive or negative integers called the signed cycle-type of \( w \). It is easy to see that two elements of \( W(B_d) \) are conjugate if and only if they have the same signed cycle-type, [5].

Let us denote by \( G_1 \) the subgroup of \( W(B_d) \) generated by the reflections with respect to the long roots \( \varepsilon_i - \varepsilon_j, 1 \leq i < j \leq d \). The homomorphism \( \tau_1 : G_1 \to S_d \) that corresponds to the action of \( G_1 \) over the set \( \{\varepsilon_i : i = 1, \ldots, d\} \) is bijective and it sends \( s_{\varepsilon_i - \varepsilon_j} \) to \( (i \ j) \). Let \( G_2 \) be the subgroup of \( W(B_d) \) generated by the reflections with respect to the short roots \( \varepsilon_i, i = 1, \ldots, d \), and let \( (Z_2)^d \) be the set of the functions from \( \{1, \ldots, d\} \) into \( Z_2 \). Throughout this paper we denote by \( \bar{1}_j \) and by \( z_{ij} \) the functions of \( (Z_2)^d \) so defined

\[
\bar{1}_j(j) = 1 \quad \text{and} \quad \bar{1}_j(h) = 0 \quad \text{for each} \quad h \neq j
\]

and

\[
z_{ij}(i) = z_{ij}(j) = 0 \quad \text{and} \quad z_{ij}(h) = \bar{0} \quad \text{for each} \quad h \neq i, j \quad \text{and} \quad z \in Z_2.
\]

Moreover we denote by \( \bar{1}_{i, \ldots, h} \) the function of \( (Z_2)^d \) that sends to \( \bar{1} \) only the indexes \( i, \ldots, h \). Let \( \tau_2 : G_2 \to (Z_2)^d \) be the homomorphism that maps \( s_{\varepsilon_i} \) into \( \bar{1}_i \). It is easy to prove that \( \tau_2 \) is an isomorphism. Let \( \Phi \) be the homomorphism from \( S_d \) in \( \text{Aut}((Z_2)^d) \) which assigns to \( t \in S_d \) \( \Phi(t) \in \text{Aut}((Z_2)^d) \) where

\[
[\Phi(t) z'](j) := z'(j^t) \quad \text{for each} \quad z' \in (Z_2)^d.
\]

Let \( (Z_2)^d \times^s S_d \) be the semidirect product of \( (Z_2)^d \) and \( S_d \) through the homomorphism \( \Phi \). Given \((z'; t_1), (z''; t_2) \in (Z_2)^d \times^s S_d \) we put

\[
(z'; t_1) \cdot (z''; t_2) := (z' + \Phi(t_1)(z'')); t_1 t_2).
\]

Let \( \Psi : (Z_2)^d \times^s S_d \to W(B_d) \) be the map so defined

\[
\Psi((z'; t_1)) = \tau_2^{-1}(z') \tau_1^{-1}(t_1).
\]

It is easy to check \( \Psi \) is an isomorphism which sends \((0; (i \ j)) \) to \( s_{\varepsilon_i - \varepsilon_j} (\bar{1}_i; id) \) to \( s_{\varepsilon_i} \) and \((\bar{1}_i; (i \ j)) \) to \( s_{\varepsilon_i} s_{\varepsilon_i - \varepsilon_j} = s_{\varepsilon_i + \varepsilon_j} \).

Let us denote by \( r_j \) the reflection with respect to \( \varepsilon_i \pm \varepsilon_j \). The image via the injective homomorphism \( \tau \) of \( (r_2 \cdots r_e) \) is a product of two e-cycles of type \( ss' \).

Since \( s_{\varepsilon_i - \varepsilon_j} s_{\varepsilon_i} = s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}, s_{\varepsilon_i}^2 = id \) and \( s_{\varepsilon_i + \varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j} \) one has

\[
(r_2 \cdots r_1) = (c_h \cdots c_k)(r'_2 \cdots r'_e)
\]

where \( r'_j \) is the reflection with respect to the long root \( \varepsilon_i - \varepsilon_j, c_h, \ldots, c_k \) are reflections with respect to the short roots \( \varepsilon_{i_h}, \ldots, \varepsilon_{i_k} \) and the indexes \( i_h, \ldots, i_k \subseteq \{i_1, \ldots, i_r\} \) are an even number. Hence \( \Psi^{-1}(r_2 \cdots r_e) = (\bar{1}_{i_h \cdots i_k} (i_{1 \cdots e})), \) where \( \bar{1}_{i_h \cdots i_k} \in (Z_2)^d \) sends to \( \bar{1} \) only an even number of indexes.

Throughout this paper we denote by \((a; \xi)\) an element in \( (Z_2)^d \times^s S_d \) such that \( \xi \) is a product of \( r \) disjoint cycles, \( \xi_1, \ldots, \xi_r \), with \( \xi_i \) \( e_i \)-cycle and \( a \in (Z_2)^d \) a
function which sends to \( \bar{1} \) only an even number of indexes moved by each \( \xi_i \). The element \((a; \xi)\) corresponds in \( S_{2d} \) to a product of \( 2r \) disjoint cycles of the form \( s_1 s'_1 \cdots s_r s'_r \). Note that an element of type \( c_1 r_2 \cdots r_e \), instead, corresponds in \( S_{2d} \) to a \( 2e \)-cycle of type \((i_1 \pm i_2 \pm \cdots \pm \varepsilon_{i_1} \mp i_2 \pm \cdots \mp \varepsilon_{i_e})\) and in \( (\mathbb{Z}_2)^d \times S_d \) to \((\bar{1}_{i_1 \cdots i_e}; (i_1 \cdots i_e))\) where \( \bar{1}_{i_1 \cdots i_e} \in (\mathbb{Z}_2)^d \) sends to \( \bar{1} \) only an odd number of indexes moved by \((i_1 \cdots i_e)\). From now on let us denote by \((a'; \xi) \in (\mathbb{Z}_2)^d \times S_d \) an element satisfying the following: \( \xi \) is product of \( r \) disjoint cycles \( \xi_1, \ldots, \xi_r \), with \( \xi_i \) \( e_i \)-cycle, and \( a' \in (\mathbb{Z}_2)^d \) is a function which sends to \( \bar{1} \) an even number of indexes moved by each \( \xi_i, i \notin \{j_1, \ldots, j_v\} \subset \{1, \ldots, r\} \), and an odd number of indexes moved by each cycle \( \xi_j, j \in \{j_1, \ldots, j_v\} \).

**Observation 1.2.** – Let \((z_{ij}; (ij)), (z'_{jh}; (ih)), (\bar{1}_i; id), (a; \xi) \in (\mathbb{Z}_2)^d \times S_d \), with \( \xi = (\ldots h_i \ldots) (\ldots k \ldots) \ldots \). Then

\[(z_{ij}; (ij))(z'_{jh}; (ih))(z_{ij}; (ij))^{-1} = (z_{ij} + z'_{jh}; (ij)(ih))(z_{ij}; (ij))
= (z_{ij} + z'_{jh} + z_{hi}; (ij)(ih)) = ((z + z')_{jh}; (jh)).\]

\[(\bar{1}_i; id)(z_{ij}; (ij))(\bar{1}_i; id)^{-1} = (\bar{1}_i + z_{ij}; (ij))(\bar{1}_i; id) = (\bar{1}_i + z_{ij} + \bar{1}_j; (ij))
= ((1 + z)_{ij}; (ij)).\]

\[(z_{ij}; (ij))(1_{ij}; id)(z_{ij}; (ij))^{-1} = (z_{ij} + 1_{ij}; (ij))(z_{ij}; (ij)) = (z_{ij} + 1_{ij} + z_{ij}; id)
= (1_{ij}; id).
\]

\[(\bar{1}_i; id)(a; \xi)(\bar{1}_i; id)^{-1} = (1_i + a; \xi)(\bar{1}_i; id) = (1_i + a + \bar{1}_k; \xi).
\]

\[(z_{jk}; (jk))(a; \xi)(z_{jk}; (jk))^{-1} = (z_{jk} + \Phi((jk)a; (jk)\xi)(z_{jk}; (jk))
= (z_{jk} + \Phi((jk)a; z_{jk}(jk))\xi(jk))\]

where \( \Phi((jk)a) \) is a function which sends to \( \bar{1} \) the same number of indexes sent to \( \bar{1} \) by \( a \). Therefore \( z_{jk} + \Phi((jk)a) + z_{jk} \) sends to \( \bar{1} \) only an even number of indexes moved by each cycles \((jk)\xi_i(jk)\).

1.3. Let \( X, X' \) and \( Y \) be smooth, connected, projective complex curves of genus \( \geq 0 \). Let \( d \geq 3 \) be an integer and let \( e = (e_1, \ldots, e_r) \) be a partition of \( d \) where \( e_1 \geq \cdots \geq e_r \geq 1 \). We associate to \( e \) the following element in \( S_{2d} \) having cycle type \( e \).

\[(12 \cdots e_1)(e_1 + 1 \cdots e_1 + e_2) \cdots ((e_1 + \cdots + e_{r-1}) + 1 \cdots d).\]

We denote by \( C_1 \) and \( C_2 \) the conjugate classes of \( W(B_2d) \) containing respectively reflections with respect to short roots, i.e., elements of type \((\bar{1}_i; id)\) and reflections with respect to long roots, i.e., elements of type \((z_{ij}; (ij))\). Moreover let \( C_e \) be conjugate classes of \( W(B_2d) \) containing respectively elements of the form \((a; \xi)\) and of the form \((a'; \xi)\), where \( \xi \) is product of \( r \) disjoint cycles \( \xi_1 \cdots \xi_r \) with \( \xi_i \) \( e_i \)-cycle.

**Definition 1.** – An ordered sequence \((t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)\) of permutations of \( S_{2d} \) such that \( t_i \neq id \) for each \( i = 1, \ldots, n \) and \( t_1 t_2 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g] \) is called a Hurwitz system. The subgroup \( G \subset S_{2d} \) generated
by $t_i, \lambda_k, \mu_k$ with $i = 1, \ldots, n$ and $k = 1, \ldots, g$ is called the monodromy group of the Hurwitz system.

Note that if $g = 0$ the Hurwitz systems $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ are of the form $(t_1, \ldots, t_n)$ and $t_1 \cdots t_n = \text{id}$.

**Definition 2.** We call $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ a Hurwitz system with values in $\mathbb{Z}^d \times S_d$ if $t_i \lambda_k, \mu_k \in (\mathbb{Z}^d \times S_d)$ for each $i$ and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$. Let us denote by $A_{(n_1, n_2, g)}$ the set of all Hurwitz systems, $(t_1, \ldots, t_{n_1 + n_2}; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$, with values in $\mathbb{Z}^d \times S_d$, with monodromy group $(\mathbb{Z}^d \times S_d)$, such that $n_1$ belong to $C_1$ and $n_2$ to $C_2$. Moreover we denote by $A_{(n_1, n_2, g)}$ the set of all Hurwitz systems, $(t_1, \ldots, t_{n_1 + n_2 + 1}; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$, with values in $\mathbb{Z}^d \times S_d$, with monodromy group all $(\mathbb{Z}^d \times S_d)$, such that $n_1$ among the $t_i$ belong to $C_1$, $n_2$ to $C_2$ and one belong to $C_3$ (resp. $C_4$). Note that when $g = 0$ we put $A_{(n_1, n_2, 0)} := A_{(n_1, n_2)}$, $A_{(n_1, n_2, 0)} := A_{(n_1, n_2, g)}$ and $A_{(n_1, n_2, [\lambda_1, \ldots, \lambda_g], 0)} := A_{(n_1, n_2, [\lambda_1, \ldots, \lambda_g])}$.

Let $g$ be the genus of $Y$ and let $b_0 \in Y$. By Riemann’s existence theorem there is a natural one-to-one correspondence between the following sets:

- the set of equivalence classes of pairs $[h : X \rightarrow Y, \varphi]$ where $h$ is a degree $2d$ covering unramified in $b_0$ and with branch locus $D$ and $\varphi : h^{-1}(b_0) \rightarrow \{-d, \ldots, -1, 1, \ldots, d\}$ a bijection, and
- the set of homomorphisms $m : \pi_1(Y - D, b_0) \rightarrow S_{2d}$ whose monodromy group is transitive.

Let $D = \{b_1, \ldots, b_n\}$, we choose loops $\gamma_i$ around $b_i$ and closed arcs $a_k, b_k$ oriented counterclockwise so that $\gamma_1, \ldots, \gamma_n, a_1, b_1, \ldots, a_g, b_g$ is a standard generating system of the fundamental group $\pi_1(Y - D, b_0)$. The images via the monodromy homomorphisms $m$ of $\gamma_1, \ldots, \gamma_n, a_1, b_1, \ldots, a_g, b_g$ determine Hurwitz systems whose monodromy groups are transitive subgroups of $S_{2d}$. Then chosen a standard generating system of $\pi_1(Y - D, b_0)$, to each class $[h : X \rightarrow Y, \varphi]$ corresponding a Hurwitz system with transitive monodromy group.

