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Kontsevich Deformation Quantization on Lie Algebras.

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Sunto. – *Consideriamo il prodotto star di Kontsevich sul duale* $\mathfrak{g}^*$ *di una algebra di Lie generale* $\mathfrak{g}$ *dotata di parentesi di Poisson lineare. Mostriamo che questo prodotto star fornisce una quantizzazione di deformazione mediante immersioni parziali nella direzione della parentesi di Poisson.*

Summary. – *We consider Kontsevich star product on the dual* $\mathfrak{g}^*$ *of a general Lie algebra* $\mathfrak{g}$ *equipped with the linear Poisson bracket. We show that this star product provides a deformation quantization by partial embeddings in the direction of the Poisson bracket.*

1. – Introduction.

Star products are associative deformations of usual product of functions [BFFLS].

The problem of existence of star products has been resolved by different steps. First M. De Wilde and P.Lecomte have proved the existence of star products on symplectic manifolds [DL]. Since then many papers dealing with different cases of Poisson manifolds were appeared. In particular very interesting geometric proofs were given by H. Omori, Y. Maeda and A. Yoshioka in [OMY] and by B. Fedosov in [F].

Recently, M. Kontsevich has resolved the problem of existence of star products on any finite dimensional Poisson manifold [K]. He has built a star product on $\mathbb{R}^d$ equipped with any Poisson structure a. This star product has been defined by considering oriented graphs.

The study of star products is used as a tool for quantization of classical mechanical systems. A classical mechanical system is given by its phase space which is a C^∞-manifold M equipped with a Poisson bracket $\{\ ,\ \}$. To quantize this system, one selects a suitable algebra A of C^∞ functions on M (for example A can be the set of compactly supported functions on M or Schwartz functions) with the product being point wise multiplication. One then deforms this product in the direction of the Poisson bracket. That is if we denote the deformation parameter by Plank's constant $\hbar$, taking real values in some interval about 0, then one tries to define a family $\star_\hbar$ of associative products in such a way that for $f, g \in A$ one has

$$f \star_\hbar g \longrightarrow fg$$

and

$$(f \star_\hbar g - g \star_\hbar f)/i\hbar \longrightarrow \{f, g\}$$

as $\hbar$ goes to zero.

This is the property characterizing how a deformation quantization is related to a given Poisson bracket. We remark that it is only an infinitesimal condition at $\hbar = 0$, so one does not expect deformations for a given Poisson bracket to be unique. We also remark that in most examples in the literature, it is not precise what kind of convergence is involved in the limit above. In this paper, we will precise a certain kind of convergence for Kontsevich star product on the dual of a general Lie algebra.

The paper is organized as follows:

In the second section , we define the linear Poisson bracket on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ and we give the Fourier transform of this Poisson structure on $\mathfrak{g}$.

In the third section, we recall the construction of Kontsevich star product on $\mathbb{R}^d$ and we give the universal integral formula of this star product on the dual of a Lie algebra.

Finally, in the last section, we give the definition of a deformation quantization by partial embeddings and we show that Kontsevich star product gives an example of such deformation quantization that is the convergence of the following limit

$$\lim_{\hbar \to 0} (f \star_\hbar g - g \star_\hbar f)/i\hbar = \{f, g\}$$

is well precise.

2. – Fourier transform of the Poisson bracket on $\mathfrak{g}^*$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. On the space $C^\infty(\mathfrak{g}^*)$ of C^∞ functions on $\mathfrak{g}^*$, we consider the Poisson structure given in [W].

Let $u \in C^\infty(\mathfrak{g}^*)$, its differential $du(\xi)$ at $\xi \in \mathfrak{g}^*$ is a linear function on the tangent space to $\mathfrak{g}^*$ at ξ. This space is identified with $\mathfrak{g}^*$. Thus, we can consider $du(\xi)$ as an element of $\mathfrak{g}$ which defines a linear function on $\mathfrak{g}^*$ by

$$\langle du(\xi), \mu \rangle = \left(\frac{d}{dt}\right)_{|t=0} u(\xi + t\mu)$$

where $\langle \ , \ \rangle$ is the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Then for $u, v \in C^\infty(\mathfrak{g}^*)$, we can define the Poisson bracket by

$$\{u, v\}(\xi) = \langle [du(\xi), dv(\xi)], \xi \rangle$$

where [,] is the Lie bracket in $\mathfrak{g}$.

Now, we will consider the Fourier transform of this Poisson bracket to obtain the corresponding Poisson bracket in $\mathfrak{g}$ (see $[R]_1$).

