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Interior $C^{1,\alpha}$-Regularity of Weak Solutions to the Equations of Stationary Motions of Certain Non-Newtonian Fluids in Two Dimensions


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**Interior $C^{1,a}$-Regularity of Weak Solutions to the Equations of Stationary Motions of Certain Non-Newtonian Fluids in Two Dimensions.**

**J. Wolf**

**Sunto.** – Si dimostra l’hölderianità del gradiente di ogni soluzione debole di un sistema di equazioni degenerate, che descrivono il moto di un fluido incompressibile non-newtoniano in due dimensioni, sotto condizioni usuali di monotonia e di andamento all’infinito di ordine $q - 1 (1 < q < 2)$.

**Summary.** – In the present work we prove the interior Hölder continuity of the gradient matrix of any weak solution of equations, which describes the motion of non-Newtonian fluid in two dimensions, restricting ourself to the shear thinning case $1 < q < 2$.

1. – Introduction. Statement of the main result.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The stationary motion of an incompressible fluid through $\Omega$ is governed by the following two equations

$$
(1.1) \quad -\text{div} S + u \cdot \nabla u = -\nabla p + f \quad \text{in} \quad \Omega,
$$

$$
(1.2) \quad \text{div} u = 0 \quad \text{in} \quad \Omega,
$$

where

$$
S = \{S_{ij}\} = \text{deviatoric stress tensor} \,(^1),
$$

$$
p = \text{pressure},
$$

$$
u = \{u_1, u_2\} = \text{velocity},
$$

$$
f = \{f_1, f_2\} = \text{external force}.
$$

($^1$) Throughout Latin subscripts take the values 1, 2. Repeated subscripts imply summation over 1, 2.
Assuming the condition of adherence on the boundary of \( \Omega \) we have
\[
(1.3) \quad u = 0 \quad \text{on} \quad \partial \Omega.
\]

In addition \( S \) may depend on the «rate of strain tensor» \( D = \{D_{ij}\} \), which is defined by
\[
D_{ij} = D_{ij}(u) := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i,j = 1,2.
\]
(cf. [2], [3], [10]).

In the present paper we consider constitutive laws of the following both types
\[
S = v(D_{II})^{(q-2)/2}D, \quad 1 < q < 2
\]
\[
S = v(1 + D_{II})^{(q-2)/2}D, \quad 1 < q < 2 \quad (v = \text{const} > 0)
\]
as special cases, where
\[
D_{II} = \frac{1}{2} D_{ij} D_{ij} = \text{second invariant of } D
\]
(cf. [2], [4], [14]). A fluid which is determined by the first of these constitutive laws is said «pseudoplastic» or «shear thinning». This motivates us to impose the following conditions on \( S \):
\[
(1.4) \quad \sum_{i,j,k,l=1}^{2} \left| \frac{\partial S_{ij}}{\partial \xi_{kl}} (\xi) \right| \leq c_0 (\mu + \| \xi \|^q)^{q-2} \quad \forall \xi \in M_{\text{sym}}^4 \setminus \{0\};
\]
\[
(1.5) \quad \frac{\partial S_{ij}}{\partial \xi_{kl}} (\xi) \eta_{kl} \eta_{ij} \geq v_0 (\mu + \| \xi \|^q)^{q-2} \| \eta \|^2 \quad \forall \xi, \eta \in M_{\text{sym}}^4 \setminus \{0\},
\]
\( (c_0 > 0, v_0 > 0) \). Here \( 1 < q < 2 \) and \( \mu \geq 0 \) are fixed numbers.

Clearly, (1.4) implies
\[
(1.6) \quad \| S(\xi) \| \leq \frac{c}{q-1} (\mu + \| \xi \|^q)^{q-1} + \| S(0) \| \quad \forall \xi \in M_{\text{sym}}^4.
\]

**Weak solution to** (1.1), (1.2). By \( W^{1,\sigma}(\Omega) \) and \( W_0^{1,\sigma}(\Omega) (1 \leq \sigma < +\infty) \) we denote the usual Sobolev spaces.