Let us denote by $H_{W(B_0), (n_1, n_2)}(Y, b_0)$ be the Hurwitz space that parametrizes equivalence classes of pairs $[h : X \rightarrow Y, \varphi]$ where $h$ is a degree $2d$ covering unramified in $b_0$, with $n_1 + n_2$ branch points and Hurwitz system belonging to $A_{(n_1, n_2, g)}$. By $H_{W(B_0), (n_1, n_2, 0)}(Y, b_0)$ and $H_{W(B_0), (n_1, n_2, [\lambda_1, \ldots, \lambda_g])}(Y, b_0)$ we denote the Hurwitz spaces that parametrize equivalence classes of pairs $[h : X \rightarrow Y, \varphi]$ as above, only this time $h$ is branched in $n_1 + n_2 + 1$ points and has Hurwitz system belonging respectively to $A_{(n_1, n_2, g)}$ and to $A_{(n_1, n_2, [\lambda_1, \ldots, \lambda_g])}$. In reality in this paper we work with sequences of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where $\pi$ is a branched cov-
ering of degree 2 and f is a degree d branched covering with monodromy group $S_d$. We denote by $D_\pi$ the branch locus of $\pi$, by $D_f$ the one of $f$ and by $D$ we denote the branch locus of $f \circ \pi$.

**Definition 3.** A sequence of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ where $\pi$ is a branched covering of degree 2 and $f$ is a degree d branched covering with monodromy group $S_d$ is called:

a) of type $(n_1; n_2)$ if $\pi$ is branched in $n_1$ points, $f$ is a simple covering with $n_2$ branch points and $f(D_\pi) \cap D_f = \emptyset$;

b) of type $(n_1; n_2, c)$ if $f$ is branched in $n_2 + 1$ points, $n_2$ of which are points of simple branching and one is a special point $c$ whose local monodromy has cyclic type $c$, and again $\pi$ has $n_1$ branch points and $f(D_\pi) \cap D_f = \emptyset$;

c) of type $(n_1; n_2, [j_1, \ldots, j_v])$ if $f$ is a covering as in b), $\pi$ is a covering with $n_1 + v > 1$ branch points such that $D_\pi \cap f^{-1}(c) = \{c_{j_1}, \ldots, c_{j_v}\}$ and moreover $D_\pi$ is not contained in $f^{-1}(c)$.

Note that throughout this paper we will work with sequences such that if $x \in D_\pi$ and $f(x) = y$ with $y \neq c$ then $f^{-1}(y) \cap D_\pi = \{x\}$.

Let $b_0 \in Y - D$ and let $[f \circ \pi, \phi]$ be the equivalence class of a sequence of coverings $f \circ \pi$ and a bijection $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \ldots, -1, 1, \ldots, d\}$ such that if $f^{-1}(b_0) = \{y_1, \ldots, y_d\}$ and $\pi^{-1}(y_i) = \{x_i, x_{-i}\}$, $\phi(x_i) = i$ and $\phi(x_{-i}) = -i$. We want to understand what is the Hurwitz system associated to a class of this type.

At first we suppose that $f \circ \pi$ is a sequence of type $(n_1; n_2)$. Let $[\gamma] \in \pi_1(Y - D, b_0)$ and let $\gamma$ be a closed arc that bounds a region containing an unique point $b \in D$. If $b \in f(D_\pi)$, lifting $\gamma$ through $f$ we obtain $d$ closed arcs, one of which bounds a point in $D_\pi$. The lifting of this arc through $\pi$ is an arc $\tilde{\gamma}$ with $\tilde{\gamma}(0) = x_i$ and $\tilde{\gamma}(1) = x_{-i}$ for some $i \in \{1, \ldots, d\}$. So $m(\gamma) \in S_{2d}$ is a transposition that sends $i$ to $-i$, i.e., the local monodromy $m(\gamma)$ is a reflection with respect to the short root $\epsilon_i$. If instead $b \in D_f$, lifting $\gamma$ through $f$ we obtain $d - 2$ closed arcs and one arc $\tilde{\gamma}$ with $\tilde{\gamma}(0) = y_i$ and $\tilde{\gamma}(1) = y_j$. Lifting $\tilde{\gamma}$ through $\pi$ we obtain two distinct arcs of $X$ having starting points in the set $\{x_i, x_{-i}\}$ and ending points in $\{x_j, x_{-j}\}$. So $m(\gamma)$ is a permutation of $S_{2d}$ that transforms the set $\{i, -i\}$ into the set $\{j, -j\}$, i.e., the local monodromy $m(\gamma)$ is a reflection with respect to a long root. Then the Hurwitz system $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_d, \mu_d)$ associated to the equivalence class $[f \circ \pi, \phi]$ is such that $n_1$ among the $t_i$ belong to $C_1$ and $n_2$ to $C_2$. Moreover since the monodromy group of $f$ is all $S_d$ and at least one among the $t_i$ is of the form $(1; id)$, the monodromy group of $f \circ \pi$ is all $(\mathbb{Z}_2)^d \times S_d$. This implies that the Hurwitz system of a pair $[f \circ \pi, \phi]$ where $f \circ \pi$ is a sequence of type $(n_1; n_2)$ belong to $A_{(n_1; n_2), g}$.

Now we suppose that $f \circ \pi$ is a sequence of type $(n_1; n_2, c)$. Again let $[\gamma] \in \pi_1(Y - D, b_0)$ with $\gamma$ closed arc that bounds a region containing an unique point $b \in D$. Let $b = c$ and let $f^{-1}(c) = \{c_1, \ldots, c_r\}$ where $c_i$ has multiplicity $e_i$, $i = 1, \ldots, r$. Since $c_i$ has multiplicity $e_i$, there are $e_i$ lifting of $\gamma$ in $X'$.
\(\gamma_1, \gamma_2, \ldots, \gamma_{e_i}, \) such that \(\gamma_j(0) = y_{j-1}, \gamma_j(1) = y_j+1 = \gamma_{j+1}(0),\) for each \(j = 1, \ldots, (e_i-1)\) and \(\gamma_{e_i}(1) = y_1.\) Lifting each \(\gamma_j\) by \(\pi\) we obtain two disjoint arcs with starting points in \(\{x_{j-1}, x_j\}\) and ending points in the set \(\{x_{j-1}, x_j\} \in S_{2d}\) transforms \(\{j_1, -j_1\}\) into \(\{j_1, -j_1\}\) for each \(j = 1, \ldots, e_i\) and \(i = 1, \ldots, r.\) Hence \(m(\gamma)\) is product of \(2r\) disjoint cycles, \(s_1s_2^r \cdots s_{r-v}s_{r-v-1}^v q_{j_1} \cdots q_{j_v}\), satisfying the following: \(q_{j_k}\) for each \(k = 1, \ldots, v,\) is a \(2e_j\)-cycle of type \((h_1 \ldots h_{e_j} - h_1 \ldots h_{e_j})\) where the indexes \(h_2, \ldots, h_{e_j}\) can be either positive or negative and the cycles \(s_1, s_2^r\) are as above. Then \(m(\gamma)\) corresponds in \((\mathbb{Z}_2)^d \times s_{d} \) to an element of the form \((a', \xi).\) So we can assert that the Hurwitz system associated to the class \([f \circ \pi, \phi]\) where \(f \circ \pi\) is a sequence of type \((n_1; n_2, [j_1, \ldots, j_v])\) belong to \(A_{(n_1; n_2, [j_1, \ldots, j_v])}.\)

From what we have said above and by Riemann’s existence theorem we can identify the space of the pairs \([f \circ \pi, \phi]\) where \(f \circ \pi\) is a sequence of type \((n_1; n_2)\) by \(H_{W(B_d)}(n_1; n_2)(Y, b_0),\) the space of the pairs \([f \circ \pi, \phi]\) where \(f \circ \pi\) is a sequence of type \((n_1; n_2, e)\) by \(H_{W(B_d)}(n_1; n_2, e)(Y, b_0)\) and the space of the pairs \([f \circ \pi, \phi]\) where \(f \circ \pi\) is a sequence of type \((n_1; n_2, [j_1, \ldots, j_v])\) by \(H_{W(B_d)}(n_1; n_2, [j_1, \ldots, j_v])(Y, b_0)\).

Let \(H_{W(B_d)}(n_1; n_2)(Y), H_{W(B_d)}(n_1; n_2, e)(Y)\) and \(H_{W(B_d)}(n_1; n_2, [j_1, \ldots, j_v])(Y)\) be the Hurwitz spaces that parametrize equivalence classes \([f \circ \pi, \phi]\) where \(f \circ \pi\) is a sequence of branched coverings of type respectively \((n_1; n_2),\) \((n_1; n_2, e)\) and \((n_1; n_2, [j_1, \ldots, j_v]).\)

Let \(Y^{(n)}\) be the \(n\)-fold symmetric product of \(Y\) and let \(\Delta\) be the codimension 1 locus of \(Y^{(n)}\) consisting of non simple divisors. Let \(\delta_1 : H_{W(B_d)}(n_1; n_2)(Y, b_0) \rightarrow (Y - b_0)^{(n_1 + n_2)} - \Delta, \delta_2 : H_{W(B_d)}(n_1; n_2, e)(Y, b_0) \rightarrow (Y - b_0)^{(n_1 + n_2 + 1)} - \Delta\) and \(\delta_3 : H_{W(B_d)}(n_1; n_2, [j_1, \ldots, j_v])(Y, b_0) \rightarrow (Y - b_0)^{(n_1 + n_2 + 1)} - \Delta\) be the maps which assign to each \([f \circ \pi, \phi]\) the branch locus \(D\) of \(f \circ \pi.\) By Riemann’s existence theorem we can identify the fiber of \(\delta_1, \delta_2, \delta_3\) over \(D\) respectively with \(A_{(n_1; n_2, g), A_{(n_1; n_2, e), g}, A_{(n_1; n_2, [j_1, \ldots, j_v]), g}}.\) There is an unique topology on \(H_{W(B_d)}(n_1; n_2)(Y, b_0), H_{W(B_d)}(n_1; n_2, e)(Y, b_0)\) and \(H_{W(B_d)}(n_1; n_2, [j_1, \ldots, j_v])(Y, b_0)\) such that \(\delta_1, \delta_2, \delta_3\) are topolo-
gical covering maps (see [7]). Therefore the braid group $\pi_1((Y - b_0)^{(n_1+n_2)} - \Delta, D)$ acts on $A_{(n_1+n_2),g}$ and the braid group $\pi_1((Y - b_0)^{(n_1+n_2+1)} - \Delta, D)$ acts on $A_{(n_1+n_2,}\mathfrak{s}),g)$ and on $A_{(n_1+n_2,\mathfrak{g},\mathfrak{s}),g}$.

If these actions are transitive then the Hurwitz spaces $H_{W(B_d),(n_1+n_2),Y,b_0}$, $H_{W(B_d),(n_1+n_2),Y,b_0}$ and $H_{W(B_d),(n_1+n_2,\mathfrak{g},\mathfrak{s}),Y,b_0}$ are connected.

1.4. Shortly we recall some notion on braid groups. The generators of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ are the elementary braids $\sigma_i$ with $i = 1, \ldots, n - 1$ and the braids $\rho_{jk}$, $\tau_{jk}$ with $1 \leq j \leq n$ and $1 \leq k \leq g$ (see [3], [6], [14]). The calculation of the action of the elementary braids $\sigma_i$ on Hurwitz systems is due to Hurwitz [9].

The elementary moves $\sigma'_i$, relative to the elementary braids $\sigma_i$, bring

$$(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$$

to

$$(t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

Therefore their inverses bring $(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ to

$$(t_1, \ldots, t_{i-1}, t_i^{-1}, t_i t_{i+1} t_i^{-1}, t_i, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

The braid moves that correspond to the generators $\rho_{ik}$, $\tau_{ik}$ were studied by Graber, Harris, Starr in [8] and by Kanev in [10]. We make use of some results proved in [10]. In this paper to each generator $\rho_{ik}$ or $\tau_{ik}$ is associated a pair of braid moves $\rho'_{ik}$, $\rho''_{ik} = (\rho'_{ik})^{-1}$ and $\tau'_{ik}$, $\tau''_{ik} = (\tau'_{ik})^{-1}$ respectively.

Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system. The braid move $\rho'_{ik}$ leaves unchanged $\lambda_l$ for each $l$, $t_j$ for each $j \neq i$ and $\mu_l$ for each $l \neq k$, while changing $t_i$ and $\mu_k$. Analogously the braid move $\tau''_{ik}$ changes $t_i$ and $\lambda_k$, leaving unchanged $\mu_l$ for each $l$, $\lambda_l$ for each $l \neq k$ and $t_j$ for each $j \neq i$.

We use the following result.

**Proposition 1** ([10] Corollary 1.9). - Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system. Let $u_k = [\lambda_1, \mu_1] \ldots [\lambda_k, \mu_k]$ for $k = 1, \ldots, g$ and let $u_0 = \text{id}$. The following formulae hold:

i) For $\rho'_1k$:

$$\rho'_1k : \mu_k \rightarrow \mu'_k = (b_1^{-1} t_i^{-1} b_1) \mu_k,$$

where $b_1 = u_{k-1} \lambda_k$

ii) For $\tau''_{1k}$:

$$\tau''_{1k} : \lambda_k \rightarrow \lambda''_k = (u_{k-1}^{-1} t_i^{-1} u_{k-1}) \lambda_k.$$

In particular

$$\tau''_{11} : \lambda_1 \rightarrow t_1^{-1} \lambda_1.$$
2. – Irreducibility of $H_{W(B_d),(n_1;n_2)}(Y, b_0)$, $H_{W(B_d),(n_1;n_2, e)}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,...,j_s])}(Y, b_0)$.