PROPOSITION 2.1. – *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\mathfrak{g}^*$ be its dual. The Fourier transform of the Poisson bracket on the space $\mathcal{S}(\mathfrak{g}^*)$ of Schwartz functions on $\mathfrak{g}^*$ is given for $f, g \in \mathcal{S}(\mathfrak{g})$ by*

$$\{f, g\}(X) = -2i\pi \int_{\mathfrak{g}} f(Y)\ (\langle [Y, X], dg(X - Y) \rangle + g(X - Y) tr(adY))\ dY.$$

PROOF. – We first recall the definition of the Fourier transform $\hat{\phi}$ of $\phi \in \mathcal{S}(\mathfrak{g}^*)$

$$\hat{\phi}(X) = \int_{\mathfrak{g}^*} \phi(\xi) e^{-2i\pi\langle X, \xi \rangle} d\xi.$$

Then the inverse Fourier transform $\overset{\vee}{f}$ of $f \in \mathcal{S}(\mathfrak{g})$ is defined by

$$\overset{\vee}{f}(\xi) = \int_{\mathfrak{g}} f(X) e^{2i\pi\langle X, \xi \rangle} dX.$$

For $f, g \in \mathcal{S}(\mathfrak{g})$, one has

$$\{f, g\}(X) = \{\overset{\vee}{f}, \overset{\vee}{g}\}^\wedge(X).$$

That is

$$\{f, g\}(X) = \int_{\mathfrak{g}^*} \langle [d\overset{\vee}{f}(\xi), d\overset{\vee}{g}(\xi)], \xi \rangle e^{-2i\pi\langle X, \xi \rangle} d\xi.$$

A direct calculation shows that

$$\langle d\overset{\vee}{f}(\xi), \mu \rangle = \int_{\mathfrak{g}} 2i\pi f(Y) \langle Y, \mu \rangle e^{2i\pi\langle Y, \xi \rangle} dY.$$

Then

$$d\overset{\vee}{f} = 2i\pi (Yf(Y))^\vee.$$

Thus, we can write

$$\begin{aligned}\{f,g\}(X) &= -4\pi^2 \int_{\mathfrak{g}^*} \langle [(Yf(Y))^{\vee}(\xi), (Zg(Z))^{\vee}(\xi)], \xi \rangle e^{-2i\pi\langle X,\xi\rangle} d\xi \\ &= -4\pi^2 \int_{\mathfrak{g}^*} e^{-2i\pi\langle X,\xi\rangle} \int_{\mathfrak{g}^2} f(Y)g(Z)\langle [Y,Z], \xi\rangle e^{2i\pi\langle Y+Z,\xi\rangle} dY dZ d\xi \\ &= -4\pi^2 \int_{\mathfrak{g}^*} e^{-2i\pi\langle X,\xi\rangle} \int_{\mathfrak{g}} \left\langle \int_{\mathfrak{g}} f(Y)g(Z-Y)[Y,Z]dY, \xi \right\rangle e^{2i\pi\langle Z,\xi\rangle} dZ d\xi.\end{aligned}$$

Let Φ be the function in the space $\mathcal{S}(\mathfrak{g},\mathfrak{g})$ of Schwartz function on $\mathfrak{g}$ with values in $\mathfrak{g}$ defined by

$$\Phi(Z) = \int_{\mathfrak{g}} f(Y)g(Z-Y)[Y,Z]dY.$$

Let $X_1, ..., X_d$ be a basis of $\mathfrak{g}$ and $\xi_1, ..., \xi_d$ the coordinates of ξ in the dual basis. Let $\Phi_1, ..., \Phi_d$ be the components of Φ and let ∂_j denote the partial derivative in the $j^{\underline{th}}$ direction. The inverse Fourier transform of $div\, \Phi$ is given by

$$\begin{aligned}(div\, \Phi)^{\vee}(\xi) &= \int_{\mathfrak{g}} \sum_{j=1}^{d} \partial_j \Phi_j(Z) e^{2i\pi\langle Z,\xi\rangle} dZ \\ &= \sum_{j=1}^{d} \int_{\mathfrak{g}} -2i\pi\xi_j \Phi_j(Z) e^{2i\pi\langle Z,\xi\rangle} dZ \\ &= -2i\pi \int_{\mathfrak{g}} \langle \Phi(Z), \xi\rangle e^{2i\pi\langle Z,\xi\rangle} dZ.\end{aligned}$$

Thus we can write

$$\begin{aligned}\{f,g\}(X) &= -2i\pi\ ((div\, \Phi)^{\vee})^{\wedge}(X) \\ &= -2i\pi\ div \int_{\mathfrak{g}} f(Y)g(X-Y)[Y,X]dY.\end{aligned}$$

It follows that

$$\{f,g\}(X) = -2i\pi \int_{\mathfrak{g}} f(Y) \left(\sum_{j=1}^{d} (\partial_j g)(X-Y)[Y,X]_j + g(X-Y)\partial_j([Y,X]_j) \right) dY$$

where $[Y,X]_j$ is the $j^{\underline{th}}$ component of $[Y,X]$.

Now, one has

$$\sum_{j=1}^{d} \partial_j\Big([Y,X]_j\Big) = \sum_{j=1}^{d}\left[Y, \partial_j\left(\sum_{k=1}^{d} x_k X_k\right)\right]_j$$
$$= \sum_{j=1}^{d}[Y,X_j]_j = tr(adY).$$

Finally, we obtain

$$\{f,g\}(X) = -2i\pi\int_{\mathfrak{g}} f(Y)\ (\langle[Y,X],dg(X-Y)\rangle + g(X-Y)tr(adY)\ )dY.$$

3. – Kontsevich star product on dual of Lie algebras.

