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A Variational Inequality for a Degenerate Elliptic Operator Under Minimal Assumptions on the Coefficients.

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Sunto. – *In questa nota si studia un problema di esistenza e unicità di soluzioni di una disuguaglianza variazionale associata al seguente operatore degenere:*

$$(*)\qquad Lu = -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i} + d_j u\right)_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu.$$

I coefficienti dei termini di ordine inferiore e del termine noto di $(*)$ *appartengono ad una generalizzazione degenere del classico spazio di Stummel-Kato. Il peso* w*, che fornisce la degenerazione, appartiene alla classe* A_2 *di Muckenoupt.*

Summary. – *In this note we obtain the existence and the uniqueness of the solution of a variational inequality associated to the degenerate operator*

$$(*)\qquad Lu = -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i} + d_j u\right)_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

assuming the coefficients of the lower terms and the known term belonging to a suitable degenerate Stummel-Kato class. The weight w*, which gives the degeneration, belongs to the Muckenoupt class* A_2*.*

1. – Introduction.

In [4] Chiarenza and Frasca studied the variational inequality

$$(1.1)\qquad u \in \mathbb{K} \;:\; a(u, v-u) \geq \langle T, v-u \rangle \quad \forall v \in \mathbb{K},$$

where $a(u,v)$ is the bilinear form associated to the elliptic degenerate operator

$$(1.2)\qquad Lu = -\sum_{i,j=1}^{n} \left(a_{ij}u_{x_i} + d_j u\right)_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu.$$

Here the principal part of the operator Lu is assumed to be degenerate in the sense that the ellipticity condition is

$$\exists \upsilon > 0 \ : \ \upsilon w|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \frac{1}{\upsilon} w|\xi|^2 \ a.e. \ in \ \Omega, \ \forall \xi \in \mathbb{R}^n,$$

with $w(x)$ a Muckenhoupt's A_2 weight (see [7]).

Chiarenza and Frasca assumed on the coefficients of the lower order terms hypotheses involving L^p spaces.

We stress that the results contained in [4] improved, at least in some instances, those obtained by M.E. Marina [11] in which the weight w giving the degeneration was assumed under convenient hypothesis such as Murthy-Stampacchia [13].

In this note we study the same kind of problem as in [4]. Our purpose is to substitute the L^p assumptions on the coefficients of the lower order terms of operator (1.2) with more general hypothesis which do not imply any high integrability.

The first major contribution in the direction outlined above, seems to have been given by Aizenman and Simon ([1]). In that paper they were able to prove a Harnack inequality for positive weak solutions of equation

$$-\Delta u + cu = 0,$$

assuming the coefficient c in the so called Stummel Kato class defined as follows

$$S = \left\{ f \in L^1(\Omega) \ : \ \sup_{x \in \Omega} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-2}} \, dy \equiv \eta(f, r) \to_{r \to 0} 0 \right\}.$$

They improved the classical works [15] and [9]. We also remember the works [3], [8], [14], [16], [17] and [18].

In this note assuming the coefficients of (1.2) in a degenerate version of the Stummel Kato class (see Definition 2.7) introduced by C. Gutierrez in [8] (see also [16]) we prove the existence and the uniqueness of the solution of (1.1).

We wish to point out that a crucial role in our proofs is played by a Fefferman's type inequality proved in Theorem 2.9.

2. – Function spaces and preliminary results.

Let $p > 1$, a function $w : \mathbb{R}^n \to]0, +\infty[$, such that $w(x)$ and $[w(x)]^{-\frac{1}{p-1}}$ belong to $L^1_{loc}(\mathbb{R}^n)$, is said an A_p weight if and only if

$$\sup_{B_r} \left(\frac{1}{|B_r|} \int_{B_r} w(x) dx \right) \left(\frac{1}{|B_r|} \int_{B_r} [w(x)]^{-\frac{1}{p-1}} dx \right)^{p-1} = C_0 < +\infty, \tag{2.1}$$

where B_r [1] is a ball in $\mathbb{R}^n$; C_0 is said the A_p constant of w.

We now recall some results about A_p weights (see [7] for the proof).

LEMMA 2.1. – *Let $w(x)$ be an A_p weight, $p \in]1, +\infty[$, set $w(B_r) = \int_{B_r} w(x)dx$, then*

a) there exists a constant $C_d > 1$ such that

$$w(B(x, 2r)) \leq C_d w(B(x, r));$$

b) there exists a positive constant $K < 1$ such that

$$w(B(x, r)) \leq K w(B(x, 2r));$$

c) for any bounded subset Ω of $\mathbb{R}^n$ there exists a positive constant $C = C(w, \Omega)$ such that

$$|B_r|^p \leq C w(B_r)$$

for any ball B_r contained in Ω.