In this section we will prove the irreducibility of the Hurwitz spaces $H_{W(B_d),(n_1;n_2)}(Y, b_0)$, $H_{W(B_d),(n_1;n_2, e)}(Y, b_0)$ and $H_{W(B_d),(n_1;n_2,[j_1,...,j_s])}(Y, b_0)$. Since these spaces are smooth in order to prove their irreducibility it suffices to prove they are connected. We observed that if $\pi_1((Y - b_0)^{(n_1+n_2)} - \Delta, D)$ acts transitively on $A_{(n_1;n_2),g}$, $H_{W(B_d),(n_1;n_2)}(Y, b_0)$ is connected. Analogously if $\pi_1((Y - b_0)^{(n_1+n_2+1)} - \Delta, D)$ acts transitively on $A_{(n_1;n_2),g}$ ($A_{(n_1;n_2,[j_1,...,j_s]),g}$) the Hurwitz space $H_{W(B_d),(n_1;n_2, e)}(Y, b_0)$ (resp. $H_{W(B_d),(n_1;n_2,[j_1,...,j_s])}(Y, b_0)$) is connected. In order to prove the transitivity of these actions we will prove that, acting by braid moves, it is possible to bring each Hurwitz system respectively in $A_{(n_1;n_2),g}$, $A_{(n_1;n_2, e),g}$, $A_{(n_1;n_2,[j_1,...,j_s]),g}$ to a given normal form.

**Definition 4.** – We call two Hurwitz systems with values in $(\mathbb{Z}_2)^d \times S_d$ braid equivalent if one is obtained from the other by a finite sequence of braid moves $\sigma_i, \rho_{jk}^i, \tau_{jk}^i, (\sigma_i)^{-1}, \rho_{jk}^{\sigma_i}, \tau_{jk}^{\sigma_i}$ where $1 \leq i \leq n - 1, 1 \leq j \leq n$ and $1 \leq k \leq g$. We denote the braid equivalence by $\sim$.

**Definition 5.** – Two ordered $n$-tuples (or sequences) of elements in $(\mathbb{Z}_2)^d \times S_d$, $(t_1, \ldots, t_n)$ and $(\tilde{t}_1, \ldots, \tilde{t}_n)$, are called braid equivalent if $(\tilde{t}_1, \ldots, \tilde{t}_n)$ is obtained from $(t_1, \ldots, t_n)$ by a finite sequence of braid moves of type $\sigma_i'$, $(\sigma_i^*)^{-1}$. Note that if $t_1 \cdots t_n = s$ then $\tilde{t}_1 \cdots \tilde{t}_n = s$.

We use the following result.

**Lemma 1** ([12] Lemma 2.4). – Let $(t'_1, \ldots, t'_n)$ be a sequence of transpositions of $S_d$ such that $G = (t'_1, \ldots, t'_n)$ is transitive. Then $(t'_1, \ldots, t'_n)$ is braid equivalent to $(\ldots, (ij))$ where $(ij)$ is an arbitrary transposition of $G$.

**Theorem 1.** – The Hurwitz space $H_{W(B_d),(n_1;n_2)}(D^1, b_0)$ is irreducible.

**Proof.** – The theorem follows if we prove that each Hurwitz system in $A_{(n_1;n_2)}$ is braid equivalent to the normal form

$$(0; (12)), (0; (12)), (0; (13)), (0; (13)), \ldots, (0; (1d - 1)), (0; (1d - 1)), (0; (1d)), \ldots, (0; (1d)), (\tilde{1}; id), \ldots, (\tilde{1}; id))$$

where each $(0; (1i))$, $2 \leq i \leq d - 1$, appears twice, $(0; (1d))$ appears $(n_2 - 2(d - 2))$-times and $(\tilde{1}; id)$ appears $n_1$-times.

**Step 1.** – We claim that each Hurwitz system $(t_1, \ldots, t_{n_1+n_2}) \in A_{(n_1;n_2)}$ is braid equivalent to a Hurwitz system of type $(\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (\tilde{1}; id), \ldots, (\tilde{1}; id))$. If among
the \(t_j\) there are elements of type \((\bar{1}_1; id)\), acting by elementary moves \(\sigma'_j\), we move them to the right, obtaining the sequence \((\bar{t}_1, \ldots, \bar{t}_{n_2}, (\bar{1}_1; id), \ldots, (\bar{1}_1; id))\). Acting again by appropriate braid moves \(\sigma'_j\), we moves to the right the other elements of our Hurwitz system of type \((1_1; id)\), obtaining the new system

\[
(\bar{t}_1, \ldots, \bar{t}_{n_2}, (1_k; id), \ldots, (1_k; id), (1_1; id), \ldots, (1_1; id)).
\]

Let \(\bar{t}_j = (z_{ij}; t'_j)\), \(j = 1, \ldots, n_2\). \((t'_1, \ldots, t'_{n_2})\) is the Hurwitz system of a covering of degree \(d \geq 3\) of \(\mathbb{P}^1\) with \(n_2\) points of simple branching and monodromy group \(S_d\). So by Lemma 1 we can replace \((\bar{t}_1, \ldots, \bar{t}_{n_2})\) by a new braid equivalent sequence which has at the place \(n_2\) \((z_{1k}; (1h))\). Applying \(\sigma'_{n_2}\) twice, by (iii) and (ii), one has

\[
(\ldots, (z_{1k}; (1h)), (1_1; id), \ldots) \sim (\ldots, (1_1; id), (z'_{1k}; (1h)), \ldots) \sim
\]

\[
(\ldots, (z'_{1k}; (1h)), (\bar{1}_1; id), \ldots).
\]

Now we move the new element of type \((\bar{1}_1; id)\) so obtained near by the others. Proceeding in this way for each \((1_1; id) \neq (1_1; id)\) we obtain the claim.

**Step 2.** – Starting by \((t_1, \ldots, t_{n_1+n_2})\) and applying Step 1 we obtain \((\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (\bar{1}_1; id), \ldots, (1_1; id))\). Because \(H_d, n_2(\mathbb{P}^1)\) is irreducible (see [9]), applying appropriate braid moves \(\sigma'_j\) and their inverses, we can bring the sequence \((\tilde{t}_1, \ldots, \tilde{t}_{n_2})\) to the form

\[
((a_{12}^0; (12)), (b_{12}^0; (12)), \ldots, (a_{id-1}^d; (1d - 1)), (b_{id-1}^d; (1d - 1)), (c_{id}^1; (1d)),
\]

\[
(c_{id}^2; (1d)), \ldots, (c_{id}^h; (1d))
\]

where \(h = (n_2 - 2(d - 2))\) and \(a^j, b^j, c^k \in \mathbb{Z}_2\). Seeing that

\[
(a_{12}^0; (12))(b_{12}^0; (12)) \cdots (a_{id-1}^d; (1d - 1))(b_{id-1}^d; (1d - 1))(c_{id}^1; (1d))(c_{id}^2; (1d)) \cdots (c_{id}^h; (1d))(\bar{1}_1; id) \cdots (1_1; id) = (0; id),
\]

one has

\[
(a_{12}^0 + b_{12}^0 + \cdots + a_{id-1}^d + b_{id-1}^d + c_{id}^1 + \cdots + c_{id}^h + \bar{1}_1 + \cdots + 1_1) = 0.
\]

That implies

\[
a^j + b^j \equiv 0 \pmod{2} \quad \text{for each} \quad j = 2, \ldots, d - 1 \quad \text{and so} \quad a^j = b^j, \quad \text{and}
\]

\[
c^1 + \cdots + c^h \equiv 0 \pmod{2}, \quad \text{then the} \ c^k = \bar{1} \text{are an even number.}
\]

If in our Hurwitz system there are elements of type \((0; (1d))\), applying braid moves of type \(\sigma'_j\) we moves them near by the elements \((\bar{1}_1; id)\). Since the number of \((0; (1d))\) is even, by elementary moves \(\sigma'_j\) we can move to the left one element of type \((\bar{1}_1; id)\) obtaining the Hurwitz system

\[
((a_{12}^0; (12)), (a_{12}^0; (12)), \ldots, (a_{id-1}^d; (1d - 1)), (a_{id-1}^d; (1d - 1)), (\bar{1}_{id}; (1d)), \ldots,
\]

\[
(\bar{1}_{id}; (1d)), (\bar{1}_1; id), (0; (1d)), \ldots, (0; (1d)), (\bar{1}_1; id), \ldots,
\]

where the elements \((\bar{1}_{id}; (1d))\) occupy the places \(2(d - 2) + 1, \ 2(d - 2) + 2, \ldots, k\).
Applying successively the braid moves \((\sigma'_k)^{-1}, (\sigma'_{k-1})^{-1}, \ldots, (\sigma'_{2(d-2)+1})^{-1}\) by (ii) we obtain
\[
((\bar{1}_{1d}; (1d)), \ldots, (\bar{1}_{1d}; (1d)), (\bar{1}; id)) \sim ((\bar{1}_{1d}; (1d)), \ldots, (\bar{1}_{1d}; (1d)), (\bar{1}; id), (0; (1d))) \sim \cdots \sim ((\bar{1}; id), (0; (1d)), \ldots, (0; (1d))).
\]
Now if \(a'^{-1}_{d-1} = \bar{1}\), acting by \((\sigma'_{2(d-2)})^{-1}, (\sigma'_{2(d-2)-1})^{-1}\), one has
\[
((\bar{1}_{1d-1}; (1d-1)), (\bar{1}_{1d-1}; (1d-1)), (\bar{1}; id)) \sim ((\bar{1}; id), (0; (1d-1)), (0; (1d-1))).
\]
If instead \(a'^{-1}_{d-1} = 0\), we use \(\sigma'_{2(d-2)}, \sigma'_{2(d-2)-1}\) so we replace
\[
((0; (1d-1)), (0; (1d-1)), (\bar{1}; id)) \quad \text{by} \quad ((\bar{1}; id), (0; (1d-1)), (0; (1d-1))).
\]
Proceeding in this way for each \(a'^{j}, j = 2, \ldots, d-2\), after \((d-2)\)-steps we obtain the system
\[
((\bar{1}; id), (0; (12)), (0; (12)), (0; (13)), (0; (13)), \ldots, (0; (1d-1)), (0; (1d-1)), (0; (1d)), \ldots, (0; (1d)), (\bar{1}; id), \ldots, (\bar{1}; id)).
\]
Now the proof follows by applying in the order the sequence of elementary moves \((\sigma'_{1})^{-1}, (\sigma'_{2})^{-1}, \ldots, (\sigma'_{n})^{-1}\).

**Remark 2.1.** – The irreducibility of \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1, b_0)\) also follows by observing the map \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1) \rightarrow H_{d,n_2}(\mathbb{P}^1)\) which sends \([X \xrightarrow{\pi} X' \xrightarrow{f} \mathbb{P}^1]\) to \([X' \xrightarrow{\pi'} \mathbb{P}^1]\) has fibers given by Hurwitz spaces of type \(H_{2,n_1}(X')\). Since the spaces \(H_{d,n_2}(\mathbb{P}^1)\) and \(H_{2,n_1}(X')\) are irreducible, \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1)\) is irreducible. \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1)\) parametrizes equivalence classes of coverings of \(\mathbb{P}^1\) with monodromy group \(G = W(B_d)\). Conjugating by elements of \(G\) we leave each braid orbit invariant (see [1] or [16] Lemma 9.4). This implies that the irreducible components of \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1)\) are in one-to-one correspondence by ones of \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1, b_0)\) and so the Hurwitz space \(H_{W(B_d),(n_1,n_2)}(\mathbb{P}^1, b_0)\) is irreducible.

2.2. Let \((t'_1, \ldots, t'_n)\) be a sequence of transpositions of \(S_d\) such that \(t'_1 \cdots t'_n = s\) and \(\langle t'_1, \ldots, t'_n \rangle\) is transitive. Let \(s = s_1 \cdots s_r\) be a factorization of \(s\) into a product of independent cycles and let \(I'_1, \ldots, I'_r\) be the domains of transitivity of \(s\). If \(\#I'_i = e_i\) for each \(1 \leq i \leq r\) and \(1_i\) is the minimal number in \(I'_i\), then we write \(s_i = (1_i, 2_i \ldots (e_i)_i)\). Let us order the \(I'_i\) so that \(1_1 < 1_2 < \cdots < 1_r\) and denote by \(Z_i\) the sequence \(((1_1, 2_i), (1_1, 3_i), \ldots, (1_1, (e_i)_i))\). Let \(Z\) be the concatenation \(Z_1Z_2\ldots Z_r\). The sequence \(Z\) consists of \(N = \sum_{i=1}^{r} (e_i - 1)\) transpositions. We use the following result.