**Definition 1.1.** – Assume (1.6). Let \( f \in [L^2(\Omega)]^2 \). A vector-valued function \( u \in [W^{1,q}(\Omega)]^2 \) with \( \text{div} \ u = 0 \) is called a weak solution to (1.1), (1.2) if the fol
lowing integral identity is fulfilled for all $\varphi \in [C_c^\infty(\Omega)]^2$ with $\text{div} \varphi = 0$: 

\begin{equation}
\int_\Omega S_{ij}(D(u))D_{ij}(\varphi) \, dx + \int_\Omega u_i \frac{\partial u_j}{\partial x_i} \varphi_j \, dx = \int_\Omega f_j \varphi_j \, dx.
\end{equation}

Our main result is the following

**THEOREM. Assume (1.4), (1.5). Let $u \in [W^{1,q}(\Omega)]^2$ be a weak solution to (1.1), (1.2). Suppose**

\begin{equation}
\text{there exists } \sigma > 2: \quad \tilde{f} = f - u \cdot \nabla u \in [L^\sigma_{\text{loc}}(\Omega)]^2.
\end{equation}

**Then there exists a number** $0 < a < 1$ **such that**

\begin{equation}
u \in [C^{1,a}(\Omega)]^2.
\end{equation}

In the case $\frac{3}{2} < q < 2$ the existence of the second weak derivatives in $L^s_{\text{loc}}(\Omega) (q \leq s < 2)$ of any weak solution $u \in [W^{1,q}(\Omega)]^2$ to (1.1), (1.2) has been proved in [13]. Thus, by means of Sobolev’s imbedding theorem it is readily seen that the assumption (1.8) in the Theorem is always fulfilled in this particular case. Therefore replacing this condition by

\begin{equation}
\frac{3}{2} < q < 2, \quad f \in [L^\sigma_{\text{loc}}(\Omega)]^2 \quad (\sigma > 2)
\end{equation}

implies the

**COROLLARY I. Assume (1.4), (1.5) and (1.8'). Let $u \in [W^{1,q}(\Omega)]^2$ be a weak solution to (1.1), (1.2). Then $\frac{\partial u_j}{\partial x_i} \ (i,j = 1, 2)$ are (locally) Hölder continuous functions.**

In addition neglecting the convective term and making use of [11], where the author has proved the existence of the second derivatives for those weak solutions we also have

**COROLLARY II. Assume (1.4), (1.5) and $f \in [L^\sigma_{\text{loc}}(\Omega)]^2 \ (\sigma > 2)$. Let $u \in [W^{1,q}(\Omega)]^2$ satisfy the identity**

\begin{equation}
\int_\Omega S_{ij}(D(u))D_{ij}(\varphi) \, dx = \int_\Omega f_j \varphi_j \, dx
\end{equation}

**for all** $\varphi \in [C_c^\infty(\Omega)]^2$ **with** $\text{div} \varphi = 0$. **Then** $\frac{\partial u_j}{\partial x_i} \ (i,j = 1, 2)$ **are (locally) Hölder continuous functions.**
Remark 1. – The existence of a strong solution $u \in [C^{1,\alpha}(\Omega)]^2 \left( q > \frac{6}{5} \right)$ resp. 
$u \in [C^{1,\alpha}(\Omega)]^2 \left( q > \frac{3}{2} \right)$ to the system (1.1)-(1.3) has been proved by Kaplický, 
Málek, Stará in [9] imposing conditions on the constitutive law which are slightly 
more restrictive then ours. In contrast to [9] here the boundary condition (1.3) 
does not play an essential role for the proof of the Hölder continuity of $\nabla u$. 
Therefore our result is applicable to weak solutions to (1.1), (1.2) fulfilling, instead 
of (1.3), any boundary condition.