Recently Kontsevich has resolved the problem of existence of star products on any finite dimensional Poisson manifold. In fact, he has built a star product on $\mathbb{R}^d$ equipped with any Poisson structure α [K].

In this section, we will recall briefly the construction of this star product and we will give its universal integral formula on dual of Lie algebras.

We consider $\mathbb{R}^d$ equipped with a Poisson structure α. The Poisson bracket of two C^∞ functions u and v is denoted by

$$\alpha(u,v) = \sum_{1\le i,j\le d} \alpha^{ij}\partial_i u \partial_j v$$

The Kontsevich star product is constructed by considering for any integer $n\geqslant 0$ a family of oriented graphs G_n.

DEFINITION 3.1. [K]. – *An oriented graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $V_\Gamma \times V_\Gamma$.*

Elements of V_Γ are vertices of Γ, elements of E_Γ are edges of Γ. If $e=(v_1,v_2)\in E_\Gamma \subset V_\Gamma\times V_\Gamma$ is an edge then we say that e starts at v_1 and ends at v_2 and we write $e=\overrightarrow{v_1v_2}$.

We say that an oriented graph Γ belongs to G_n if:

1) Γ has $n+2$ vertices and $2n$ edges.

2) The set of vertices V_Γ is $\{1,...,n\}\cup\{L,R\}$, where L,R are just two symbols (Left and Right).

3) Edges of Γ are labelled by symbols $e_1^1, e_1^2, e_2^1, e_2^2, ..., e_n^1, e_n^2$.

4) For every $k\in\{1,...,n\}$ edges e_k^1 and e_k^2 start at the vertex k.

5) For any $v\in V_\Gamma$ the ordered pair (v,v) is not an edge of Γ.

To each graph $\Gamma \in G_n$, Kontsevich associates a bidifferential operator $B_{\Gamma,a}$:

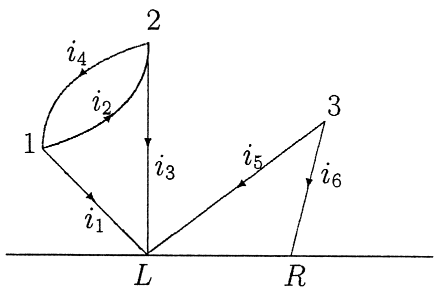
$$B_{\Gamma,a} : A \times A \longrightarrow A$$
$$(u,v) \longmapsto B_{\Gamma,a}(u,v)$$

where $A = C^\infty(O)$, O is an open domain in $\mathbb{R}^d$ and for all vertex k, $1 \leqslant k \leqslant n$, one associates the components of Poisson tensor, u is associated to the vertex L and v to the vertex R. Each edge starting at the vertex $k \in \{1,...,n\}$, acts by partial differentiation on the vertex k' on which it arrives.

The best manner to illustrate the above correspondence $\Gamma \longmapsto B_{\Gamma,a}$ is to draw an example.

Let $n = 3$. We consider the graph Γ of G_3. Γ has 3 vertices of first kind (on the upper half-plane) $\{1,2,3\}$ and 2 vertices of second kind (on the real axis) $\{L,R\}$. Γ possess 6 edges

$$(e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2) = ((1,L),(1,2),(2,L),(2,1),(3,L),(3,R))$$



In the picture the edges are indexed by $1 \leqslant i_1, i_2, ..., i_6 \leqslant d$ instead of the symbol e_*^*.

The operator $B_{\Gamma,a}$ corresponding to this graph is

$$(u,v) \longmapsto \sum_{i_1 \dots i_6} \partial_{i_4} a^{i_1 i_2} \partial_{i_2} a^{i_3 i_4} a^{i_5 i_6} \partial_{i_1 i_3 i_5} u \partial_{i_6} v.$$

The general formula for the operator $B_{\Gamma,a}$ is:

$$B_{\Gamma,a}(u,v) = \sum_{I:E_\Gamma \to \{1,\dots,d\}} \left[\prod_{k=1}^{n} \left(\prod_{\substack{e \in E_\Gamma \\ e=(.,k)}} \partial_{I(e)} \right) a^{I(e_k^1) I(e_k^2)} \right] \times \left(\prod_{\substack{e \in E_\Gamma \\ e=(.,L)}} \partial_{I(e)} \right) u \times \left(\prod_{\substack{e \in E_\Gamma \\ e=(.,R)}} \partial_{I(e)} \right) v$$

In the next step one associates a weight $w_\Gamma \in \mathbb{R}$ with each graph $\Gamma \in G_n$. This weight is defined by a geometric construction.

Let $p,\ q,\ p \neq q$ be two points on the standard upper half-plane

$$H = \{Z \in \mathbb{C} / \ Im(Z) > 0\}.$$

One denotes by $\Phi^h(p,q) \in \mathbb{R}/2\pi\mathbb{Z}$ the angle at p formed by two lines, $l(p,q)$ and $l(p,\infty)$ passing through p and q, and through p and the point ∞ on the absolute.