Let Ω be an open bounded set in $\mathbb{R}^n$. Because of the local character of our results it is sufficient to assume $\Omega \equiv B(0, R)$.

Let $w(x)$ be an A_2 weight. We give the definitions of the spaces $L^p(\Omega, w)$, $H^{1,p}(\Omega, w)$, $H^{1,p}_{loc}(\Omega, w)$, $H^{1,p}_0(\Omega, w)$, $H^{-1,p}(\Omega, w)$, $p \in [1, +\infty[$ (see also [5]).

$L^p(\Omega, w)$ is the space of measurable u in Ω, such that

$$\|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}} < +\infty.$$

Lip $(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. $\mathrm{Lip}_0(\Omega)$ denotes the class of functions $\phi \in \mathrm{Lip}(\overline{\Omega})$ with compact support contained in Ω. If ϕ belongs to Lip $(\overline{\Omega})$ we can define the norm

$$\|\phi\|_{H^{1,p}(\Omega,w)} := \|\phi\|_{L^p(\Omega,w)} + \sum_{i=1}^n \|\phi_{x_i}\|_{L^p(\Omega,w)}. \tag{2.2}$$

$H^{1,p}(\Omega, w)$ denotes the closure of Lip $(\overline{\Omega})$ under the norm (2.2). We say that $u \in H^{1,p}_{loc}(\Omega, w)$ if $u \in H^{1,p}(\Omega', w)$ for every $\Omega' \subset\subset \Omega$.

$H^{1,p}_0(\Omega, w)$ denotes the closure of $\mathrm{Lip}_0(\Omega)$ under the norm (2.2).

$H^{-1,p'}(\Omega, w)$ is the dual space of $H^{1,p}_0(\Omega, w)$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We have $T \in H^{-1,p'}(\Omega, w)$ if $\exists f_i : \dfrac{f_i}{w} \in L^{p'}(\Omega, w)$, $i = 1, 2, , n$, with $T = \sum_{i=1}^n (f_i)_{x_i}$.

[1] In this paper we will write $B(x, r)$ to denote the ball centered at x with radius r. Whenever x is not relevant we will write B_r.

Let $T \in H^{-1,2}(\Omega, w)$. We consider the equation

$$Lu = T \tag{2.3}$$

where the coefficients $a_{ij}(x)$ are measurable functions such that

$$a_{ij}(x) = a_{ji}(x) \qquad i, j = 1, 2, , n \tag{2.4}$$

and

$$\exists \upsilon > 0 : \upsilon^{-1} w(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \upsilon w(x)|\xi|^2 \, a.e. \, x \in \Omega, \, \forall \xi \in \mathbb{R}^n, \tag{2.5}$$

with the weight w belonging to the A_2 class. We say that $u \in H^{1,2}_{loc}(\Omega, w)$ is a *local weak solution* of the equation (2.3) if

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} u_{x_i} \psi_{x_j} dx = \langle T, \psi \rangle \quad \forall \psi \in C_0^{\infty}(\Omega). \tag{2.6}$$

We now define a different class of solutions. Let μ be a bounded variation measure in Ω . We say that $u \in L^1(\Omega, w)$ is a *very weak solution vanishing on $\partial\Omega$* of the equation

$$Lu = \mu \tag{2.7}$$

if

$$\int_{\Omega} u(x) L\psi(x) dx = \int_{\Omega} \psi(x) d\mu$$

for every $\psi \in H_0^{1,2}(\Omega, w) \cap C^0(\overline{\Omega})$ such that $L\psi \in C^0(\overline{\Omega})$. We observe that the class of test functions is not empty by Theorem 2.3.15 in [6].

REMARK 2.2. – For any bounded variation measure μ in Ω, there exists a unique very weak solution u of $Lu = \mu$ in Ω such that $u = 0$ on $\partial\Omega$ (see [5] Proposition 2.1). Moreover it is not difficult to show that if $u \in H_0^{1,2}(\Omega, w)$ is a weak solution of the equation $Lu = \mu$, i.e.

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} u_{x_i} \psi_{x_j} dx = \int_{\Omega} \psi(x) d\mu \quad \forall \psi \in C_0^{\infty}(\Omega)$$

then u is a very weak solution of the same equation.