**Proposition 2** ([11] or [12] pp. 369-370). – Let \((t'_1, \ldots, t'_n)\) be a sequence of transpositions such that \(t'_1 \cdots t'_n = s\) and \(\langle t'_1, \ldots, t'_n \rangle\) is transitive. Then \((t'_1, \ldots, t'_n)\) is braid equivalent to
\[
(Z, t''_N+1, \ldots, t''_n)
\]
where \( n - N \equiv 0(\text{mod } 2) \) and

(i) if \( r = 1 \) \( t''_i = (1_1 \ 2_1) \) for each \( i \geq N + 1 \)

(ii) if \( r > 1 \) then

\[
(t''_{N+1}, \ldots, t''_n) = ((1_1 \ 1_2), (1_1 \ 1_2), (1_1 \ 1_3), (1_1 \ 1_3), \ldots, (1_1 \ 1_r), \ldots, (1_1 \ 1_r))
\]

where each \((1_1, 1_i)\) appears twice if \( 2 \leq i \leq r - 1 \) and \((1_1, 1_r)\) appears an even number of times.

From now on let us denote the permutation Eq. (1) by

\[
\varepsilon = (1_1 \ 2_1 \ldots (e_1) \ldots (1_2 \ 2_2 \ldots (e_2) \ldots (1_r \ 2_r \ldots (e_r) \ldots\).\]

**Theorem 2.** – The Hurwitz space \( H_{W(B_d), (n_1; \ldots; n_d)}(\mathbb{P}^1, b_0) \) is irreducible.

**Proof.** – To prove the irreducibility of \( H_{W(B_d), (n_1; \ldots; n_d)}(\mathbb{P}^1, b_0) \) it is sufficient to check that each Hurwitz system in \( A_{(n_1; \ldots; n_d)} \) is braided equivalent to the normal form

\[
((0; (1_1 \ 2_1)), (0; (1_1 \ 3_1)), \ldots, (0; (1_r e_1)), (0; (1_2 \ 2_2)), \ldots, (0; (1_r e_2)), \ldots,
\]

\[
(0; (1_1 \ 2_1)), \ldots, (0; (1_r e_2)), (0; t''_{N+1}), \ldots, (0; t''_n), (1_1 \ 1_2), \ldots, (1_r \ 1_i, (1_1 \ 1_i) (1_r \ 1_i), (0; \varepsilon^{-1}))
\]

where \((1_1; 1_i)\) appears \( n_1 \)-times, \( N = \sum_{i=1}^n e_i - r, n_2 - N \equiv 0(\text{mod } 2) \) and

i) if \( r = 1 \) \((0; t''_j) = (0; (1_1 \ 2_1))\) for each \( j = N + 1, \ldots, n_2, \)

ii) if \( r > 1 \),

\[
((0; t''_{N+1}), \ldots, (0; t''_n)) = ((0; (1_1 \ 1_2)), (0; (1_1 \ 1_2)), (0; (1_1 \ 1_3)), (0; (1_1 \ 1_3)), \ldots,
\]

\[
(0; (1_1 \ 1_r)), \ldots, (0; (1_r \ 1_i)))
\]

where each \((0; (1_1 \ 1_i)), 2 \leq i \leq r - 1, \) appears twice and \((0; (1_1 \ 1_r))\) an even number of times.

**Step 1.** – We prove that, applying elementary moves \( \sigma_j \) and their inverses, it is possible replace \((t_1, \ldots, t_{n_1+n_2+1}) \in A_{(n_1; \ldots; n_d)} \) by a new system which has \((0; \varepsilon^{-1})\) at the place \((n_1 + n_2 + 1)\). Using inverses of braided moves \( \sigma_j \) we bring to the first \( n_2 \) places of our Hurwitz system the \( t_j \) of type \((z_{ik}; (i_k))\) and to the place \((n_2 + 1)\) the element \((a; \xi)\), obtaining

\[
(\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (a; \xi), (1_k; id), \ldots, (1_r; id)).
\]

Let \( \tilde{t}_j = (z_{ik}; t''_j), (t'_{1}, \ldots, t'_{n_2}, \xi) \) is the Hurwitz system of a covering of degree \( d \geq 3 \) of \( \mathbb{P}^1 \), with monodromy group \( S_d \), branched in \( n_2 + 1 \) points, \( n_2 \) of whose are points of simple branching and one is a special point which local monodromy has cycle type \( \varepsilon \). Since \( H_{d, n_2, \xi}(\mathbb{P}^1, b_0) \) is irreducible (see [13], [11] or [12]), acting by appropriate braid moves \( \sigma_j^{-1}, (\sigma_j')^{-1} \), we can replace the sequence \((t'_{1}, \ldots, t'_{n_2}, \xi) \) by a new sequence of type \((t''_{1}, \ldots, t''_{n_2}, \varepsilon^{-1})\). In this way our Hurwitz system results braided equivalent to \((\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (b; \varepsilon^{-1}), \ldots)\), where \( \varepsilon^{-1} = q_1 \cdots q_r, \) with the \( q_i \)
disjoint cycles, and \( b \in (\mathbb{Z}_2)^d \) is, by (v), a function that sends to \( \bar{1} \) only an even number of indexes moved by each cycle \( q_i \). By braid moves \( \sigma'_j \) we move \((b; e^{-1})\) to the last place of our Hurwitz system.

Let \( i^1, \ldots, i^s \) the indexes moved by \( q_i \), \( i = 1, \ldots, r \), which \( b \) sends to \( \bar{1} \). We assume these indexes are in the order \((i^1, \ldots, i^2, \ldots, i^s, \ldots)\).

Since \( (t''_1, \ldots, t''_{n_2}, e^{-1}) = S_d \) and \( t''_1 \cdot \cdots \cdot t''_{n_2} = e \), \( (t''_1, \ldots, t''_{n_2}) = S_d \). Note that we can ever assume that among the elements of type \((1_\nu; \nu \in \mathbb{Z}_2)\) in our Hurwitz system there is \((\bar{1}_\nu; \nu)\), for each \( 1 \leq i \leq r, 1 \leq j \leq s_i \). In fact, applying Lemma 1 we can replace \((\ldots, t_{n_2}, (\bar{1}_k; \nu), \ldots, (\bar{1}_\nu; \nu), (b; e^{-1}))\) by

\[
(\ldots, (z_{k'; \nu} ; (k'; \nu')) , (\bar{1}_k; \nu), \ldots, (1_\nu; \nu), (b; e^{-1})).
\]

In the end using the elementary move \( \sigma'_{n_2} \), by (iii), we obtain a new Hurwitz system with \((1_\nu; \nu)\) at the place \( n_2 \).

At first we check that our Hurwitz system is braid equivalent to a system which has \((b; e^{-1})\) at the last place, where \( \bar{b} \in (\mathbb{Z}_2)^d \) sends to \( \bar{1} \) all indexes moved by \( q_1 \) and the indexes moved by \( q_i \), \( i = 2, \ldots, r \), distinct by \( i^1, \ldots, i^s \). By elementary moves \( \sigma'_j \) we move \((1_{n_1}; \nu)\) to the left of \((b; e^{-1})\) and after we apply \( \sigma'_{n_1+n_2} \) obtaining by (iv) the Hurwitz system

\[
(\ldots, (\bar{1}_{1n_1} + b + 1_\nu; e^{-1}), (\bar{1}_{1n_1}; \nu))
\]

where \( * \) is the index that precedes \( 1^s_\nu \) in \( q_1 \).

If \( * = 1^{s_i-1}_\nu \), the only indexes moved by \( q_1 \) that \( b' = \bar{1}_{1n_1} + b + 1_\nu \) sends to \( \bar{1} \) are \( 1^1_\nu, \ldots, 1^{s_i-2}_\nu \). If instead \( * \neq 1^{s_i-1}_\nu \), \( b' \) sends to \( \bar{1} \) also \( 1^{s_i-1}_\nu \) and \( * \).

Let \( l \) be the number of indexes moved by \( q_1 \) included between \( 1^{s_i-1}_\nu \) and \( 1^{s_i}_\nu \). Reasoning as above after \((l + 1)\)-steps we obtain a new system having at the last place \((b; e^{-1})\) where \( \bar{b} = 1_\nu + \bar{b} + \bar{1}_{1n_1} \), \( * \) is the index which follows \( 1^{s_i-1}_\nu \) in \( q_1 \) and the only indexes moved by \( q_1 \) sent to \( \bar{1} \) by \( \bar{b} \) are \( 1^1_\nu, \ldots, 1^{s_i-1}_\nu, * \).

In this way we replaced \( b \) by a new function of \((\mathbb{Z}_2)^d \) which sends to \( \bar{1} \) the same indexes sent to \( \bar{1} \) by \( b \), except the two indexes \( 1^{s_i-1}_\nu \) and \( 1^{s_i}_\nu \). So proceeding also for \((\bar{1}_{1n_1}; \nu), (\bar{1}_{1n_1}; \nu), \ldots, (1_{1n_2}; \nu)\), after \( s_{1/2} \) steps, we obtain a new system having as required at the \((n_1 + n_2 + 1)\) place \((b; e^{-1})\).

Reasoning so also with the indexes moved by \( q_2, \ldots, q_r \) that \( \bar{b} \) sends to \( \bar{1} \) after \((\sum_{i=2}^{r} s_{1/2})\) steps we obtain the claim.

**Step 2.** – Starting by the Hurwitz system \((t_1, \ldots, t_{n_1+n_2+1})\) and applying Step 1 we have obtained the system \((\tilde{t}_1, \ldots, \tilde{t}_{n_1+n_2}, (0; e^{-1}))\). Since the group generated by transpositions corresponding to the elements \((z_{ih}; (ih))\) is all \( S_d \), we can proceed as in Step 1, Theorem 1 and so we can bring our system to the form

\[
(\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (\bar{1}_1; \nu), \ldots, (\bar{1}_1; \nu), (0; e^{-1})).
\]
By Proposition 2 this system is braid equivalent to the system
\[
( (a^1_{1,2}; (1_2 2_1)), (b^1_{1,3}; (1_3 3_1)), \ldots, (e^1_{1,1}; (1_1,1_1)), (a^2_{1,2}; (1_2 2_1)), \ldots,
( e^2_{1,2}; (1_2 2_1)), \ldots, (a^r_{1,2}; (1_2 2_r)), \ldots, (e^r_{1,1}; (1_r, e_r)), (z^{N+1}, t^r_{N+1}), \ldots,
( z^{n_2}; t^r_{n_2}), (1_{1}; i\!d), \ldots)
\]
where \(a^i, b^i, \ldots, e^i, z^j \in \mathbb{Z}_2\).

At first we analyze the case \( r > 1 \). Put
\[
(z^{N+1}_1, \ldots, z^{n_2}) = (z^2_{1,12}, (z^2_{1,12})', \ldots, z^{r-1}_{1,12}, (z^{r-1})'_{1,12}, z^r_{1,12}, \ldots, (z^r)_{1,12}).
\]
Seeing that
\[
( (a^1_{1,2}; (1_2 2_1)) \cdots (e^1_{1,1}; (1_1,1_1)) \cdots (a^r_{1,2}; (1_r 2_r)) \cdots (e^r_{1,1}; (1_r, e_r)),
( z^2_{1,12}; (1_2 1_2)) \cdots (z^r_{1,12}; (1_r 1_2)) \cdots (z^r)_{1,12}; (1_r 1_2)) (1_{1}; i\!d) \cdots
\]
one has
\[
( (a^1_{1,2}; b^1_{2,3}) + \cdots + e^1_{1,1}; (1_1,1_1) + \cdots + a^r_{1,2} + \cdots + e^r_{1,1}; (1_r, e_r) + z^2_{1,12} +
( z^2)'_{1,12} + \cdots + z^r_{1,12} + \cdots + (z^r)'_{1,12} + (z^r)_{1,12} = 0.
\]
That implies
\[
a^i = 0 \quad \text{for each } i \quad \text{and so } b^i = c^i = \ldots = e^i = 0 \quad \text{for each } i,
\]
\[
z^j + (z^j)' \equiv 0 \pmod{2} \quad \text{for each } j = 2, \ldots, r-1 \text{ and then } z^j = (z^j)',
\]
\[
z^r + (z^r) + \cdots + (z^r)' \equiv 0 \pmod{2} \quad \text{and therefore the } (z^r)^b = 1 \text{ are an even number.}
\]

If in our Hurwitz system there are elements of type \((0; (1_1,1_r))\), acting by suitable elementary moves \(\sigma_j^r\), we move them to the right obtaining
\[
((0; (1_2 2_1)), \ldots, (0; (1_r e_r)), (z^2_{1,12}; (1_2 1_2)), \ldots,
( z^2_{1,12}; (1_2 1_2)), \ldots, (z^{r-1}_{1,12}; (1_1,1_r)), \ldots, (z^{r-1}_{1,12}; (1_2,1_r)), \ldots,
(1_{1,1}; (1_1,1_r)), (0; (1_1,1_r)), (1_{1}; i\!d), \ldots)
\]
where the elements \((1_{1,1}; (1_1,1_r))\) appear at the places \((\Sigma; e^i + r - 4) + 1, \ldots, k\).