The function $\Phi^{\hbar}(p,q)$ can be defined by continuity also in case $p,q \in H \cup \mathbb{R}$, $p \neq q$. Denote by H_n the space of configurations of n distinct points on H

$$H_n = \{(p_1, ..., p_n)/p_k \in H \ p_k \neq p_l \text{ for } k \neq l\}$$

$H_n \subset \mathbb{C}^n$ is a non-compact smooth $2n$-dimensional manifold. Every edge e of the graph $\Gamma \in G_n$ defines a pair (p,q) of points on $H \cup \mathbb{R}$, thus an angle $\Phi^{\hbar}_e = \Phi^{\hbar}(p,q)$.

We define the weight of Γ as follows

$$w_\Gamma = \frac{1}{(2\pi)^{2n} n!} \int_{H_n} \bigwedge_{k=1}^{n} (d\Phi^{\hbar}_{e_k^1} \wedge d\Phi^{\hbar}_{e_k^2})$$

Finally, Kontsevich defines his star product in the following theorem.

THEOREM 3.2. [K] – *For any Poisson structure a on $\mathbb{R}^d$, the map*

$$u \star_a v = \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,a}(u,v)$$

defines an associative product.

In [ABM], the authors consider star products $a \longmapsto \star_a$ for linear Poisson structures a. They call them Kontsevich star products, since they are generalizations of the star product coming from the explicit formality $\mathcal{U}$ on $\mathbb{R}^d$ described by Kontsevich in [K]. When a is linear, $(\mathbb{R}^d, a)$ is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with structure constants $C^k_{ij} = \partial_k \ a^{ij}$.

In [ABM], the authors showed that each of these star products is given by an universal integral formula of the form

$$(u \star_a v)(\xi) = \int_{\mathfrak{g}^2} \hat{u}(X)\hat{v}(Y) \frac{F(X)F(Y)}{F(X \times_a Y)} \ e^{2i\pi\langle X \times_a Y, \xi\rangle} dXdY$$

where u,v are polynomial functions on $\mathfrak{g}^*$ $(u,v \in S(\mathfrak{g}))$ or such that $\hat{u}$, $\hat{v}$ are smooth functions and compactly supported, $X \times_a Y$ is the Baker-Campbell-Hausdorff formula in X,Y:

$$exp \ (X \times_a Y) = exp \ X \ . \ exp \ Y$$

and F is a formal series of the form

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{s_1,...,s_p \\ |s_1+...+s_p|=2n}} a_{s_1 \dots s_p} Tr(ad \ 2i\pi X)^{s_1} ... Tr(ad \ 2i\pi X)^{s_p}.$$

The expansions of all series correspond to the power series in the tensor a, each term of order n is an n-linear mapping. We extend $\mathbb{C}$-linearly these mappings. From now, all our expressions hold for complex structures a.

In [S] Shoikhet compared Kontsevich's star product and the Duflo formula in the case of linear Poisson brackets. An immediate consequence of his results is that Kontsevich's star product can be given by the following integral formula

$$(u \star_a v)(\xi) = \int_{\mathfrak{g}^2} \hat{u}(X)\hat{v}(Y) \frac{J(X)J(Y)}{J(X \times_a Y)} e^{2i\pi\langle X \times_a Y, \xi\rangle} dXdY$$

for all u, v polynomial functions on $\mathfrak{g}^*$ or such that $\hat{u}, \hat{v}$ are smooth functions and compactly supported, where

$$J(X) = det\left(\frac{sh(ad\ X/2)}{ad\ X/2}\right)^{1/2}.$$

4. – Deformation quantization by partial embeddings.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let G be the Lie group associated to $\mathfrak{g}$. Let exp be the exponential map from $\mathfrak{g}$ to G. There exists an open neighborhood $\mathcal{O}$ of $0 \in \mathfrak{g}$ on which exp is a diffeomorphism into G and such that $\mathcal{O} = -\mathcal{O}$. We will identify $\mathcal{O}$ with $exp\ \mathcal{O}$. Let ϑ be an open, convex neighborhood of $0 \in \mathfrak{g}$ such that $\vartheta^3 \subseteq \mathcal{O}$ in G and such that $\vartheta = -\vartheta$. We know that the function J given in the previous section is holomorphic near zero. We choose ϑ such that J is holomorphic on ϑ (that is ϑ is chosen contained in the ball of radius 2π).

Let $\mathfrak{g}_\hbar$ be the Lie algebra $\mathfrak{g}$ as an additive group but equipped with the Lie bracket $[\ ,\ ]_\hbar$ defined by

$$[\ ,\ ]_\hbar = \hbar[\ ,\ ] \quad \text{if} \quad \hbar \neq 0 \quad \text{and} \quad [\ ,\ ]_0 = [\ ,\ ].$$

The Lie group $G_\hbar$ of $\mathfrak{g}_\hbar$ is isomorphic to G. Let $\mathcal{O}_\hbar = \hbar^{-1}\mathcal{O}$, then we can also identify $\mathcal{O}_\hbar$ with $exp\ \mathcal{O}_\hbar$ in $G_\hbar$. Let $\times_\hbar$ denote the (partially defined) group product on $\mathcal{O}_\hbar$ from $G_\hbar$

$$X \times_\hbar Y = \hbar^{-1}((\hbar X).(\hbar Y))$$

where "." is the product (of the group G) on $\mathcal{O}$. Then $\times_\hbar$ is the Campbell-Hausdorff formula on $\mathcal{O}_\hbar$. Finally let $\vartheta_\hbar = \hbar^{-1}\vartheta$ then $\vartheta_\hbar^3 \subseteq \mathcal{O}_\hbar$ (for the product $\times_\hbar$) and

$$X \times_\hbar Y = X + Y + \frac{1}{2}[X, Y]_\hbar + O(\hbar^2)$$

where $O(\hbar^2)$ goes to 0 as $\hbar$ goes to 0.