Let $y \in \Omega$. Denote by $g_{\Omega}(x, y)$ the very weak solution vanishing on $\partial\Omega$ of the equation

$$Lu = \delta_y,$$

where δ_y is the Dirac mass at y. We call it the Green function relative to the operator L in Ω.

We now recall some results, concerning the Green function, proved in [5].

THEOREM 2.3 (see [5] Proposition 2.4). – *Let $B_r \subseteq \Omega$ be a ball and $g_{B_r}(x,y)$ be the Green's function of L in B_r, then $g_{Br}(.,y) \in H^{1,2}(B_r \setminus B(y,\varepsilon),w)$ for any $\varepsilon > 0$.*

THEOREM 2.4 (see [5] Lemma 2.7). – *Let $B_r \subseteq \Omega$ be a ball and $g_{B_r}(x,y)$ be the Green's function of L in B_r, then*

$$u(x) = \int_{B_r} g_{B_r}(x,y)d\mu(y)$$

is the very weak solution vanishing on ∂B_r of (2.6) *in B_r.*

Denoting by $g_{B(0,4R)}$ the Green's function in $B(0,4R)$ we have the following result

THEOREM 2.5. – *Let $x, y \in B(x_0, r) \subseteq \Omega$. Then there exist two positive constants C_1 and C_2, independent of x and y, such that*

$$C_1 < \frac{g_{B(x_0,4r)}(x,y)}{\int\limits_{|x-y|}^{4r} \frac{s^2}{w(B(x,s))}\frac{ds}{s}} < C_2. \tag{2.8}$$

Moreover

$$g_{\Omega''} \le g_{\Omega'} \le g_{B(0,4R)} \tag{2.9}$$

for any $\Omega'' \subseteq \Omega' \subseteq B(0,4R)$.

PROOF. – For (2.8) see Theorem 3.3 in [5]. (2.9) follows by maximum principle. □

We now define some function spaces.

DEFINITION 2.7 (see [8]). – *We set*

$$S(\Omega,w) = \left\{ f \in L^1(\Omega,w) : \sup_{x\in\Omega} \int_{\{y\in\Omega : |x-y|<r\}} |f(y)| \int_{|x-y|}^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s} w(y)dy \equiv \eta(f,r) \right\}.$$

Moreover we put

$$\eta(f) \equiv \sup_{0\le r\le\delta} \eta(f,r),$$

where δ denotes the diameter of Ω.

REMARK 2.8. – We note that in the nondegenerate case, i.e. $w = 1$, $S(\Omega, w)$ is the Kato-Stummel class (see [1], [2], [3] and [14]).

Subsequently we will use the following results

THEOREM 2.9. – *Let* $\dfrac{f}{w} \in S(\Omega, w)$. *Then there exists a positive constant* c, *independent of* u, *such that*

$$\int_{B(x_0,\rho)} |f| u^2\,dx \le c\eta(\rho) \int_{B(x_0,\rho)} |\nabla u|^2 w\,dx,$$

for every $u \in H_0^{1,2}(\Omega, w)$, *where* $\operatorname{spt} u \subset B(x_0, \rho)$.

PROOF. – To prove our thesis we suppose $f(y) \ge 0$ and put

$$F(x) = \int_{B(x_0,\rho)} f(y) g_{B(x_0,\rho)}(x, y)\,dy, \quad x \in B(x_0, \rho).$$

Then, from Theorem 2.4, $F(x)$ is the very weak solution in $B(x_0, \rho)$ of equation

$$-\sum_{i,j=1}^{n} (a_{ij} F_{x_i})_{x_j} = f.$$

Then, as in the proof of Theorem 4.8 of [5], we can conclude that $\chi_{B(x_0,\rho)} f \in H^{-1,2}(B(x_0, \rho), w)$ and, for Proposition 2.3 in [5], $F \in H_0^{1,2}(B(x_0, \rho), w)$. Therefore

$$\text{(2.9)} \quad \int_{B(x_0,\rho)} f u^2\,dx = 2 \int_{B(x_0,\rho)} \sum_{i,j=1}^{n} a_{ij} F_{x_i} u u_{x_j}\,dx \le$$

$$\le 2\upsilon \left(\int_{B(x_0,\rho)} |\nabla F|^2 u^2 w\,dx \right)^{\frac{1}{2}} \left(\int_{B(x_0,\rho)} |\nabla u|^2 w\,dx \right)^{\frac{1}{2}}.$$