Since the elements \((0; (1_1,1_r))\) are an even number, we can move using elementary moves \(\sigma_j^r\) one element \((1_{1}; i\!d)\) near by the elements of type \((1_{1,1}; (1_1,1_r))\) and then we apply the braid moves \((\sigma^r_1)^{-1}, (\sigma^r_{k-1})^{-1}, \ldots, (\sigma^r_{(\Sigma; e^i + r - 4) + 1})^{-1}\). So by (ii) we can replace \(((1_{1,1}; (1_1,1_r)), \ldots, (1_{1,1}; (1_1,1_r)), (1_{1}; i\!d), \ldots)\) by
\[
((1_{1}; i\!d), (0; (1_1,1_r)), \ldots, (0; (1_1,1_r))).
\]
Now if \(z^{r-1} = 1\) we act by the braid moves \((\sigma^r_{(\Sigma; e^i + r - 4)})^{-1}, (\sigma^r_{(\Sigma; e^i + r - 5)})^{-1}\), if instead \(z^{r-1} = 0\) we use \(\sigma^r_{(\Sigma; e^i + r - 4)}, \sigma^r_{(\Sigma; e^i + r - 4)}^{-1}\).

The proof follows by proceeding in the same way for each \(z^j, j = 2, \ldots, r - 2\) and then applying in order the elementary braid moves
\[
(\sigma^r_{(\Sigma; e^i - r + 1)})^{-1}, (\sigma^r_{(\Sigma; e^i - r + 2)})^{-1}, \ldots, (\sigma^r_{n_2})^{-1}.
\]
Let now $r = 1$. This time we have
\[
(a_{1,2}^1 + b_{2,3}^1 + \ldots + e_{(e_1-1)(e_1)}^1 + z_{1(e_1)}^{N+1} + \ldots + z_{1(e_1)}^{n_2} + \bar{1}_{(e_1)} + \ldots + \bar{1}_{(e_1)} = 0.
\]

This implies
\[
a^1 + z^{N+1} + \ldots + z^{n_2} \equiv 0 \pmod{2},
\]
\[
a^1 + b^1 \equiv 0 \pmod{2} \text{ and then } a^1 = b^1,
\]
\[
b^1 + c^1 \equiv 0 \pmod{2} \text{ and so } b^1 = c^1,
\]
\[
(e - 1)^1 \equiv 0 \pmod{2} \text{ and then } (e - 1)^1 = e^1.
\]

Therefore $a^1 = b^1 = c^1 = \ldots = e^1$. If $a^1 = b^1 = c^1 = \ldots = e^1 = 0$ the elements $z^j$ equal to 1 are an even number. So to obtain the wanted normal form we act as in the case $r > 1$.

If instead $a^1 = b^1 = c^1 = \ldots = e^1 = 1$, the number of $z^j = 1$ is odd. By appropriate elementary moves we bring our system to the form
\[
((\bar{1}_{1,2}^1; (1_2)(1_1)), \ldots, (\bar{1}_{1(e_1)}^1; (1_1(e_1)^1)), (\bar{1}_{1,2}^1; (1_2)(1_1)), \ldots, (\bar{1}_{1,2}^1; (1_2)(1_1)), (\bar{1}_{2,1}^1; id), (0; (1_2)(1_1)), \ldots, (0; (1_2)(1_1)), (\bar{1}_{1,2}^1; id), (0; e^{-1})))
\]

where the elements of type $(0; (1_2)(1_1))$, that are an odd number, occupy the places $k + 1, \ldots, n_2$. Now acting in the order by $\sigma_{n_2}^1, \ldots, \sigma_{k+1}^1$, by the (iii), we obtain the system
\[
((\bar{1}_{1,2}^1; (1_2)(1_1)), \ldots, (\bar{1}_{1,2}^1; (1_2)(1_1)), (\bar{1}_{2,1}^1; id), (0; (1_2)(1_1)), \ldots, (0; (1_2)(1_1)), (\bar{1}_{1,2}^1; id), (0; e^{-1})))
\]

Applying the braid moves $(\sigma_j^{-1})^1, \ldots, (\sigma_{(e_1)}^{-1})^1$ we can replace the sequence $((\bar{1}_{1(e_1)}^1; (1_1(e_1)^1)), (\bar{1}_{2,1}^1; (1_2)(1_1)), \ldots, (\bar{1}_{1,2}^1; (1_2)(1_1)), (\bar{1}_{2,1}^1; id))$ by
\[
((\bar{1}_{1(e_1)}^1; (1_1(e_1)^1)), (\bar{1}_{2,1}^1; id), (0; (1_2)(1_1)), \ldots, (0; (1_2)(1_1))).
\]

Now we move $(\bar{1}_{2,1}^1; id)$ to the right of $(\bar{1}_{1,2}^1; (1_2)(1_1))$ and then we use $\sigma_1^1$ obtaining the system
\[
((\bar{1}_{1}^1; id), (\bar{1}_{1,2}^1; (1_2)(1_1)), \ldots, (\bar{1}_{1(e_1)}^1; (1_1(e_1)^1)), (0; (1_2)(1_1)), \ldots).
\]

The theorem follows by applying the elementary moves
\[
\sigma_1^1, \ldots, \sigma_{e_1-1}^1, (\sigma_{e_1}^{-1})^1, \ldots, (\sigma_{n_2}^{-1})^1.
\]

**Theorem 3.** The Hurwitz space $H_{W(B)}(n_1,n_2,(j_1,\ldots,j_\nu))_{(b_1, b_0)}$ is irreducible.

**Proof.** The theorem follows if we prove that, acting by elementary moves $\sigma_j^1$ and their inverses, it is possible to bring any Hurwitz system of $A_{(n_1,n_2,(j_1,\ldots,j_\nu))}$ to
the normal form

\[(0; (1_{1} 2_{1})), \ldots, (0; (1_{1}(e_{1} 1_{j}))); (0; (1_{2} 2_{2})), \ldots, (0; (1_{2}(e_{2} 1_{j}))); \ldots, (0; (1_{r}(e_{r} 1_{j}))),\]

\[(0; (1_{r}(e_{r} 1_{j})), (z^{N+1}; t_{1}^{n_{1}}), \ldots, (z^{n_{2}}; t_{2}^{n_{2}}), (1_{1}; id), \ldots, (1_{1}; id), (1_{1}; \ldots 1_{r}; \varepsilon^{-1})),\]

where \((1_{1}; id)\) appears \(n_{1}\)-times, \(N = \sum_{i=1}^{r} e_{i} - r, n_{2} - N = 0 \mod 2\) and

i) if \(r = 1\) then \(v = 1, j_{1} = 1\) and \((z^{j}; t_{j}^{n}) = (0; (1_{1} 2_{1}))\) for each \(j = N + 1, \ldots, n_{2},\)

ii) if \(r > 1,\)

\[
(z_{1}^{1};(1_{1} 1_{j})), (z_{1}^{2};(1_{1} 1_{j})), \ldots, (z_{1}^{r};(1_{1} 1_{j})), \ldots, \]

where if \(j \in \{j_{1}, \ldots, j_{r}\}\)

\[
(z_{1}^{\cdot};(1_{1} 1_{j})), (z_{1}^{\cdot};(1_{1} 1_{j})) = (1_{1};(1_{1} 1_{j})), (0; (1_{1} 1_{j}))\]

if instead \(j \notin \{j_{1}, \ldots, j_{r}\}, 2 \leq j \leq r,\)

\[
(z_{1}^{\cdot};(1_{1} 1_{j})), (z_{1}^{\cdot};(1_{1} 1_{j})) = (0; (1_{1} 1_{j})), (0; (1_{1} 1_{j})).\]

Moreover \((z^{m}_{1};(1_{1} 1_{j})); (0; (1_{1} 1_{j})))\) for each \(m = 2, \ldots, l.\)

**Step 1.** We claim that, acting by elementary moves and their inverses, it is possible to replace \((t_{1}; \ldots, t_{n_{1}+n_{2}+1}) \in A_{(n_{1}; n_{2}; [j_{1}, \ldots, j_{r}])}\) by a new Hurwitz system having \((1_{1}; \ldots 1_{r}; \varepsilon^{-1})\) at the last place.

Proceeding as in Step 1 of Theorem 2, we can replace our Hurwitz system by one braid equivalent which has at the last place \((1_{h_{1}} \ldots h_{j_{1}}; \varepsilon^{-1})\), where \(\varepsilon^{-1} = q_{1} \ldots q_{r}\), with \(q_{k} e_{k}\)-cycle, \(h_{i}, i = 1, \ldots, v,\) is an index moved by the cycle \(q_{i}\), and \(1_{h_{1}} \ldots h_{j_{1}} \in (\mathbb{Z}_{2})^{d}\) sends to \(1\) only the indexes \(h_{i}, \ldots, h_{j_{1}}\). The claim follows if \(h_{i} = 1_{j_{i}}\) for each \(i = 1, \ldots, v\). So we suppose \(h_{i} \neq 1_{j_{i}}\). Applying suitable elementary moves and then Lemma 1, we obtain a new system in which there is \((1_{h_{1}}; id)\). By braid moves of type \(\sigma_{j}\) we move \((1_{h_{1}}; id)\) to the left of \((1_{h_{1}} \ldots h_{j_{1}}; \varepsilon^{-1})\) and then we apply the braid move \(\sigma_{m_{1}+n_{2}}\). So we have

\[
(\ldots, (1_{h_{1}}; id), (1_{h_{1}} \ldots h_{j_{1}}; \varepsilon^{-1}))) \sim (\ldots, (1_{h_{1}} + 1_{h_{1}} \ldots h_{j_{1}} + 1_{(h-1)_{j_{1}}}; \varepsilon^{-1}), (1_{h_{1}}; id))
\]

where \((h-1)_{j_{1}}\) is the index that comes first of \(h_{j_{1}}\) in \(q_{j_{1}}\) and \(a = 1_{h_{1}} + 1_{h_{1}} \ldots h_{j_{1}} + 1_{(h-1)_{j_{1}}}\) is a function that sends to \(1\) only the indexes \((h-1)_{j_{1}}, h_{j_{2}}, \ldots, h_{j_{v}}\). Let \(w\) be the indexes of \(q_{j_{1}}\) between \(h_{j_{1}}\) and \(1_{j_{1}}\). Reasoning as above after \((w+1)\)-steps we obtain a new system with \((1_{h_{1}} h_{j_{1}} \ldots h_{j_{w}}; \varepsilon^{-1})\) at the last place. Proceeding in this way for each \(h_{j_{1}} \neq 1_{j_{1}}\), after a finite number of steps, we have the claim.

**Step 2.** Starting by a Hurwitz system \((t_{1}; \ldots, t_{n_{1}+n_{2}+1})\) and applying Step 1 we obtain the system \((\tilde{t}_{1}; \ldots, \tilde{t}_{n_{1}+n_{2}}, (1_{h_{1}} \ldots h_{j_{1}}; \varepsilon^{-1}))\). Now acting by suitable elementary moves and applying Lemma 1 (see Theorem 1, Step 1) we can replace
our Hurwitz system with one braid equivalent of the form
\[ \text{(l}_1, \ldots, \text{l}_{n_2-1}; (\text{i}_1; \text{id}), \ldots, (\text{i}_1; \text{id}), (\text{i}_{1_{l_1+1}} \cdots \text{l}_{n_2-1}; \text{e}^{-1})). \]

By Proposition 2 this Hurwitz system is braid equivalent to
\[
((a^1_1; (1_2^1)), (b^{1}_{1_3}; (1_3^1)), \ldots, (e^{1}_{1_1}; (1_1^1)), (a^{2}_{1_2}; (1_2^2)), \ldots,
(e^{2}_{1_2}; (1_2^2)), \ldots, (a^{r}_{1_r}; (1_r^r)), \ldots, (e^{r}_{1_r}; (1_r^r)), (z^{N+1}, t^{n_2}_N), \ldots,
(z^{n_2}; t^{n_2}_N), (\text{i}_1; \text{id}), \ldots, (\text{i}_1; \text{id}), (\text{i}_{1_{l_1+1}} \cdots \text{l}_{n_2-1}; \text{e}^{-1})),
\]
where \(a^i, b^i, \ldots, e^i, z^j \in \mathbb{Z}_2\).