Now, if we consider the function J for the Lie bracket $[\ ,\ ]_\hbar$ in $\mathfrak{g}$ (which we will denote by $J_\hbar$) we can write

$$J_\hbar(X) = det\left(\frac{sh(ad_\hbar X/2)}{ad_\hbar X/2}\right)^{1/2}$$

where $ad_\hbar X = \hbar\ adX = ad\ \hbar X$.

Then we obtain

$$J_\hbar(X) = J(\hbar X).$$

We remark that $J_\hbar$ is holomorphic on $\vartheta_\hbar$ as J is holomorphic on ϑ.

Thus, for all $\hbar$ in $\mathbb{R}$, we can write the integral formula of Kontsevich star product $\star_\hbar$ for all u, v polynomial functions or such that $\hat{u}, \hat{v}$ are smooth functions, compactly supported as follows

$$(u \star_\hbar v)(\xi) = \int_{\mathfrak{g}^2} \hat{u}(X)\hat{v}(Y)\frac{J_\hbar(X)J_\hbar(Y)}{J_\hbar(X \times_\hbar Y)} e^{2i\pi\langle X\times_\hbar Y,\xi\rangle} dXdY.$$

We recall (see [V]) that the differential of the exponential map $exp^\hbar$ on $\mathfrak{g}_\hbar$ is given by

$$d\ exp^\hbar X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(ad_\hbar X)^n = d\ exp\ \hbar X$$

Let us put

$$\omega_\hbar(X) = \omega(\hbar X) = det(d\ exp\ \hbar X)$$

Then, $\omega_\hbar(X)dX$ is the restriction on $\mathcal{O}_\hbar$ of a left Haar measure on $G_\hbar$, where dX is the Lebesgue measure on $\mathfrak{g}$.

In the following, we will consider for each $\hbar$ the C^*-norm from the reduced group C^*-algebra $C^*_r(G_\hbar)$ (see [P] for the definition) and we remark that $C_c^\infty(\vartheta_\hbar)$ is embedded in $C_c^\infty(G_\hbar)$. Now, we will give the definition of a deformation quantization by partial embeddings in which the notion of convergence is well precise. In this definition A, $P_\hbar$ and $Q_\hbar$ will play the roles of $C_c^\infty(\mathfrak{g})$, $C_c^\infty(\mathcal{O}_\hbar)$ and $C_c^\infty(\vartheta_\hbar)$ respectively.

DEFINITION 4.1. – *Let A be a commutative algebra equipped with a C^*-norm and a Poisson bracket $\{\ ,\ \}$. A deformation quantization of A by partial embeddings in the direction of $\{\ ,\ \}$, is an interval I of real numbers with 0 as center, together with, for each $\hbar \in I$, a C^*-algebra $A_\hbar$ and linear subspaces $P_\hbar \supseteq Q_\hbar$ of A, together with a linear embedding $\ell_\hbar$ of $P_\hbar$ into $A_\hbar$ such that:*

1) *A_0 is the C^*-completion of A, with ℓ_0 the inclusion map, and $P_0 = Q_0 = A$.*

2) *$P_{-\hbar} = P_\hbar$ and $Q_{-\hbar} = Q_\hbar$.*

3) *If $|\hbar| \leqslant |\hbar_0|$ then $P_\hbar \supseteq P_{\hbar_0}$ and $Q_\hbar \supseteq Q_{\hbar_0}$.*

4) *For every* $a \in A$ *there is an* $\hbar \neq 0$ *such that* $a \in Q_\hbar$.

5) *If* $a, b \in Q_\hbar$, *there is a (unique) element,* $a \star_\hbar b$, *in* $P_\hbar$ *such that*

$$\ell_\hbar(a)\ell_\hbar(b) = \ell_\hbar(a \star_\hbar b).$$

6) *For every* $a, b \in A$

$$\|(a \star_\hbar b - b \star_\hbar a)/i\hbar - \{a, b\}\|_\hbar$$

converges to zero as $\hbar$ *goes to zero, where* $\| \quad \|_\hbar$ *denotes the norm of* $A_\hbar$ *pulled back to* $P_\hbar$ *by* $\ell_\hbar$.

7) *For any* $a \in A$, *the function* $\hbar \longmapsto \|a\|_\hbar$ *is lower semi-continuous for the* $\hbar'$*s for which* $a \in Q_\hbar$.

Now, we will show that Kontsevich's star product provides a deformation quantization by partial embeddings as defined above.