Moreover, since

$$0 \le \int_{B(x_0,\rho)} f F^{-1} u^2\,dx = \int_{B(x_0,\rho)} \sum_{i,j=1}^{n} a_{ij} F_{x_i} u (F^{-1} u^2)_{x_j}\,dx$$

we have

$$\upsilon^{-1} \int_{B(x_0,\rho)} |\nabla F|^2 F^{-2} u^2 w\,dx \le 2\upsilon \left(\int_{B(x_0,\rho)} |\nabla F|^2 u^2 F^{-2} w\,dx \right)^{\frac{1}{2}} \left(\int_{B(x_0,\rho)} |\nabla u|^2 w\,dx \right)^{\frac{1}{2}},$$

and then

$$\left(\int_{B(x_0,\rho)} |\nabla F|^2 u^2 F^{-2} w\,dx\right)^{\frac{1}{2}} \le 2v^2\left(\int_{B(x_0,\rho)} |\nabla u|^2 w\,dx\right)^{\frac{1}{2}}. \tag{2.10}$$

Note that for $x \in B(x_0, \rho)$, using Theorem 2.5, we have

$$F(x) \le c(n)\eta(r). \tag{2.11}$$

Hence, from (2.9), (2.10) and (2.11), we obtain the desired conclusion. □

LEMMA 2.10. – *Let $f \in S(\Omega, w)$. Then*

$$\forall \varepsilon > 0 \quad \exists \sigma > 0 \quad : \quad E \subset \Omega, \quad |E| < \sigma \quad \Rightarrow \quad \eta(f\chi_E) < \varepsilon.$$

PROOF. – Let $\rho > 0$ be such that $\eta(f, \rho) < \dfrac{\varepsilon}{4}$. The absolute continuity of the integral implies that there exists $\sigma > 0$ such that

$$E \subseteq \Omega, \quad |E| < \sigma \Rightarrow \int_E f(y)w(y)\,dy < \frac{\varepsilon}{2}\left(\int_\rho^{4R} \frac{s^2}{w(B(x,s))}\frac{ds}{s}\right)^{-1}.$$

Hence

$$\eta(f\chi_E) \le \sup_{0<r\le\rho} \eta(f\chi_E, r) + \sup_{\rho<r} \eta(f\chi_E, r) \le$$

$$\le \eta(f,\rho) + \sup_{\rho<r}\sup_{x\in\Omega} \int_{|x-y|<r} |f(y)\chi_E(y)| \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))}\,ds\,w(y)\,dy \le$$

$$\le \eta(f,\rho) + \sup_{\rho<r}\sup_{x\in\Omega}\left\{\int_{|x-y|<\rho} |f(y)| \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))}\,ds\,w(y)\,dy + \right.$$

$$\left. + \int_{\rho<|x-y|\le r} |f(y)\chi_E(y)| \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))}\,ds\,w(y)\,dy\right\} \le$$

$$\le 2\eta(f,\rho) + \int_\rho^{4R} \frac{s}{w(B(x,s))}\,ds \int_E |f(y)|\,w(y)\,dy < \varepsilon.$$

□

LEMMA 2.11. – *Let $f \in S(\Omega, w)$ and $\varepsilon > 0$. Then there exist two functions f_1 and f_2 such that*

$$f = f_1 + f_2; \quad f_2 \in L^\infty(\Omega); \quad \eta(f_1) < \varepsilon.$$

PROOF. – Let $\rho > 0$ be such that $\eta(f, \rho) < \frac{\varepsilon}{4}$. Setting $A_k = \{x \in \Omega : |f(y)| > k\}$ we have

$$\lim_{k\to\infty} \int_{A_k} |f(y)|\, dy = 0,$$

then there exists $k' > 0$ such that

$$\int_{A_{k'}} |f(y)|\, dy < \frac{\varepsilon}{2} \left(\int_{\rho}^{4R} \frac{s^2}{w(B(x,s))} \frac{ds}{s} \right)^{-1}.$$

Now, setting $f_1 = f\chi_{A_{k'}}$ and $f_2 = f - f_1$, we have

$$f_2 \in L^\infty(\Omega), \qquad \|f_2\|_{L^\infty(\Omega)} \le k'$$

and, proceeding as in Lemma 2.10 we have $\eta(f_1) < \varepsilon$. □

LEMMA 2.12. – *Let* $\frac{f}{w} \in S(\Omega, w)$. *Then, for any* $\varepsilon > 0$, *there exists* $K(\varepsilon) > 0$ *such that*

$$\int_\Omega |f| u^2\, dx \le \varepsilon \int_\Omega |\nabla u|^2 w\, dx + K(\varepsilon) \int_\Omega u^2 w\, dx,$$

for every $u \in C_0^\infty(\Omega)$.