2. The Hölder continuity of $\nabla u$, where $u$ is any weak solution of an elliptic 
system with coefficients satisfying conditions similar to (1.4), (1.5) has been 
proved in [12] via higher integrability of the function $(1 + |\nabla u|^{(q-2)/2} \nabla^2 u)^3$. In 
order to achieve the necessary reverse Hölder inequality in the appendix of this 
paper, based on the well known Poincaré inequality, the authors show that there 
exists $4 \in \mathbb{R}^4$ such that

$$
\int_{B_r} (1 + |\nabla u| + |\nabla u - A|^2 |\nabla u - A|^2 \, dx \leq \frac{c r^2}{B_r} (1 + |\nabla u|^{q-2} |\nabla^2 u|^2 \, dx,
$$

where $c = \text{const} > 0$ depends only on $q$.

Unfortunately such an argument seems not to work equally if one replaces 
the gradient $\nabla u$ by the symmetric gradient $D(u)$. However in the present 
paper by an entirely different method we are able to establish an appropriate reverse Hölder inequality which will imply the higher integrability of 
$(\mu + |D(u)|^{(q-2)/2} \nabla D(u)) (\mu \geq 0)$ (cf. section 4) based on two different Caccioppoli inequalities (cf. Theorem 3.1 resp. Theorem 3.2) and the Poincaré inequality.

In the appendix of our paper, dealing with the special case $\mu = 0$, we prove that $|D(u)|^{(q-2)/2} \nabla D(u) \in [L^2(B_r)]^3$ for each $B_r \subset \subset \Omega$. In addition we derive an appropriate estimate of the $L^2(B_r)$- norm of $|D(u)|^{(q-2)/2} \nabla D(u)$ which is needed in section 3.

2. – Preliminaries.

This section is devoted to some preliminary lemmas which will be used in the following sections.

Lemma 2.1. – Let $N$ be an integer $\geq 1$. Let $0 < a < 1$. Then the following 
inequality holds for all $\xi, \eta \in \mathbb{R}^N \setminus \{0\}$

$$
(\mu + |\xi| + |\eta|)^{-a} \leq \int_0^1 (\mu + |\xi + t\eta|)^{-a} \, dt \leq \frac{3^a}{1-a} (\mu + |\xi| + |\eta|)^{-a}.
$$

(3) Here $\nabla^2 u$ denotes the matrix of second derivatives of $u$. 
PROOF. – The first inequality of (2.1) is trivially fulfilled. It only remains to prove the second.

First let us consider the case \( \max\{\|\zeta\|, |\eta|\} \leq \mu \). Here we easily see that
\[
\int_0^1 (\mu + |\zeta + t\eta|)^{-a} dt \leq 3^a (\mu + \mu + \mu)^{-a} \leq 3^a (\mu + |\zeta| + |\eta|)^{-a}.
\]
Next, assume that \( |\zeta| \leq |\eta| \) and \( |\eta| > \mu \). Set \( \tau := \frac{|\zeta|}{|\eta|} \). Then \( 0 < \tau \leq 1 \) and we estimate
\[
\int_0^1 (\mu + |\zeta + t\eta|)^{-a} dt \leq \int_0^\tau (\mu + (\tau - t)|\eta|)^{-a} dt + \int_\tau^1 (\mu + (t - \tau)|\eta|)^{-a} dt
\]
\[
= \frac{1}{(1 - a)|\eta|} (\mu + \tau|\eta|)^{1-a} + (\mu + (1 - \tau)|\eta|)^{1-a} - 2\mu
\]
\[
\leq \frac{1}{(1 - a)|\eta|} |\eta|^{-a} \leq \frac{3^a}{1 - a} (\mu + |\zeta| + |\eta|)^{-a}.
\]
Finally, we consider the case \( |\eta| \leq |\zeta| \) and \( |\zeta| > \mu \). Here we estimate
\[
\int_0^1 (\mu + |\zeta + t\eta|)^{-a} dt \leq \int_0^1 (\mu + (1 - t)|\zeta|)^{-a} dt
\]
\[
= \frac{1}{(1 - a)|\zeta|} (\mu + |\zeta|)^{1-a} - \mu
\]
\[
\leq \frac{1}{(1 - a)|\zeta|} |\zeta|^{-a} \leq \frac{3^a}{1 - a} (\mu + |\zeta| + |\eta|)^{-a}.
\]