THEOREM 4.2. – *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra and let* $\{\ ,\ \}$ *be the Poisson bracket on* $A = C_c^\infty(\mathfrak{g})$ *(the set of* C^∞ *functions compactly supported on* $\mathfrak{g}$*) obtained by Fourier transform of the linear Poisson bracket on* $\mathfrak{g}^*$.

Using the previous notations, let $P_\hbar = C_c^\infty(\mathcal{O}_\hbar)$ *and* $Q_\hbar = C_c^\infty(\vartheta_\hbar)$ *with their embeddings in* $C_r^*(G_\hbar)$. *Then this structure together with Kontsevich star product* $\star_\hbar$ *provides a deformation quantization of* A *by partial embeddings in the direction of* $-(2\pi)^{-1}\{\ ,\ \}$.

PROOF. – Let us first remark that the five first properties are easily obtained by using our hypothesis. Let u and v be two functions on $\mathfrak{g}^*$ such that $\hat{u}$ and $\hat{v}$ are smooth and compactly supported. We consider the $\hbar'$s sufficiently small so that the supports of $\hat{u}$ and $\hat{v}$ are in $\vartheta_\hbar$.

We define the star product on $\mathfrak{g}$ by Fourier transform. More precisely, if $f = \hat{u}$ and $g = \hat{v}$, we put

$$f \star_\hbar g = \hat{u} \star_\hbar \hat{v} = (u \star_\hbar v)^\wedge.$$

Then, we can write

$$(f \star_\hbar g)(Z) = \int_{\mathfrak{g}^*} (u \star_\hbar v)(\xi) e^{-2i\pi\langle Z, \xi\rangle} d\xi.$$

Let $Z = X \times_\hbar Y$, then $Y = X^{-1} \times_\hbar Z$ and $\omega_\hbar(Z)dZ = \omega_\hbar(Y)dY$. Thus, we obtain

$$\begin{aligned} f \star_\hbar g(Z) &= \int_{\mathfrak{g}} f(X)\, g(X^{-1} \times_\hbar Z)\, \frac{J_\hbar(X) J_\hbar(X^{-1} \times_\hbar Z)}{J_\hbar(Z)}\, \frac{\omega_\hbar(Z)}{\omega_\hbar(X^{-1} \times_\hbar Z)}\, dX \\ &= \frac{\omega_\hbar}{J_\hbar}(Z) \int_{\mathfrak{g}} \left(f\ \frac{J_\hbar}{\omega_\hbar}\right)(X)\ \left(g \frac{J_\hbar}{\omega_\hbar}\right)(X^{-1} \times_\hbar Z) \omega_\hbar(X)\, dX \end{aligned}$$

We remark that if we have $f(X) \neq 0$ and $g(X^{-1} \times_\hbar Z) \neq 0$ then $X \in \mathcal{V}_\hbar$ and $X^{-1} \times_\hbar Z \in \mathcal{V}_\hbar$. Then $Z \in \mathcal{V}_\hbar^2$ (by using $\times_\hbar$ as a product). Now as $\mathcal{V}_\hbar^3 \subseteq \mathcal{O}_\hbar$, for all $Z \in \mathcal{V}_\hbar^2$, the above integral is well defined. Thus if we consider that the integral vanishes when $X^{-1} \times_\hbar Z$ is not defined, we can see that $f \star_\hbar g \in C_c^\infty(\mathfrak{g})$. We also notice that $\omega_\hbar$ and $J_\hbar$ don't vanish on $\mathcal{V}_\hbar$ since $\mathcal{V}_\hbar$ is contained in the ball of radius $2\pi\hbar^{-1}$.

It follows that

$$(f \star_\hbar g)(Z)\left(\frac{J_\hbar}{\omega_\hbar}\right)(Z) = \left(f \frac{J_\hbar}{\omega_\hbar}\right) *_\hbar \left(g \frac{J_\hbar}{\omega_\hbar}\right)(Z) ,$$

where $*_\hbar$ is the convolution of functions in $C_c^\infty(\mathcal{V}_\hbar)$ for the left Haar measure $\omega_\hbar(X)dX$.

By a change of variables we obtain

$$\begin{aligned}\left((f \star_\hbar g - g \star_\hbar f)\left(\frac{J_\hbar}{\omega_\hbar}\right)(Z)\right)/\hbar &= \frac{1}{\hbar}\int_{\mathfrak{g}} \left(f\frac{J_\hbar}{\omega_\hbar}\right)(-X)\left(g\frac{J_\hbar}{\omega_\hbar}\right)(X \times_\hbar Z)\,\Delta_\hbar(X)^{-1}\omega_\hbar(X)dX \\ &\quad - \frac{1}{\hbar}\int_{\mathfrak{g}} \left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(Z \times_\hbar X)\left(f\ \frac{J_\hbar}{\omega_\hbar}\right)(-X)\ \omega_\hbar(X)\ dX\end{aligned}$$

where $\Delta_\hbar(X)$ is the modular function for $G_\hbar$ defined by

$$\Delta_\hbar(X) = e^{-\hbar tr(adX)}$$

Let

$$A_\hbar(Z,X) = \frac{1}{\hbar}\left(f\ \frac{J_\hbar}{\omega_\hbar}\right)(-X)\left(\left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(X \times_\hbar Z) - \left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(X+Z)\right),$$

$$B_\hbar(Z,X) = \frac{1}{\hbar}\left(f\ \frac{J_\hbar}{\omega_\hbar}\right)(-X)\left(\left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(X+Z)(\Delta_\hbar^{-1}(X)-1)\right),$$

and

$$C_\hbar(Z,X) = \frac{1}{\hbar}\left(f\ \frac{J_\hbar}{\omega_\hbar}\right)(-X)\left(\left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(Z \times_\hbar X) - \left(g\ \frac{J_\hbar}{\omega_\hbar}\right)(Z+X)\right).$$