PROOF. – Using Lemma 2.11 and Theorem 2.9 we have

$$\int_\Omega |f(y)| u^2(y)\, dy \le \int_\Omega |f_1(y)| u^2(y)\, dy + \int_\Omega |f_2(y)| u^2(y)\, dy \le$$

$$\le c\varepsilon \int_\Omega |\nabla u(y)|^2 w(y)\, dy + K(\varepsilon) \int_\Omega u^2(y) w(y)\, dy.$$

□

3. – Uniqueness and existence results for degenerate elliptic operators.

Let Ω be a bounded open set in $\mathbb{R}^n$ and $w \in A_2$. We consider the linear differential operator

$$Lu = -\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i} + d_j u\right)_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu \tag{3.1}$$

where a_{ij}, d_j, b_i and c $(i, j = 1, \ldots, n)$ are measurable functions such that

$$a_{ij} = a_{ji} \tag{3.2}$$

$$\exists \upsilon > 0 \; : \; \upsilon w|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \frac{1}{\upsilon} w|\xi|^2 \; a.e. \; in \; \Omega, \; \forall \xi \in \mathbb{R}^n, \tag{3.3}$$

$$\left(\frac{d_i}{w}\right)^2, \left(\frac{b_i}{w}\right)^2 \in S(\Omega, w), \quad \frac{c}{w} \in S(\Omega, w) \tag{3.4}$$

$$\int_\Omega \left(\sum_{i=1}^{n} d_i \varphi_{x_i} + c\varphi \right) dx \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega), \; \varphi \geq 0 \quad \text{in}\, \Omega. \tag{3.5}$$

Now we define for $u, v \in H_0^{1,2}(\Omega, w)$ the bilinear form

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n} (a_{ij}u_{x_i} + d_j u)v_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} v + cuv \right) dx.$$

Given $\psi \in H_0^{1,2}(\Omega, w)$, $\psi \leq 0$ in $\partial\Omega$ and $T \in H^{-1,2}(\Omega, w)$, in the non-empty, closed and convex set

$$\mathbb{K} = \left\{ v \in H_0^{1,2}(\Omega, w) \; : \; v \geq \psi \, a.e. \, \Omega \right\},$$

we consider the problem

$$u \in \mathbb{K} \; : \; a(u, v - u) \geq \langle T, v - u \rangle \;\; \forall v \in \mathbb{K}. \tag{3.6}$$

We have the following

THEOREM 3.1. – *Let u be a solution of* (3.6). *We assume that* (3.2), (3.3), (3.4) *and* (3.5) *hold. Then*

$$\|u\|_{H^{1,2}(\Omega,w)} \leq k\big(\|T\|_{H^{-1,2}(\Omega,w)} + \|\psi\|_{H^{1,2}(\Omega,w)}\big),$$

where k is a positive constant depending on n, υ, the S_w modulus and the L^1 norm of $\dfrac{|b-d|^2}{w^2} = \sum_{i=1}^{n} \dfrac{|b_i - d_i|^2}{w^2}$.

PROOF. – Taking $v = u^+$ in (3.6), we have

$$a(u, u^-) \geq \langle T, u^- \rangle;$$

since $a(u^+, u^-) = 0$, we have

$$a(u^-, u^-) \leq -\langle T, u^- \rangle$$

and for (3.5) it results

$$\int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}(u^-)_{x_i}(u^-)_{x_j} + \sum_{i=1}^{n} (b_i - d_i)(u^-)_{x_i} u^- \right\} dx \leq -\langle T, u^- \rangle. \tag{3.7}$$

From (3.7), in a standard way, using Theorem 2.9 we have

$$\|u^-\|_{H^{1,2}(\Omega,w)} \le c(\upsilon,n,w)\big(\|T\|_{H^{-1,2}(\Omega,w)} + \|u^-\|_{L^2(\Omega,w)}\big).$$

Keeping in mind that $u^- \le \psi^-$ a.e. in Ω, we get

$$\|u^-\|_{L^2(\Omega,w)} \le \|\psi^-\|_{L^2(\Omega,w)} \le \|\psi\|_{L^2(\Omega,w)} \le \|\psi\|_{H^{1,2}(\Omega,w)}.$$

Then thesis holds for u^-. Now, taking $z = u^+ - \psi^+$ in Ω, we need only to estimate $\|z\|_{H^{1,2}(\Omega,w)}$ to obtain the conclusion. We begin introducing some notations. For $h \in \mathbb{R} \cup \{+\infty\}$, $h > 0$ and $k \in \mathbb{R}$, $0 \le k \le h$, we set

$$\Omega(k,h) = \{x \in \Omega \ : \ k \le z(x) < h,\ |\nabla u| > 0\}. \tag{3.8}$$