**Lemma 2.2.** Let \( \phi : [a, b] \to [0, +\infty[ (-\infty < a < b < +\infty) \) be bounded. Assume that there are constants \( A, B, a \) and \( 0 < \varepsilon < 1 \), such that
\begin{equation}
(2.2) \quad \phi(\rho) \leq A(R - \rho)^{-a} + B + \varepsilon \phi(R) \quad \forall a \leq \rho \leq R \leq b.
\end{equation}
Then there exists a positive constant \( c = c(a, \varepsilon) \) such that
\begin{equation}
(2.3) \quad \dot{\phi}(\rho) \leq c (A(R - \rho)^{-a} + B) \quad \forall a \leq \rho \leq R \leq b.
\end{equation}
For the proof of Lemma 2.3 see for instance [6, chap. III]. \( \square \)

3. – Caccioppoli-type inequalities.

The aim of this section is the proof of two Caccioppoli inequalities, which play an essential role for the proof of the Theorem. To begin with, for \( \mu \geq 0 \) we define
\[
\begin{align*}
V_\mu(\xi) &:= (\mu + ||\xi||)^{(q-2)/2} \quad \text{for} \quad \xi \in \mathbf{M}^1_{\text{sym}} \setminus \{0\} \\
V_\mu(0) &:= \mu^{(q-2)/2} \quad \text{if} \quad \mu > 0, \\
V_\mu(0) &:= 0 \quad \text{if} \quad \mu = 0.
\end{align*}
\]
Theorem 3.1. – Let \( u \in [W^{1, q}(\Omega)]^2 \) be a weak solution to the equations (1.1), (1.2). Let all assumptions of the Theorem be fulfilled. Then

\[
V_p(\|D(u)\|) \frac{\partial^2 u_i}{\partial x_k \partial x_l} \in L^2_{\text{loc}}(\Omega) \quad (i, k, l = 1, 2)
\]

and for each \( x_0 \in \Omega, 0 < r < \text{dist}(x_0, \partial \Omega) \) and \( \lambda \in \mathbb{R}^2 \times \mathbb{R}^2 (\lambda^+ \neq 0) \) there holds

\[
\sum_{k=1}^{2} \int_{B_{r/2}} V_p^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 dx \leq \frac{c}{r^2} \left( \mu + \|\lambda^+\|^{\frac{q_0}{q_0 - 2}} \int_{B_r} \|D(u)\|^q dx \right)^{\frac{2-q}{q_0-2}} \int_{B_r} |\nabla u - \lambda|^2 dx \quad (4)
\]

\[
+ c \int_{B_r} \left( 1 + \|D(u)\|^{2 - \frac{q_0 - 2}{q_0 - 2}} + |A u|^{(2 + \sigma)/\sigma} + |f|^{(2 + \sigma)/2} \right) dx,
\]

where

\[
\lambda_{ij}^+ := \frac{1}{2}(\lambda_{ij} + \lambda_{ji}) \quad (i, j = 1, 2),
\]

and \( c = \text{const} > 0 \) depending on \( c_0/v_0 \) and \( q \) only.

To prove Theorem 3.1 we will make essential use of the following two lemmas, which can be proved by an elementary calculus.

Lemma 3.1. – For each function \( w \in [W^{2, 1}(\Omega)]^2 \) there holds

\[
\left| \frac{\partial^2 w_i}{\partial x_k \partial x_l} \right| \leq \sum_{j=1}^{2} \left\| \frac{\partial}{\partial x_j} D(w) \right\| \quad \text{a.e. in} \quad \Omega \quad (i, k, l = 1, 2).
\]