Then

$$\left((f \star_\hbar g - g \star_\hbar f)\left(\frac{J_\hbar}{\omega_\hbar}\right)(Z)\right)/\hbar = \int_{\mathfrak{g}} (A_\hbar(Z,X)\Delta_\hbar^{-1}(X) + B_\hbar(Z,X) - C_\hbar(Z,X))\omega_\hbar(X)dX.$$

And

$$((f \star_\hbar g - g \star_\hbar f)(Z))/\hbar = \left(\frac{\omega_\hbar}{J_\hbar}\right)(Z)\int_{\mathfrak{g}} (A_\hbar(Z,X)\Delta_\hbar^{-1}(X) + (B_\hbar - C_\hbar)(Z,X))\omega_\hbar(X)\,dX.$$

One has

$$X \times_\hbar Z = X + Z + \frac{1}{2}\,\hbar\,[X,Z] + \hbar^2 R(\hbar, Z, X)$$

where R is a continuous function. Let

$$M(\hbar, Z, X) = \frac{1}{2}\,[X,Z] + \hbar\, R(\hbar, Z, X)$$

If we write the Taylor formula for the function $g\dfrac{J_\hbar}{\omega_\hbar}$ about $X+Z$, we obtain

$$\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X \times_\hbar Z) = \left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z) + \hbar\left\langle M(\hbar,Z,X),\ d\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z)\right\rangle +$$
$$\left(\frac{\hbar^2}{2}\right) d^2\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z+\tau\hbar\, M(\hbar,Z,X))\ (M(\hbar,Z,X),\ M(\hbar,Z,X))$$

where $\tau \in [0,1]$ and depends on $\hbar, Z$ and X.

Then

$$A_\hbar(Z,X) - \left(f\,\frac{J_\hbar}{\omega_\hbar}\right)(-X)\left\langle\frac{1}{2}\,[X,Z],\ d\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z)\right\rangle =$$
$$\hbar\left(f\,\frac{J_\hbar}{\omega_\hbar}\right)(-X)\left\langle R(\hbar,Z,X),\ d\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z)\right\rangle +$$
$$\frac{\hbar}{2}\left(f\,\frac{J_\hbar}{\omega_\hbar}\right)(-X)\ d^2\left(g\,\frac{J_\hbar}{\omega_\hbar}\right)(X+Z+\tau\hbar\, M(\hbar,Z,X))\ (M(\hbar,Z,X),\ M(\hbar,Z,X)).$$

Now, as $\omega_\hbar$ and $J_\hbar$ are holomorphic functions on $\vartheta_\hbar$, we can see that $\dfrac{J_\hbar}{\omega_\hbar}$ converges uniformly on compact subsets to 1 and $g\,\dfrac{J_\hbar}{\omega_\hbar}$ converges uniformly on compact subsets to g. Then $d\left(g\,\dfrac{J_\hbar}{\omega_\hbar}\right)$ and $d^2\left(g\,\dfrac{J_\hbar}{\omega_\hbar}\right)$ also converge uniformly on compact subsets respectively to dg and $d^2 g$. We also remark that $d^2 g$, R and M are continuous functions, so they are bounded on compact sets. It follows that $A_\hbar(Z,X)$ converges uniformly on compact subsets of $\vartheta_\hbar^2 \times \vartheta_\hbar$ to

$$f(-X)\left\langle\frac{1}{2}[X,Z],\ dg(X+Z)\right\rangle$$

if $\hbar$ goes to zero.

In the same way since $\Delta_\hbar$ and $\omega_\hbar$ are entire functions on $\mathfrak{g}$, they converge to 1 uniformly on compact subsets and so we can deduce that

$$A_\hbar(Z,X)\Delta_\hbar(X)^{-1}\omega_\hbar(X)$$

also converges uniformly on compact subsets to

$$f(-X)\left\langle \frac{1}{2}[X,Z],\ dg(X+Z)\right\rangle.$$

We can also show that $C_\hbar(Z,X)\omega_\hbar(X)$ converges uniformly on compact subsets to

$$f(-X)\left\langle \frac{1}{2}[Z,X],\ dg(Z+X)\right\rangle.$$

Finally, it is clear that $B_\hbar(Z,X)$ converges uniformly on compact subsets to

$$f(-X)\ g(X+Z)\ tr(adX).$$

Then $B_\hbar(Z,X)\omega_\hbar(X)$ also converges to the same limit. Now, it is clear that $\frac{\omega_\hbar}{J_\hbar}$ converges uniformly on compact sets to 1. Thus, we can deduce that

$$((f\star_\hbar g)(Z)-(g\star_\hbar f)(Z))/i\hbar$$

converges uniformly on compact subsets to

$$\frac{1}{i}\int_{\mathfrak{g}} f(-X)(\langle[X,Z],\ dg(Z+X)\rangle+g(Z+X)tr(adX))dX$$

as $\hbar$ goes to zero. But if we compare the above formula to the results of the second section, we obtain

$$((f\star_\hbar g)(Z)-(g\star_\hbar f)(Z))/i\hbar$$

converges uniformly on compact sets to $-(2\pi)^{-1}\{f,g\}(Z)$.