For fixed h we have that the not increasing function

$$k \to |\Omega(k,h)|$$

is continuous in $[0,h]$ and it results

$$\lim_{k\to h^-} |\Omega(k,h)| = 0.$$

Moreover we set

$$V^h(k) = c^{\frac{1}{2}}(n,w)\left[\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k,h)}\right)\right]^{\frac{1}{2}}\upsilon^{-1},$$

where $c(n,w)$ is the same constant as in Theorem 2.9. Function $V^h(k)$ is not increasing in $[0,h[$ and

$$\lim_{k\to h^-} V^h(k) = 0.$$

$V^h(k)$ is a continuous function in $[0,h[$. In fact

$$\left|\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k+s,h)}\right) - \eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k,h)}\right)\right| \le \eta\left(\frac{|b-d|^2}{w^2}|\chi_{\Omega(k+s,h)} - \chi_{\Omega}^{(k,h)}|\right),$$

and, by Lemma 2.10 we obtain that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\chi_{\Omega(k+s,h)} - \chi_{\Omega}^{(k,h)}| < \delta$ then

$$\eta\left(\frac{|b-d|^2}{w^2}|\chi_{\Omega(k+s,h)} - \chi_{\Omega}^{(k,h)}|\right) < \varepsilon.$$

Now set $k_0 = +\infty$ we obtain t non negative real numbers in the following way. If $V^{k_0}(0) \le \frac{1}{2}$ we take $k_1 = 0$, on the contrary we will fix k_1 such that $V^{k_0}(k_1) = \frac{1}{2}$. If $k_1 > 0$, for any $k_2 \in [0,k_1[$ we consider function $V^{k_1}(k_2)$. Again, if $V^{k_1}(0) \le \frac{1}{2}$ we take $k_2 = 0$, otherwise we choose k_2 such that $V^{k_1}(k_2) = \frac{1}{2}$. Iterating this process we are able to prove that there exists a positive number

$$t \le 1 + 16\upsilon^{-2} \int_{\rho}^{4R} \frac{s}{w(B(x,s))}\, ds \int_{\Omega} \frac{|b-d|^2}{w^2}\, dx,$$

such that $k_t = 0$, where ρ is such that

$$\left[\eta\left(\frac{|b-d|^2}{w^2};\rho\right)\right]^{\frac{1}{2}} \le \frac{\upsilon}{8[c(n,w)]^{\frac{1}{2}}}. \tag{3.9}$$

In fact if $V^{k_0}(0) \le \frac{1}{2}$ we take $t = 1$, otherwise there exists $m \in \mathbb{N}$, $m > 1$ such that $V^{k_{i-1}}(k_i) = \frac{1}{2}$, $\forall i = 1, 2, \ldots, m$ and

$$\frac{m}{2} = [c(n,w)]^{\frac{1}{2}} \upsilon^{-1} \sum_{i=1}^{m} \left[\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})}\right)\right]^{\frac{1}{2}}. \tag{3.10}$$

Fixing $\rho > 0$ as in (3.9), we have, as in the proof of Lemma 2.10

$$\left[\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})}\right)\right]^{\frac{1}{2}} \le$$

$$\le \left[\sup_{0<r\le\rho} \eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})};r\right)\right]^{\frac{1}{2}} + \left[\sup_{r>\rho} \eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})};r\right)\right]^{\frac{1}{2}} \le$$

$$\le 2\left[\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})};\rho\right)\right]^{\frac{1}{2}} + \left(\int_{\rho}^{4R} \frac{s}{w(B(x,s))}\, ds \int_{\Omega(k_i,k_{i-1})} \frac{|b-d|^2}{w^2}\, dy\right)^{\frac{1}{2}},$$

and then

$$\sum_{i=1}^{m} \left[\eta\left(\frac{|b-d|^2}{w^2}\chi_{\Omega(k_i,k_{i-1})}\right)\right]^{\frac{1}{2}} \le \frac{m\upsilon}{4[c(n,w)]^{\frac{1}{2}}} + \left(\int_{\rho}^{4R} \frac{s}{w(B(x,s))}\, ds \int_{\Omega} \frac{|b-d|^2}{w^2}\, dy\right)^{\frac{1}{2}}.$$