Proof. – Assertion (3.3) immediately follows from the the following equations:

\[
\frac{\partial^2 w_1}{\partial x_1^2} = \frac{\partial}{\partial x_1} D_{11}(w), \quad \frac{\partial^2 w_1}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2} D_{11}(w),
\]

\[
\frac{\partial^2 w_2}{\partial x_2^2} = \frac{\partial}{\partial x_2} D_{22}(w), \quad \frac{\partial^2 w_2}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1} D_{22}(w),
\]

\((4)\) For \( A \subset \mathbb{R}^2 \), \( A \) being Lebesgue measurable and \( \varphi \in L^1(A) \) define.

\[
\int_A \varphi \, dx = \varphi_A = \frac{1}{\text{mes}(A)} \int_A \varphi \, dx.
\]
\[
\frac{\partial^2 w_1}{\partial x_2^2} = \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_2^2} \frac{\partial^2 w_2}{\partial x_2^2} - \frac{\partial^2 w_2}{\partial x_2^2} = 2 \frac{\partial}{\partial x_2} D_{12}(w) - \frac{\partial}{\partial x_1} D_{12}(w),
\]

\[
\frac{\partial^2 w_2}{\partial x_1^2} = \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_1^2} \frac{\partial^2 w_1}{\partial x_1^2} - \frac{\partial^2 w_1}{\partial x_1^2} = 2 \frac{\partial}{\partial x_1} D_{12}(w) - \frac{\partial}{\partial x_2} D_{11}(w).
\]

Lemma 3.2. There exists a positive constant \(c\), such that for any given ball \(B_R = B_R(x_0) \subset \mathbb{R}^2\) we have

\[
\left\{ \begin{array}{l}
\text{for every } v \in W^{1,2}(B_R(x_0)); \\
\left( \int_{B_R} |v|^q \, dx \right)^{2/q} \leq c \varepsilon^{\frac{q-2}{2}} \int_{B_R} |v|^2 \, dx + \varepsilon \int_{B_R} |\nabla v|^2 \, dx \quad \forall \varepsilon > 0.
\end{array} \right.
\]

Proof. 1) First, let us consider the case \(x_0 = 0\) and \(R = 1\). By the aid of Sobolev's embedding theorem we find

\[
\|v\|_{L^\infty(B_1)} \leq c(q) \left( \|v\|_{L^2(B_1)}^2 + \|
abla v\|_{(L^2(B_1))^2}^2 \right)^{1/2} \quad \forall v \in W^{1,2}(B_1).
\]

Set \(\theta := 2 - q\). Then \(0 < \theta < 1\) and

\[
\frac{1 - \theta}{2} + \frac{\theta}{2q'} = \frac{1}{q'}.
\]

Thus, by interpolation, making use of (3.5) and Young's inequality applied with \(\varepsilon > 0\) arbitrarily chosen gives

\[
\|v\|_{L^q(B_1)}^2 \leq \|v\|_{L^2(B_1)}^{2(1-\theta)} \|v\|_{L^2(B_1)}^{2\theta} \leq \|v\|_{L^2(B_1)}^{2(1-\theta)} \left( \|v\|_{L^2(B_1)}^2 + \|
abla v\|_{L^2(B_1)}^2 \right)^{\theta} \leq c \varepsilon^{\theta/(1-\theta)} \|v\|_{L^2(B_1)}^2 + \varepsilon \|
abla v\|_{L^2(B_1)}^2 \leq c \varepsilon^{(q-2)/(q-1)} \|v\|_{L^2(B_1)}^2 + \varepsilon \|
abla v\|_{L^2(B_1)}^2.
\]

2) Second, for any given ball \(B_R(x_0) \subset \mathbb{R}^2\) the estimate (3.4) easily follows from 1) using an elementary rescaling argument.

Proof of Theorem 3.1. The proof is divided into four steps.