For f and g fixed, if $\hbar$ varies in a compact interval it is clear that there exists a compact set which contains the supports of all the $f\star_\hbar g$. Then, we can deduce that

$$(f\star_\hbar g-g\star_\hbar f)/i\hbar$$

converges to $-(2\pi)^{-1}\{f,g\}$ for the inductive limit topology and so that it converges also for the L^1-norm for dX. Then

$$(f\star_\hbar g-g\star_\hbar f)\omega_\hbar/i\hbar$$

converges to $-(2\pi)^{-1}\{f,g\}$ for the L^1-norm for dX.

Since the L^1 norm for $\omega_\hbar(X)dX$ dominates the norms of $C^*_r(G_\hbar)$, then if we denote these norms by $\|\ \ \|_\hbar$ we obtain

$$\|(f\star_\hbar g-g\star_\hbar f)/i\hbar+(2\pi)^{-1}\{f,g\}\|_\hbar$$

converges to zero as $\hbar$ goes to zero. Thus, we conclude the proof of property 6) of the definition 4.1.

Finally, to prove property 7), we will use the C^*-norms of the reduced groups C^*-algebras $C_r^*(G_\hbar)$. These norms are given by the left regular representations on $L^2(G_\hbar)$. We first prove that we have continuity on any $\hbar_0 \neq 0$. Let $f \in C_c^\infty(\vartheta_{\hbar_0})$. We consider the $\hbar'$s for which $f \in C_c^\infty(\vartheta_\hbar)$. f defines the measure $f(X)\omega_\hbar(X)dX$ on $G_\hbar$. We can transfer the finite measure from $G_\hbar$ to G by the isomorphism $\sigma_\hbar : X \longmapsto \hbar X$. In particular, we obtain an isometric isomorphism from $L^1(G_\hbar)$ to $L^1(G)$ and so an isomorphism between the corresponding C^*-algebras. A simple computation, shows that $f(X)\omega_\hbar(X)dX$ is transferred to $\hbar^{-1}f(\hbar^{-1}X)\omega(X)dX$ and so we obtain

$$\|f\|_\hbar = \|\hbar^{-1}f(\hbar^{-1}X)\|_1$$

where $\| \quad \|_1$ is the norm in $L^1(G)$ for the measure $\omega(X)dX$. Now, as $\hbar$ goes to $\hbar_0$, it is clear that $\hbar^{-1}f(\hbar^{-1}X)$ converges uniformly to $\hbar_0^{-1}f(\hbar_0^{-1}X)$ with supports contained in a fixed compact set and so we have also convergence in $L^1(G)$ and in $C^*(G)$. Thus $\|f\|_\hbar$ converges to $\|f\|_{\hbar_0}$.

Now, we will prove the lower semi-continuity on $\hbar = 0$. Let $\| \quad \|_2^\hbar$ be the norm in $L^2(G_\hbar)$ corresponding to the Haar measure $\omega_\hbar(X)dX$. Let $f \in C_c^\infty(\mathfrak{g})$ and $\varepsilon > 0$. Then there exists ϕ in $C_c^\infty(\mathfrak{g})$ such that $\|\phi\|_2^0 = 1$ and $\|f * \phi\|_2^0 \geqslant \|f\|_0 - \varepsilon$ where $*$ is the usual convolution of functions.

By similar arguments as above, we can show that $f \star_\hbar \phi$ converges to $f * \phi$ uniformly as $\hbar$ goes to zero with supports contained in a compact set. Then, $|f \star_\hbar \phi|^2\omega_\hbar$ converges to $|f * \phi|^2$ and $\|f \star_\hbar \phi\|_2^\hbar$ converges to $\|f * \phi\|_2^0$. Thus, for $\hbar$ sufficiently small

$$\|f \star_\hbar \phi\|_2^\hbar \geqslant \|f * \phi\|_2^0 - \varepsilon \ ,$$

and

$$\|\phi\|_2^\hbar \leqslant \|\phi\|_2^0 + \varepsilon = 1 + \varepsilon \ .$$

Finally, we obtain

$$\|f\|_\hbar(1+\varepsilon) \geqslant \|f\|_\hbar\|\phi\|_2^\hbar \geqslant \|f \star_\hbar \phi\|_2^\hbar \geqslant \|f * \phi\|_2^0 - \varepsilon \geqslant \|f\|_0 - 2\varepsilon \ .$$

Then

$$\|f\|_\hbar \geqslant \frac{(\|f\|_0 - 2\varepsilon)}{1+\varepsilon} \ .$$

This proves the lower semi-continuity.

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