From (3.10) we obtain

$$m \le 16\,\upsilon^{-2} c(n,w) \int_{\rho}^{4R} \frac{s}{w(B(x,s))}\, ds \int_{\Omega} \frac{|b-d|^2}{w^2}\, dy.$$

Now we set

$$\Omega_1 = \Omega(k_1, \infty),$$

and

$$z_1 = (z - k_1)^+ = \max(z - k_1, 0).$$

Since $z_1 \in H_0^{1,2}(\Omega, w)$ and $u - z_1 \in \mathbb{K}$, assuming $v = u - z_1$ in (3.6), it follows

$$a(u^+, z_1) \le \langle T, z_1 \rangle \tag{3.11}$$

By (3.5) we have

$$\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} z_{x_i} (z_1)_{x_j} + \sum_{i=1}^n (b_i - d_i)(u^+)_{x_i} z_1 \right\} dx \le \langle T, z_1 \rangle -$$

$$- \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} (\psi^+)_{x_i} (z_1)_{x_j} + \sum_{i=1}^n (b_i - d_i)(\psi^+)_{x_i} z_1 \right\} dx,$$

from which, by assumptions (3.3), (3.4) and Theorem 2.9, we obtain

$$\upsilon \|z_1\|^2_{H^{1,2}(\Omega,w)} \le \Bigg[\|T\|_{H^{-1,2}(\Omega,w)} + \frac{1}{\upsilon} \|\psi\|_{H^{1,2}(\Omega,w)} +$$

$$+ [c(n,w)]^{\frac{1}{2}} \|\psi\|_{H^{1,2}(\Omega,w)} \eta \left(\frac{|b-d|^2}{w^2} \right) \Bigg] \|z_1\|_{H^{1,2}(\Omega,w)} +$$

$$+ [c(n,w)]^{\frac{1}{2}} \left[\eta \left(\frac{|b-d|^2}{w^2} \chi_{\Omega_1} \right) \right]^{\frac{1}{2}} \int_\Omega |\nabla z_1|^2 w \, dx$$

and

$$\upsilon \|z_1\|^2_{H^{1,2}(\Omega,w)} \le B \|z_1\|_{H^{1,2}(\Omega,w)} + \upsilon V^\infty(k_1),$$

where

$$B = \|T\|_{H^{-1,2}(\Omega,w)} + \frac{1}{\upsilon} \|\psi\|_{H^{1,2}(\Omega,w)} + [c(n,w)]^{\frac{1}{2}} \|\psi\|_{H^{1,2}(\Omega,w)} \, \eta \left(\frac{|b-d|^2}{w^2} \right).$$

Remembering that $V^\infty(k_1) = \frac{1}{2}$, we have

$$\|z_1\|_{H^{1,2}(\Omega,w)} \le \frac{2}{\upsilon} B.$$

Now for any $q = 2, \dots, t$ we set

$$\Omega_q = \Omega(k - q, k_{q-1})$$

and

$$z_q = \begin{cases} 0 & \textit{if } u < k_q \\ z - k_q & \textit{if } k_q \le z < k - q - 1 \\ k_{q-1} - k_q & \textit{if } z \ge k_{q-1}. \end{cases}$$

For any $q = 2, \dots, t$, by (3.5) we have

$$\int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(u^+)_{x_i}(z_q)_{x_j} + \sum_{i=1}^{n} (b_i - d_i)(u^+)_{x_i} z_q \right\} dx \le \langle T, z_q \rangle.$$

Now, proceeding as above,we obtain

$$\|z_q\|_{H^{1,2}(\Omega,w)} \le \frac{2}{\upsilon} B, \qquad \forall q = 2, \dots, t.$$

Moreover we have

$$\int_{\Omega} (b_i - d_i) z_{x_i} z_q \, dx = \sum_{h=1}^{q} \int_{\Omega_h} (b_i - d_i)(z_h)_{x_i} z_q \, dx$$

and by (3.3) and Theorem 2.9 we get

$$\upsilon \|z_q\|_{H^{1,2}(\Omega,w)} \le B + \upsilon V^{k_{q-1}}(k_q) \|z_q\|_{H^{1,2}(\Omega,w)} + \upsilon \sum_{h=2}^{q-1} V^{k_{h-1}}(k_h) \|z_h\|_{H^{1,2}(\Omega,w)}.$$