1° Interior differentiability of the weak solution. Let \(\lambda \in \mathbb{R}^2 \times \mathbb{R}^2 (\lambda^+ \neq 0)\) be fixed. Taking into account (1.8) from [11] it follows that the second weak derivatives exist and

\[
(1 + \|D(u)\|)^{q-2/2} \frac{\partial}{\partial x_k} D(u) \in [L^2_{\text{loc}}(\Omega)]^4 \quad (k = 1, 2).
\]
Then by $\lambda^+ \neq 0$,

$$\left(\mu + \|D(u)\| + \|D(u) - \lambda^+\|\right)^{q^*-2} \frac{\partial}{\partial x_k} D(u) \in [L^2_{\text{loc}}(\Omega)]^4 \quad (k = 1, 2) \tag{*}.$$ 

Using the product- and chain rule gives

$$\frac{\partial}{\partial x_k} \left(\mu + \|D(u)\| + \|D(u) - \lambda^+\|\right)^{q^*-2} (D(u) - \lambda^+)$$

$$= (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q^*-2} \frac{\partial}{\partial x_k} D(u)$$

$$+ (q - 2) (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q^*-3} (D(u) - \lambda^+)$$

$$\times \left(\frac{\partial}{\partial x_k} D_{ij}(u) \left(\frac{D_{ij}(u)}{\|D(u)\|} + \frac{D_{ij}(u) - \lambda^+_{ij}}{\|D(u) - \lambda^+\|}\right)\right)$$

a.e. in $\Omega$. Hence

$$(3.6) \quad (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q^*-2} (D(u) - \lambda^+) \in [W^{1,2}_{\text{loc}}(\Omega)]^4,$$

and

$$(3.7) \quad \left\| \frac{\partial}{\partial x_k} \left(\mu + \|D(u)\| + \|D(u) - \lambda^+\|\right)^{q^*-2} (D(u) - \lambda^+) \right\|$$

$$\leq 2 (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q^*-2} \left\| \frac{\partial}{\partial x_k} D(u) \right\|$$

a.e. in $\Omega \ (k = 1, 2)$.

**2. Local existence of the pressure and interior pressure estimates.** Let $x_0 \in \Omega, 0 < R < \text{dist}(x_0, \partial \Omega)$ and $\lambda \in \mathbb{R}^2 \times \mathbb{R}^2$ with $\lambda^+ \neq 0$ be arbitrarily chosen. Consulting [5; Th. III 3.1, Th. III 5.2] one gets a pressure $\hat{p} \in L^q(B_R)/\mathbb{R}$, such that for any $p \in \hat{p}$,

$$(3.8) \quad \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) D_{ij}(\varphi) \, dx = \int_{B_R} f_j \varphi_j \, dx + \int_{B_R} p \, \text{div} \, \varphi \, dx$$

for all $\varphi \in [W^{1,q}_{0}(B_R)]^2$. In addition, observing

$$(q^*)' = \frac{q^*}{q^* - 1} = \frac{2q}{3q - 2} < 2$$

(*) Here we have made use of the inequality

$$\mu + \|D(u)\| + \|D(u) - \lambda^+\| \geq \frac{1}{2} (\|\lambda^+\| + \|D(u)\|).$$
it follows
\[
\|p - p_{B_R}\|_{L^{q'}(B_R)} \leq c \left( \sum_{i,j=1}^{2} \|s_{ij}(D(u)) - s_{ij}(\lambda^+)^{q-2}\|_{L^{q'}(B_R)} + \|\tilde{f}\|_{L^{q'(q-2)}(B_R)} \right) \forall p \in \hat{p} \quad (6).
\]

Then by the aid of (1.4) applying Lemma 2.1 and Hölder’s inequality gives
\[
\left( \int_{B_R} |p - p_{B_R}|^{q'} dx \right)^{2/q'} \leq c \left( \int_{B_R} [(\mu + \|D(u)\| + \|D(u) - \lambda^+\|^{q-2}\|D(u) - \lambda^+\|^{q-2}) dx \right)^{2/q'}
\]
\[
+ c R^{4/q} \int_{B_R} |\tilde{f}|^2 dx \quad \forall p \in \hat{p}
\]