At first we suppose \(r > 1\). Put
\[ (z^{N+1}, \ldots, z^{n_2}) = (z^{2}_{1_1}, (z^{2}_{1_1})_{1_{l_1+1}}, \ldots, z^{r-1}_{1_1}, (z^{r-1}_{1_1})_{1_{l_1+1}}, (z^{r})_{1_1}). \]
Since
\[
(a^{1}_{1_2}; (1_2^1)) \cdots (e^{1}_{1_1}; (1_1^1)) \cdots (a^{r}_{1_r}; (1_r^r)) \cdots (e^{r}_{1_r}; (1_r^r)) (\text{i}_1; \text{id}) \cdots
\]
one has
\[
(a^{1}_{1_2} + b^{1}_{1_3} + \ldots + e^{1}_{1_1} + a^{2}_{1_2} + b^{2}_{1_3} + \ldots + e^{2}_{1_2} + \ldots + a^{r}_{1_r} + b^{r}_{1_r} + \ldots + e^{r}_{1_r} + z^{r}_{1_1} + (z^{r})_{1_{l_1+1}} + \text{i}_1 + \ldots + \text{i}_1) = (\text{i}_{1_{l_1+1}} \cdots \text{l}_{n_2-1}; \text{e}^{-1}).
\]
This implies
\[ a^i = 0 \] for each \(i\), and so \(b^i = c^i = \ldots = e^i = 0 \) for each \(i\),
\[ z^j + (z^j)^1 \equiv 0 \pmod{2} \] for \(j \notin \{j_1, \ldots, j_v\}, 2 \leq j \leq r - 1, \) and then \(z^j = (z^j)^1, z^j + (z^j)^1 \equiv 1 \pmod{2} \) for \(j \in \{j_1, \ldots, j_v\}, 2 \leq j \leq r - 1, \) so either \(z^j = 1\) and \((z^j)^1 = 0\) or \((z^j)^1 = 1\) and \(z^j = 0, z^r + (z^r)^1 + \ldots + (z^r)^{r} \equiv 0 \pmod{2} \) if \(r \notin \{j_1, \ldots, j_v\}\) and so the number of \((z^r)^{r} = 1\) is even, while if \(r \in \{j_1, \ldots, j_v\}, z^r + (z^r)^1 + \ldots + (z^r)^{r} \equiv 1 \pmod{2} \) and then the \((z^r)^{r} = 1\) are an odd number.

Note that, applying the braid move \(\sigma'_h\), it is possible to replace \((\text{l}_h = (0; (1_1^1)), \text{l}_{h+1} = (1_{1_1}^1; (1_1^1)))\) by \((1_{1_1}^1; (1_1^1)), (0; (1_1^1)),\) so we can suppose \(z^j = 1, (z^j)^1 = 0\) for each \(j \in \{j_1, \ldots, j_v\}, 2 \leq j \leq r - 1.\)

By braid move \(\sigma'_j\) we move the elements of type \((0; (1_1^1))\) near by ones of type \((\text{i}_1; \text{id})\). So we obtain a new system in which the elements \((\text{i}_{1_{l_1+1}}; (1_1^1))\) are at the places \((n \cdot e^i + r - 4) + 1, \ldots, k.\)

Applying the moves \(\sigma'_{n_2}, \ldots, \sigma'_{k+1}\), we bring to the \((k + 1)\)-place \((\text{i}_1; \text{id})\) if \(r \notin \{j_1, \ldots, j_v\}, \text{i}_1; \text{id}\) if \(r \in \{j_1, \ldots, j_v\}\).
If \(r \notin \{j_1, \ldots, j_v\},\) we act by \((\sigma_j')^{-1}, \ldots, (\sigma_j')^{-1})^{-1}, \) to replace the sequence \(((\text{i}_{1_{l_1+1}}; (1_1^1)), \ldots, (\text{i}_{1_{l_2+1}}; (1_1^1)), (\text{i}_1; \text{id})\) by
\[ ((\text{i}_1; \text{id}), (0; (1_1^1)), \ldots, (0; (1_1^1))). \]
If instead \( r \in \{j_1, \ldots, j_r\} \) we apply the sequence of braid moves 
\((\sigma'_{k})^{-1}, \ldots, (\sigma'_{(\Sigma', e_i, r-4)+2})^{-1}, (\sigma'_{(\Sigma', e_i, r-4)+1})^{-1}\), so we have
\[
((\bar{1}_{1,1}; (1_{1,1}r)), \ldots, (\bar{1}_{1,1}; (1_{1,1}r)), (\bar{1}_{i}; id)) \sim ((\bar{1}_{1}; id), (\bar{1}_{1,1}; (1_{1,1}r)),
(0; (1_{1,1}r)), \ldots, (0; (1_{1,1}r)))).
\]
Now if \( z^{r-1} = (z^{r-1})^1 = \bar{1} \), applying \((\sigma'_{(\Sigma', e_i, r-4)})^{-1}, (\sigma'_{(\Sigma', e_i, r-4)+1})^{-1}\) one has
\[
((\bar{1}_{1,1-r}; (1_{1,1-r}r)), (\bar{1}_{1,1-r}; (1_{1,1-r}r)), (\bar{1}_{i}; id)) \sim
((\bar{1}_{1}; id), (0; (1_{1,1-r}r)), (0; (1_{1,1-r}r)))).
\]
If instead either \( z^{r-1} = (z^{r-1})^1 = 0 \) or \( z^{r-1} = \bar{1} \) and \( (z^{r-1})^1 = 0 \) we use the moves
\(
(\sigma'_{(\Sigma', e_i, r-4)}, (\sigma'_{(\Sigma', e_i, r-4)+1})\) to replace 
\((z^{j}_{1,1-r}; (1_{1,1-r}r)), (z^{j}_{1,1-r}; (1_{1,1-r}r)), (\bar{1}_{i}; id))\) by the sequence
\[
((\bar{1}_{1}; id), (z^{j}_{1,1-r}; (1_{1,1-r}r)), (z^{j}_{1,1-r}; (1_{1,1-r}r)))).
\]
Proceeding in the same way for each \( j = 2, \ldots, r-2 \) and then applying successively the braid moves 
\((\sigma'_{(\Sigma', e_i, r-4)+1})^{-1}, (\sigma'_{(\Sigma', e_i, r-4)+2})^{-1}, \ldots, (\sigma'_{n_2})^{-1}\) we obtain the theorem.

Now let \( r = 1 \). This time one has
\[
(a_1^{1,2} + b_1^{1,2} + \cdots + e_1^{1,2} + \bar{1}_{1,1}(e_i) + \cdots + z^{N+1}_{1,1} + \cdots + z^{n_2}_{1,1} + \bar{1}_{1,1} + \cdots + \bar{1}_{1,1} = \bar{1}_{1,1}h).
\]
Therefore \( a^1 = b^1 = c^1 = \cdots = e^1 = 0 \) and \( e^1 + z^{N+1} + \cdots + z^{n_2} + 1 + \cdots + 1 \equiv 1 \) (mod 2) (see Theorem 2, Step 2, case \( r = 1 \)). Moreover \( e^1 + z^{N+1} + \cdots + z^{n_2} + 1 + \cdots + 1 \equiv 1 \) (mod 2) and then the number of elements \((\bar{1}_{1}; id)\) in our system is odd.

Observe that if \( a^1 = b^1 = c^1 = \cdots = e^1 = 0 \), the \( z^j = 1 \) are even, if instead \( a^1 = b^1 = c^1 = \cdots = e^1 = 1 \) the \( z^j = \bar{1} \) are odd.

By suitable braid moves we bring our Hurwitz system to the form
\[
((a_1^{1,2}; (1_{1,2}1)), (b_1^{1,2}; (1_{1,2}1)), (c_1^{1,2}; (1_{1,2}1)), (e_1^{1,2}; (1_{1,2}1)), (\bar{1}_{1,2}; (1_{1,2}1)), \ldots,
(\bar{1}_{1,1}; (1_{1,1}r)), (0; (1_{1,1}r)), (0; (1_{1,1}r)), (\bar{1}_{1}; id), \ldots, (\bar{1}_{1}; id), (\bar{1}; e^{-1}))
\]
where the elements \((0; (1_{1,1}r))\) occupy the places \( k + 1, \ldots, n_2 \).

Using in order the elementary moves \( \sigma'_{n_2}, \ldots, \sigma'_{k+1} \) we bring to the place \( (k+1) \) \((\bar{1}_{1}; id)\) if \( a^1 = \cdots = e^1 = 0 \), \((\bar{1}_{2}; id)\) if \( a^1 = \cdots = e^1 = 1 \). Now if \( a^1 = \cdots = e^1 = 0 \) we obtain the desired form applying the sequence of inverse braid moves \((\sigma'_{k})^{-1}, \ldots, (\sigma'_{(\Sigma', e_i, r-4)+2})^{-1}, (\sigma'_{(\Sigma', e_i, r-4)+1})^{-1}\). When instead \( a^1 = \cdots = e^1 = \bar{1} \) the proof follows by applying the moves \((\sigma'_{k})^{-1}, \ldots, (\sigma'_{(\Sigma', e_i, r-4)+2})^{-1}, (\sigma'_{(\Sigma', e_i, r-4)+1})^{-1}\) (see Theorem 2, Step 2, case \( r = 1 \)).

The next result follows from Theorem 2 and Theorem 3.

**Theorem 4.** – *The Hurwitz spaces* \( H_{W(B_{n_2}), (n_1; n_2, 2)}(\mathbb{P}^1) \) *and* \( H_{W(B_{n_2}), (n_1; n_2, j_1, \ldots, j_r)}(\mathbb{P}^1) \) *are irreducible.*
Proof. – The map forgetful $H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1, b_0) \rightarrow H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1)$ given by $[f \circ \pi, \varphi] \rightarrow [f \circ \pi]$ is a morphism and it has image given by a dense subset $H$ of $H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1)$. Since, by Theorem 2, $H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1, b_0)$ is irreducible also $H$ is irreducible and then $H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1)$ is irreducible.

Analogously, using Theorem 3, one prove that also the Hurwitz space $H_{W(B_d),(n_1; n_2)(j_1, \ldots, j_l)}(\mathbb{P}^1)$ is irreducible.

2.3. Let $Y$ be a smooth, connected, projective, complex curve of genus $g \geq 1$. Let $b_0 \in Y$ and let $|\mathcal{E}| = \sum_{i=1}^r (e_i - 1)$.

Theorem 5. – If $n_2 \geq 2d - 2$ the Hurwitz space $H_{W(B_d),(n_1; n_2)}(Y, b_0)$ is irreducible.

Proof. – To prove the irreducibility of $H_{W(B_d),(n_1; n_2)}(Y, b_0)$ it is sufficient to check that each Hurwitz system $(t_1, \ldots, t_{n_1+n_2}; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \in A_{W,B_d}(n_1, n_2, g)$ is braid equivalent to a system of the form $(\tilde{t}_1, \ldots, \tilde{t}_{n_1+n_2}; (0; id), \ldots, (0; id))$. In fact $(\tilde{t}_1, \ldots, \tilde{t}_{n_1+n_2})$ is the Hurwitz system of a covering in $H_{W(B_d),(n_1; n_2)}(\mathbb{P}^1, b_0)$, so the proof follows by Theorem 1.

With inverses of elementary moves $\sigma_j'$ we move to the left the $t_j$ of type $(z_{ik}; (ih))$, obtaining the system

$$(\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (1_{k-1}; id), \ldots, (1_{k+1}; id); \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

Let $\lambda_k = (a_k; \lambda_k', \mu_k = (b_k; \mu_k')$ and $\tilde{t}_j = (z_{ik}; t'_j)$ where $t'_j = (ih)$. Note that $(t'_1, \ldots, t'_{n_2}; \lambda_1', \mu_1', \ldots, \lambda_g', \mu_g')$ is the Hurwitz system of a covering of degree $d \geq 3$ of $Y$, with $n_2$ points of simple branching and with monodromy group $S_d$. Since for $n_2 \geq 2d - 2$ the Hurwitz space $H_{d,n_2}^n(Y, b_0)$ is irreducible (see [10]), acting by suitable braid moves $\sigma_i', \rho_{jk}', t_{jk}'$, $1 \leq i \leq n_2 - 1$, $1 \leq j \leq n_2$, $1 \leq k \leq g$ and their inverses, we can bring $(t'_1, \ldots, t'_{n_2}; \lambda_1', \mu_1', \ldots, \lambda_g', \mu_g')$ to the form $(t''_1, \ldots, t''_{n_2}; id, \ldots, id)$. Therefore our Hurwitz system is braid equivalent to the system $(\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (1_{k-1}; id), \ldots, (1_{k+1}; id); (a'_1; id), (b'_1; id), \ldots, (a'_g; id), (b'_g; id))$ where $a'_k, b'_k \in (Z_2)^d$.

The theorem follows if $a'_k = 0$ and $b'_k = 0$ for each $k = 1, \ldots, g$. So let $a'_1 \neq 0$ and let $l$ be the indexes sent to 1 by $a'_1$. Let $i$ be one of these indexes. As we saw in Theorem 2, Step 1, it is not restrictive to suppose that in our system there is $(1; id)$ and after we apply the braid move $t_{11}'$ which transforms $(a'_1; id)$ into $(a'_1; id)$ where $(a'_1; id) = (1; id)(a'_1; id)$ and $a'_i$ is a function that sends $i$ to 0. So reasoning after $(l - 1)$-steps we obtain a new Hurwitz system in which there is $(0; id)$ at the place $(n_2 + n_1 + 1)$. If $a'_1 = 0$, $b'_1 \neq 0$ and $b'_i$ sends $i$ to 1, we move to the first place of our system $(1; id)$ and after we apply the braid move $t_{11}'$ that transforms $(b'_1; id)$ into $(b'_1; id)$ where $b'_1$ sends $i$ to 0. Proceeding as above for all indexes sent to 1 by $b'_1$ we can replace $(b'_1; id)$ by $(0; id)$. Observe that if $a'_1 \neq 0$ and
\[ a'_l = b'_l = 0, \text{ for each } l \leq k - 1, \text{ we reason in the same way this time applying the braid move } \tau'_l. \] Analogously if \( b'_k \neq 0 \) and \( a'_l = b'_l = 0, \text{ for each } l \leq k - 1 \) and \( a'_k = 0 \) we apply the braid move \( \rho'_{1k} \) to transform \((b'_k; id)\) into \((0; id)\).