Since $V^{k_{q-1}}(K_q) = \dfrac{1}{2}$ and $V^{k_{h-1}}(K_h) \le \dfrac{1}{2}$ for any $h = 2, \dots, q-1$, we have

$$\|z_q\|_{H^{1,2}(\Omega,w)} \le 2q\upsilon^{-1} B$$

for any $q = 2, \dots, t$. Finally we obtain the desired conclusion noticing that

$$\|z^+\|_{H^{1,2}(\Omega,w)} \le \sum_{h=1}^{t} \|z_h\|_{H^{1,2}(\Omega,w)}.$$

□

The uniqueness of solution of variational inequality (3.6) is proved by the following

THEOREM 3.2. – *Let $u \in \mathbb{K}$ be a solution to problem* (3.6). *Let $u_1 \in H_0^{1,2}(\Omega, w)$, $u_1 \ge \psi$ a.e. in Ω, $a(u, \phi) \ge \langle T, \phi \rangle$ $\forall \phi \in H_0^{1,2}(\Omega, w)$, $\phi \ge 0$. Then $u \le u_1$ a.e. in Ω.*

PROOF. – Set $v = \min(u, u_1)$, we claim that $u = v$ a.e. in Ω. Let h be a non negative real number, we have

$$z_h = (u - v - h)^+ \in H_0^{1,2}(\Omega, w).$$

Proceeding as in [4] (Theorem 3.1) we obtain

$$\|z_h\|_{H^{1,2}(\Omega,w)} \le \left\{ \int_{\Omega_h} \frac{|b-d|^2}{w^2} z_h^2 w \, dx \right\}^{\frac{1}{2}},$$

where

$$\Omega_h = \{x \in \Omega \ : \ u - v > h \, and \, |\nabla(u - v)| > 0\}.$$

Using Theorem 2.9 instead of Hölder plus Sobolev inequalities we have the thesis. □

The previous result yields

COROLLARY 3.3. – *In the same assumptions of Theorem* 3.2, *u is the only solution of* (3.6).

Our existence result follows from Theorem 3.1 and the following compactness embedding theorem (see e.g. [4])

THEOREM 3.4. – *Let* $w \in A_2$. *There exists a constant* C_Ω, *depending on u, the* A_2 *constant of w, and* $\varepsilon > \frac{1}{2}$ *such that for some* $u \in H_0^{1,2}(\Omega, w)$ *and* $1 \le k \le \frac{n}{n - 2\varepsilon} = n'$ *we have*

$$\|u\|_{L^{2k}(\Omega,w)} \le \|u\|_{H^{1,2}(\Omega,w)}.$$

For $1 \le k < n'$ *the embedding of* $H_0^{1,2}(\Omega, w)$ *in* $L^{2k}(\Omega, w)$ *is compact.*

Precisely we obtain

THEOREM 3.5. – *Under assumptions* (3.1), (3.2), (3.3), (3.4) *and* (3.5) *there exists the solution of variational inequality* (3.6).

PROOF. – We start to note that, by Theorem 2.9, the bilinear form $a(u, v)$ is continuous in $H_0^{1,2}(\Omega, w)$.

Now, for any $v \in H_0^{1,2}(\Omega, w)$ we have

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} v_{x_i} v_{x_j} \, dx - \sum_{i=1}^{n} \int_\Omega (b_i - d_i) v_{x_i} v \, dx \le L(v, v).$$

Then, using Lemma 2.12 we have that there exists a positive constant μ, such that

$$a(v, v) + \mu \|v\|^2_{L^2(\Omega,w)} \ge \frac{\upsilon}{2} \|v\|^2_{H_0^{1,2}(\Omega,w)}, \quad \forall v \in H_0^{1,2}(\Omega, w).$$

Then (see e.g.[10]) $\forall \phi \in L^2(\Omega, w)$ the problem

$$u \in \mathbb{K} \ : \ a(u, v - u) + \mu \int_\Omega u(v - u) w \, dx \ge \langle T, v - u \rangle + \mu \int_\Omega \phi(v - u) w \, dx \ \ \forall u \in \mathbb{K}$$

have unique solution $u = S(\phi)$

In this way we have defined the operator

$$S : L^2(\Omega, w) \to L^2(\Omega, w).$$

that results continuous and compact. Moreover, if we consider $a \in [0,1]$ and ϕ such that $\phi = aS(\phi)$, assuming $u = S(\phi)$, we get

$$a(u, v-u) + \mu(1-a)\int_\Omega u(v-u)w\,dx \geq \langle T, v-u\rangle \quad \forall v \in \mathbb{K}.$$

From the apriori estimate we can estimate $\|S(\phi)\|_{H^{1,2}(\Omega,w)}$ and $\|\phi\|_{L^2(\Omega,w)}$ with constants independent from ϕ and a. Then the desired conclusion follows from Leray-Schauder theorem. □

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