**Theorem 6.** If \( n_2 + |\mathcal{E}| \geq 2d \) the Hurwitz spaces \( H_{W(B_d), (n_1, n_2, \mathcal{E})}(Y, b_0) \) and \( H_{W(B_d), (n_1, n_2, \{j_1, \ldots, j_s\})}(Y, b_0) \) are irreducible.

**Proof.** To prove the irreducibility of \( H_{W(B_d), (n_1, n_2, \mathcal{E})}(Y, b_0) \) (respectively \( H_{W(B_d), (n_1, n_2, \{j_1, \ldots, j_s\})}(Y, b_0) \)) it is sufficient to test that, acting by braid moves \( \sigma'_j, \rho'_{jk}, \tau'_j \) and their inverses, it is possible to bring each Hurwitz system in \( A_{(n_1, n_2, \mathcal{E}), g} \) (resp. \( A_{(n_1, n_2, \{j_1, \ldots, j_s\}, g}) \) to a given normal form. If we check it is possible to replace \((t_1, \ldots, t_{n_2 + 1}; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \in A_{(n_1, n_2, \mathcal{E}), g} \) (resp. \( A_{(n_1, n_2, \{j_1, \ldots, j_s\}, g}) \)) by \((\tilde{t}_1, \ldots, \tilde{t}_{n_2 + 1}; (0; id), \ldots, (0; id))\), the proof follows by Theorem 2 (resp. Theorem 3). Acting by appropriate elementary moves \( \sigma'_j \) we can bring our Hurwitz system to the form

\[
(\tilde{t}_1, \ldots, \tilde{t}_{n_2 + 1}, (\hat{1}_k; id), \ldots, (\hat{1}_k; id); \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g),
\]

where \( \lambda_k = (a_k; \lambda'_k) \), \( \mu_k = (b_k; \mu'_k) \), for some \( l \tilde{l}_l = (a_j; t'_j = \zeta) \) (resp. \( (a'_j; t'_j = \zeta) \)) and \( \tilde{t}_j = (a_j; t'_j) \) for each \( j \neq l \). \((t'_1, \ldots, t'_{n_2 + 1}; \lambda'_1, \mu'_g) \) is the Hurwitz system of a branched covering of degree \( d \geq 3 \) of \( Y \), with monodromy group \( S_d \) and with \( n_2 \) points of simple branching and one special point whose local monodromy has cycle type \( \mathcal{E} \). Since \( n_2 + |\mathcal{E}| \geq 2d \) the Hurwitz space \( H^o_{d, n_2, \mathcal{E}}(Y, b_0) \) is irreducible (see [15]). Therefore it is possible acting by braid move, to replace \((t'_1, \ldots, t'_{n_2 + 1}; \lambda'_1, \mu'_g) \) by \((t'_1, \ldots, t'_{n_2 + 1}; id, \ldots, id) \). So our Hurwitz system results braid equivalent to the system \((\tilde{t}_1, \ldots, \tilde{t}_{n_2 + 1}; (\hat{1}_k; id), \ldots, (\hat{1}_k; id); (a'_1; id), (b'_1; id), \ldots, (a'_g, id), (b'_g; id)), \)
where \( a'_k, b'_k \in (\mathbb{Z}_2)^l \). Now to obtain the desired normal form it is sufficient to proceed as in Theorem 4.

**Theorem 7.** In the same hypothesis of Theorem 5 the Hurwitz space \( H_{W(B_d), (n_1, n_2, \mathcal{E})}(Y) \) is irreducible. In the same hypothesis of Theorem 6 the Hurwitz spaces \( H_{W(B_d), (n_1, n_2, \mathcal{E})}(Y) \) and \( H_{W(B_d), (n_1, n_2, \{j_1, \ldots, j_s\})}(Y) \) are irreducible.

**Proof.** To obtain the thesis it is sufficient to proceed as in Theorem 4 by using respectively Theorem 5 and Theorem 6.

3. **The case** \( D_* \subseteq f^{-1}(c) \).

Let \( X \) and \( X' \) be smooth, connected, projective, complex curves of genus \( \geq 0 \). Let \( X \to X' \to \mathbb{P}^1 \) be a sequence of coverings satisfying the following: \( \pi \) is a branched covering of degree \( 2 \) with \( n_1 > 0 \) branch points and \( f \) is a branched covering of degree \( d \geq 3 \), with \( n_2 \) points of simple branching and one special
point whose local monodromy has cycle type $e$. Let us denote by $c$ the special point of $f$. In this section we are interested in coverings such that $D_{\pi} \subset f^{-1}(c) = \{c_1, \ldots, c_r\}$. Note that $n_1$ is even follows by Hurwitz formula, therefore $r \geq 2$. Let $D_f$ be the branch locus of $f$. Then $f \circ \pi$ is a covering of degree $2d$ of $\mathbb{P}^1$, with $n_2 + 1$ branch points and branch locus $D = D_f$. Let $D_{\pi} = \{c_{j_1}, \ldots, c_{j_{n_1}}\}$ where $c_{j_i}$ has multiplicity $e_j$, $1 \leq i \leq n_1$ and $j_1 < j_2 < \ldots < j_{n_1}$. Let us denote by $H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1)$ the Hurwitz space that parametrizes equivalence classes $[f \circ \pi]$ where $f \circ \pi$ is a covering as above. Let $\delta : H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1) \to (\mathbb{P}^1)^{n_2+1} - \Delta$ the map that sends each class $[f \circ \pi]$ to the branch locus of $f \circ \pi$.

**Definition 6.** Two Hurwitz systems, $(t_1, \ldots, t_n)$ and $(\bar{t}_1, \ldots, \bar{t}_n)$ with values in $(\mathbb{Z}_2)^d \times S_d$, are called equivalent if there exists $s \in (\mathbb{Z}_2)^d \times S_d$ such that $\bar{t}_j = s^{-1} t_j$ for each $j$. The equivalence class containing $(t_1, \ldots, t_n)$ is denoted by $[t_1, \ldots, t_n]$.

Let $A_{(n_2, j_1, \ldots, j_{n_1})}$ be the set of all equivalence classes $[t_1, \ldots, t_{n_2+1}]$ of Hurwitz systems with values in $(\mathbb{Z}_2)^d \times S_d$, with transitive monodromy group, such that $n_2$ among the $t_j$ are of type $(z_{j_k}; (ih))$ and one of type $(a'; \xi)$. By Riemann existence theorem we can identify the fiber of $\delta$ over $D$ by $A_{(n_2, j_1, \ldots, j_{n_1})}$. There is a unique topology on $H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1)$ such that $\delta$ is a topological covering map, [7]. So the braid group $\pi_1((\mathbb{P}^1)^{n_2+1} - \Delta, D)$ acts on $A_{(n_2, j_1, \ldots, j_{n_1})}$. If this action is transitive then $H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1)$ is connected.

**Theorem 8.** The Hurwitz space $H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1)$ is irreducible.

**Proof.** Since $H_{W(B_d), (n_2, j_1, \ldots, j_{n_1})}(\mathbb{P}^1)$ is smooth in order to prove its irreducibility it suffices to prove it is connected. Therefore the theorem follows if we prove that, acting by elementary moves $\sigma_j$ and their inverses, it is possible to bring each $[t_1, \ldots, t_{n_2+1}] \in A_{(n_2, j_1, \ldots, j_{n_1})}$ to the normal form

$$[(0; (1, 2)), (0; (1, 3)), \ldots, (0; (1, e_1)), (0; (1, 2 e_2)), \ldots, (0; (1, 2 e_2)), \ldots, (0; (1, e_1)), (z_{j_1 1 2}^{1, 1 2}; (1, 1 2)), ((z_{j_1 1 2}^{2, 1 2}; (1, 1 2)), (z_{j_1 1 2}^{3, 1 2}; (1, 1 2)), \ldots, ((z_{j_1 1 2}^{l, 1 2}; (1, 1 2)), (I_{n_1}; (1, 1 2)), (I_{n_1}; (1, 1 2)); (1, 1 2)),$$

where the pairs $(z_{j_1 1 2}^{1, 1 2}; (1, 1 2)), (z_{j_1 1 2}^{2, 1 2}; (1, 1 2))$, $2 \leq j \leq r$, satisfy either (2) or (3) depending on whether $j$ belongs to $\{j_1, \ldots, j_{n_1}\}$ or not. Moreover $((z_{j_1 1 2}^{m, 1 2}; (1, 1 2)) = (0; (1, 1 2))$ for each $m = 2, \ldots, l$.

**Step 1.** We check that, acting by elementary moves and their inverses, it is possible to bring each $[t_1, \ldots, t_{n_2+1}] \in A_{(n_2, j_1, \ldots, j_{n_1})}$ to $[\bar{t}_1, \ldots, \bar{t}_{n_2}, (I_{n_1}; e^{-1})]$. By elementary moves $\sigma_j$ we move to the last place of our system the element of type $(a'; \xi)$. Because $\xi$ is in the same conjugate class of $e$, there is $s \in S_d$ such that
\[ e^{-1} = s^{-1} \xi s. \] So if we conjugate each element of our system by \((0; s)\), we obtain a new system of the class having to the last place \((b'; e^{-1})\). Let \(e^{-1} = q_1 \cdots q_r\) where \(q_i\) is an \(e_i\)-cycle and let \(i^1, i^2, \ldots, i^n\) be the indexes moved by \(q_i\) which \(b'\) sends to \(1\). Let

\[ q_i = (\ldots i^1 (i^1 + 1) \ldots (i^2 - 1) i^2 \ldots i^{n-1} (i^{n-1} + 1) \ldots (i^n - 1) i^n \ldots). \]

Conjugating by \((\tilde{1}_{(i^1-1)\ldots(i^{n-1})}; id)\) we obtain

\[ [(\ldots, (\tilde{1}_{(i^1-1)\ldots(i^{n-1})}; b' + \tilde{1}_{(i^1-1); e^{-1}})] \]

where \(b'' = \tilde{1}_{(i^1-1)\ldots(i^{n-1})}; b' + \tilde{1}_{(i^1-1); e^{-1}} \in (\mathbb{Z}_2)^d\) is a function that sends to \(1\) only the indexes \(i^1, i^2, \ldots, i^{n-2}\). If \(i \notin \{j_1, \ldots, j_{n_1}\}\), \(s_i\) is even. So if we conjugate by

\[ (\tilde{1}_{(i^1-1)\ldots(i^{n-2}-1)}; id)(\tilde{1}_{(i^1-5)\ldots(i^{n-4}-1)}; id) \cdots \]

we obtain a new system of the class with \((\tilde{b}; e^{-1})\) at the place \((n_2 + 1)\), where \(\tilde{b}\) is a function which sends to \(0\) all indexes of \(q_i\). If \(i \in \{j_1, \ldots, j_{n_1}\}\), \(s_i\) is odd and then conjugating by

\[ (\tilde{1}_{(i^1-1)\ldots(i^{n-2}-1)}; id)(\tilde{1}_{(i^1-5)\ldots(i^{n-4}-1)}; id) \cdots \]

we obtain another system of the class having \((\tilde{b}; e^{-1})\) at the last place, where the only index of \(q_i\) mapped in \(1\) by \(\tilde{b}\) is \(i^1\). So if \(i^1 \neq 1\) and \(q_i = (1, e_1), \ldots (i^1 - 1, i^1) \ldots\), we conjugate by \((\tilde{1}_{(i^1-1); id}; id)\) to obtain \([\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (\tilde{b}; e^{-1})]; \tilde{b} \in (\mathbb{Z}_2)^d\) sends \(1\) to \(\tilde{1}\) and to \(0\) each other indexes moved by the cycle \(q_i\). We obtain the claim reasoning in this way for each \(i = 1, \ldots, r\).

**Step 2.** - Starting by \([t_1, \ldots, t_{n_2+1}] \in A_{\{j_1, \ldots, j_{n_1}\}}\) and applying Step 1 we obtain \([\tilde{t}_1, \ldots, \tilde{t}_{n_2}, (\tilde{1}_{1, \ldots, n_1}; e^{-1})]\). Let \(\tilde{t}_j = (z_{ik}; t_j)\). Since \(\langle t'_1, \ldots, t'_{n_2} \rangle = S_d\), \(t'_1 \cdots t'_{n_2} = e\) and \(r \geq 2\), by Proposition 2 \([\tilde{t}_1, \cdots, \tilde{t}_{n_2}, (\tilde{1}_{1, \ldots, n_1}; e^{-1})]\) is braid equivalent to

\[ (a^1_{1,2}; (1,2)), (b^1_{1,3}; (1,3)), \ldots, (e^1_{1, (e_1)}; (1, e_1)), (a^2_{1,2}; (1,2)), \ldots, (e^2_{1, (e_2)}; (1,2)), \ldots, (a^n_{1,2}; (1,2)), \ldots, (e^n_{r (e_r)}; (1, e_r)), (z^2_{1,1}; (1,1)), (z^2_{1,2}; (1,2)), \ldots, (z^{n}_{1, r}; (1, r)), (z^{n}_{1,1}; (1,1)) \]

where \(a^i, b^j, \ldots, e^i, z^i, (z^{i})^k \in \mathbb{Z}_2\). Seeing that

\[ (a^1_{1,2}; (1,2)) \cdots (e^1_{1, (e_1)}; (1, e_1)) \cdots (a^n_{1,2}; (1,2)) \cdots (e^n_{r (e_r)}; (1, e_r)) \]

\[ (z^2_{1,2}; (1,2))(z^2_{1,1}; (1,1)) \cdots (z^r_{1, r}; (1, r)) \cdots (z^r_{1,1}; (1,1)) = (\tilde{1}_{(e_1)}(e_2) \ldots (e_{n_1}) ; e) \]
we have 
\[
(a_{1,2}^1 + b_{2,3}^1 + \cdots + e_{(e-1)(e)}^1 + a_{1,2}^r + b_{2,3}^r + \cdots + e_{(e-1)(e)}^r) + \\
+z_{(e_1)(e_2)}^2 + (z_{(e_1)(e_2)}^1 + \cdots + z_{(e_1)(e_2)}^r) + \\
= \tilde{t}_{(e_1)(e_2)} \cdot \\
\]

Therefore (see Theorem 3, Step 2, case \( r > 1 \)) \( a^i = b^i = c^i = \ldots = e^i = 0 \) for each \( i \), \( z^j = (z^j)^1 \) for each \( j \not\in \{j_1, \ldots, j_{n_1}\} \), while \( z^j = 1 \) and \( (z^j)^1 = 0 \) for each \( j \in \{j_1, \ldots, j_{n_1}\} \). Moreover the number of \( (z^r)^h = 1 \) is even if \( r \not\in \{j_1, \ldots, j_{n_1}\} \), odd otherwise.

Acting by braid moves \( \sigma_j \) we move to the left of \( (\tilde{t}_{1,2, \ldots, n_1}; \varepsilon^{-1}) \) the elements of type \((0; (1_1, r))\), obtaining so a new braid equivalent system in which the elements \((1_1, 1_1; (1_1, 1_1, r))\) are at the places \((\Sigma_i e_i + r - 4) + 1, \ldots, k \). Since \( n_1 > 0 \) and \( n_1 \) is even, \( n_1 \geq 2 \) and so \( \{j_1, \ldots, j_{n_1}\} \geq 2 \). At first we analyze the case in which there exists at least one \( j \in \{j_1, \ldots, j_{n_1}\} \) such that \( j \neq 1, r \). Since for each \( j \in \{j_1, \ldots, j_{n_1}\} \) \( j \neq 1 \), we obtain pairs of type \(( (1_1, 1_1, 1_1); (0; (1_1, 1_1)) \), in our Hurwitz system there is one pair of this type with \( j \neq r \). Let \( h \) and \( h + 1 \) be the places occupy by \((1_1, 1_1, 1_1)) \) and \((0; (1_1, 1_1)) \). With elementary moves we bring the pair \(( (1_1, 1_1, 1_1); (0; (1_1, 1_1)) \) to the left of the element \((1_1, 1_1, 1_1; (1_1, 1_1)) \) of place \((\Sigma_i e_i + r - 4) + 1 \). Now, if \( r \not\in \{j_1, \ldots, j_{n_1}\} \), we apply the moves 
\[
\sigma'(\sum_i e_{i+r-4})^{-1}, \sigma'(\sum_i e_{i+r-4}+1), \sigma'(\sum_i e_{i+r-4}), \ldots, \sigma'_{k-2}, \sigma'_{k-1}, \sigma'_{k-1}, \sigma'_{k-2}, \\
\]
by 
\[
((1_1, 1_1), (0; (1_1, 1_1)), (1_1, 1_1; (1_1, 1_1)), \ldots, (1_1, 1_1; (1_1, 1_1))) \\
\]

If instead \( r \in \{j_1, \ldots, j_{n_1}\} \), we use in the order the moves \((\sigma'(\sum_i e_{i+r-4})^{-1}, \sigma'(\sum_i e_{i+r-4})^{-1}, \sigma'(\sum_i e_{i+r-4}+1), \sigma'(\sum_i e_{i+r-4}), \ldots, \sigma'_{k-2}, \sigma'_{k-1}, \sigma'_{k-1}, \sigma'_{k-2} \) obtaining 
\[
((1_1, 1_1); (1_1, 1_1)), (0; (1_1, 1_1)), (1_1, 1_1; (1_1, 1_1)), \ldots, (1_1, 1_1; (1_1, 1_1))) \sim \\
((1_1, 1_1, 1_1; (1_1, 1_1)), (0; (1_1, 1_1)), (1_1, 1_1, 1_1; (1_1, 1_1)), (0; (1_1, 1_1)), (1_1, 1_1, 1_1; (1_1, 1_1))) \\
\]

Acting by \( \sigma'_{k-2}, \sigma'_{k-1}, \sigma'_{k-3}, \sigma'_{k-2}, \ldots, \sigma'_{(\sum_i e_{i+r-4})-1}, \sigma'_{(\sum_i e_{i+r-4})} \), we replace if \( r \not\in \{j_1, \ldots, j_{n_1}\} \) the sequence \((0; (1_1, 1_1)), \ldots, (0; (1_1, 1_1)), (1_1, 1_1; (1_1, 1_1)), (0; (1_1, 1_1)) \) by 
\[
((1_1, 1_1); (1_1, 1_1)), (0; (1_1, 1_1)), (0; (1_1, 1_1)), \ldots, (0; (1_1, 1_1)) \\
\]

If instead \( r \in \{j_1, \ldots, j_{n_1}\} \) the sequence \((1_1, 1_1; (1_1, 1_1)), (0; (1_1, 1_1)), \ldots, (0; (1_1, 1_1)), \ldots, (0; (1_1, 1_1)) \),
(0; (1,1r)), (I_1, r; (1,1r))) by

\((\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)), (\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)), \ldots, (0; (1,1r))).\)

Now if \(z^{r-1} = (z^{r-1})^1 = 1\), applying \((\sigma'_{(\Sigma^r e_i+r-4)})^{-1} \), \((\sigma'_{(\Sigma^r e_i+r-4)}-1)^{-1} \),
\((\sigma'_{(\Sigma^r e_i+r-4)-3})^{-1} \), \((\sigma'_{(\Sigma^r e_i+r-4)-2})^{-1}\) we have

\((\tilde{I}_1, I_1r; (1,1r-1)), (\tilde{I}_1, I_1r; (1,1r-1)), (\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)))\)

\((\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)), (0; (1,1r-1)), (0; (1,1r-1))).\)

If either \(z^{r-1} = (z^{r-1})^1 = 0\) or \(z^{r-1} = 1\) and \((z^{r-1})^1 = 0\), we act by
\(\sigma'_{(\Sigma^r e_i+r-4)-2} \), \(\sigma'_{(\Sigma^r e_i+r-4)-1} \), \(\sigma'_{(\Sigma^r e_i+r-4)-3} \), \(\sigma'_{(\Sigma^r e_i+r-4)-2}\) obtaining either

\((0; (1,1r-1)), (0; (1,1r-1)), (\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)))\)

\((\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r)), (0; (1,1r-1)), (0; (1,1r-1))).\)

or

\((\tilde{I}_1, I_1r; (1,1r-1)), (0; (1,1r-1)), (\tilde{I}_1, I_1r; (1,1r)), (0; (1,1r))\)

\((0; (1,1r)), (\tilde{I}_1, I_1r; (1,1r)), (\tilde{I}_1, I_1r; (1,1r-1)), (0; (1,1r-1))).\)

The proof follows by proceeding in this way for each \(j = 2, \ldots, r - 2\) and applying
\((\sigma'_{(\Sigma^r e_i+r-4)})^{-1} \), \((\sigma'_{(\Sigma^r e_i+r-4)+1})^{-1} \), \((\sigma'_{(\Sigma^r e_i+r-4)+2})^{-1} \),
\((\sigma'_{(\Sigma^r e_i+r-4)+3})^{-1} \), \((\sigma'_{(\Sigma^r e_i+r-4)+2})^{-1} \), \ldots,
\((\sigma'_{k})^{-1}, \sigma'_{k-1}^{-1} \).

Now we analyze the case in which there is not any \(j \in \{j_1, \ldots, j_{n_1}\} \) such that
\(j \neq 1, r\), this implies \(n_1 = 2\) and \(\{j_1, j_2\} = \{1, r\}\). Note that since \(r \in \{j_1, j_2\}\),
the \((z^{r})^k = 1\) are an odd number and so among the elements of type
\((z^{r})^k; (1,1r)); our Hurwitz system there is at least one \((0; (1,1r))).\) Recall
that the elements \((I_1, I_1r; (1,1r))\) occupy the places \((\Sigma^r e_i + r - 4) + 1, \ldots, k\), so
acting by the braid move \((\sigma'_{(\Sigma^r e_i+r-4)})^{-1}\) we replace the sequence

\((z^{r-1}_{1,1r-1}; (1,1r-1)), (z_{1,1r}; (1,1r)), \ldots, (I_1, I_1r; (1,1r)), (0; (1,1r))\) by

\((\tilde{I}_1, I_1r; (1,1r)), (z_{1,1r-1}; (1,1r-1)), \ldots, (I_1, I_1r; (1,1r)), (0; (1,1r)))\)

where \(z\) is either \(\tilde{0}\) or \(\tilde{1}\) depending on whether \(z^{r-1}\) is equal to \(\tilde{1}\) or \(\tilde{0}\). Applying
the elementary moves \(\sigma'_{(\Sigma^r e_i+r-4)+1} \), \(\sigma'_{k-1}^{-1}, \sigma'_{k}^{-1}\) we can replace this sequence
by

\((\tilde{I}_1, I_1r; (1,1r)), (z'_{1,1r-1}; (1,1r-1)), \ldots, (z'_{1,1r-1}; (1,1r-1)),
(0; (1,1r)), (z_{1,1r-1}; (1,1r-1)))\)

where \(z'\) is either \(\tilde{0}\) or \(\tilde{1}\) depending on whether \(z\) is equal to \(\tilde{1}\) or \(\tilde{0}\). Now using the
braid moves \(\sigma'_{(\Sigma^r e_i+r-4)+1} \), \(\sigma'_{k-2}^{-1}, \sigma'_{k}^{-1}, (\sigma'_{k-1}^{-1} \) we obtain that the sequence
above is braid equivalent to
\[
((z''_{1, 1}; (1_1 1_1 1_1)), \ldots, (z''_{1, r}; (1_{r-1} 1_1)), (z_{1, 1_{r-1}}; (1_1 1_{r-1})), \ldots, (\tilde{1}_{1, r}; (1_{r-1} 1_1)), (0; (1_{r-1} 1_1)))
\]
where \(z''\) is either \(\tilde{0}\) or \(\tilde{1}\) depending on whether \(z'\) is equal to \(\tilde{1}\) or \(\tilde{0}\). Acting by \((\sigma'_{k-2})^{-1}, \ldots, (\sigma'_{(\sum_{e_i + r - 4})})^{-1}, (\sigma'_{\sum_{e_i + r - 4}})^{-1}, \ldots, (\sigma'_{k-2})^{-1}\) we bring our sequence to the form
\[
((0; (1_1 1_1)), \ldots, (0; (1_1 1_1)), (z_{1, 1_{r-1}}; (1_1 1_{r-1})), (\tilde{1}; (1_{r-1} 1_1)), (0; (1_{r-1} 1_1))).
\]
Now if \(z = 0\) we act by \(\sigma'_{k-1}, \sigma'_{k}\), if instead \(z = \tilde{1}\) we use \(\sigma'_{k}, \sigma'_{k-1}, \sigma'_{k}\) to obtain
\[
((z_{1, 1_{r-1}}; (1_1 1_{r-1})), (\tilde{1}; (1_{r-1} 1_1)), (0; (1_{r-1} 1_1))) \sim
((\tilde{1}; (1_1 1_1)), (0; (1_1 1_1)), (z_{1, 1_{r-1}}; (1_1 1_{r-1}))).
\]
Then applying the elementary moves \((\sigma'_{k-2})^{-1}, \ldots, (\sigma'_{(\sum_{e_i + r - 4})})^{-1}, \sigma'_{k}, \ldots, \sigma'_{(\sum_{e_i + r - 4})}\) we can replace the sequence \((0; (1_1 1_1)), \ldots, (0; (1_1 1_1)), (\tilde{1}; (1_1 1_1)), (0; (1_1 1_1)), ((z_{1, 1_{r-1}}; (1_1 1_{r-1})))\) by
\[
((z'_{1, 1_{r-1}}; (1_1 1_{r-1})), (\tilde{1}; (1_1 1_1)), (0; (1_1 1_1)), \ldots, (0; (1_1 1_1))).
\]
Now to obtain the required normal form it is sufficient to proceed as in the case in which there exists at least one \(j\) belonging to \(\{j_1, \ldots, j_{n_1}\}\) such that \(j \neq 1, r\).

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