(cf. [5], [11]). Taking into account (3.7), the first integral on the right of the last inequality may be estimated by (3.4) (with \(v = (\mu + \|D(u)\| + \|D(u) - \lambda^+\|^{q-2}(D_{ij}(u) - \lambda_{ij}^+)) (ij = 1, 2))). Thus for each \(\varepsilon > 0\),
\[
\left( \int_{B_R} |p - p_{B_R}|^{q'} dx \right)^{2/q'} \leq c \varepsilon^{2-q} \int_{B_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|^{2(q-2)}) dx
\]
\[
+ c \sum_{k=1}^{2} \int_{B_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|^{2(q-2)} \|\partial_{x_k} D(u)\|^2 dx
\]
\[
+ c \int_{B_R} |\tilde{f}|^2 dx.
\]

On the other hand again appealing to [5], [11] the following estimate is true for every \(p \in \hat{p}\)
\[
\|p - p_{B_R}\|_{L^2(B_R)} \leq c \left( \sum_{i,j=1}^{2} \|s_{ij}(D(u)) - s_{ij}(\lambda^+)^{q-2}\|_{L^2(B_R)} + R\|\tilde{f}\|_{L^2(B_R)} \right).
\]

(6) To this end, \(c\) denotes a constant which may change its numerical value from line to line, but depends neither on \(R\) nor on \(u\).
Then taking into account (1.4) we obtain

$$
\int_{B_R} (p - p_{B_R})^2 \, dx \\
\leq \, c \int_{B_R} \left( \mu + \|D(u)\| + \|D(u) - \lambda^+\|^2 + 2(\mu - \lambda^+)\|D(u) - \lambda^+\|^2 \right) \, dx
$$

(3.11)

$$
+ \, c R^2 \int_{B_R} |\tilde{f}|^2 \, dx.
$$

3. Let \(x_0 \in \Omega\) and \(0 < r < \frac{1}{2} \text{dist} (x_0, \partial \Omega)\) be fixed. Let \(\frac{r}{2} \leq \rho < R \leq r\) be arbitrarily chosen. Let \(\zeta \in C_c^\infty(B_R)\) be a cut-off function for the ball \(B_\rho\), i.e. \(0 \leq \zeta \leq 1\) in \(B_R\), \(\zeta = 1\) on \(B_\rho\), such that

$$
\left| \frac{\partial \zeta}{\partial x_i} \right| \leq \frac{c}{R - \rho}, \quad \left| \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \right| \leq \frac{c}{(R - \rho)^2} \quad \text{in} \quad B_R \quad (i, j = 1, 2)
$$

\((c = \text{const independent of } r)\). It is readily seen that

$$
\phi_j = \Delta_{k_- h}(\zeta^2((\Delta_{k_- h} u_j) - \lambda_{jk} h)) \quad (0 < h < r, j = 1, 2) \quad (7)
$$

is an admissible test function in (3.8). Then inserting \(\phi\) into (3.8) and applying the transformation formula of the Lebesgue integral gives

$$
\int_{B_R} (A_{k_- h} S_{ij}(D(u))) (A_{k_- h} D_{ij}(u)) \zeta^2 \, dx
$$

$$
= - \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) \Delta_{k_- h} \left( \zeta \frac{\partial \zeta}{\partial x_i} ((\Delta_{k_- h} u_j) - \lambda_{jk} h) \right) \, dx
$$

$$
+ \zeta \frac{\partial \zeta}{\partial x_j} ((\Delta_{k_- h} u_i) - \lambda_{ik} h) \right) \, dx
$$

$$
+ \int_{B_R} \tilde{f}_i A_{k_- h} (\zeta^2 ((\Delta_{k_- h} u_i) - \lambda_{ik} h)) \, dx \quad (8)
$$

(3.12)

for all \(p \in \hat{p}\).

(7) Here \(A_{\lambda, h} v\) denotes the difference \(v(\cdot + \lambda e_k) - v\), where \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\).

(8) Notice that \(\frac{\partial u_i}{\partial x_i} = 0\).
To proceed we first note that

\begin{equation}
V_{\mu} \frac{\partial}{\partial x_k} D(u) \in [L^2_{\text{loc}}(\Omega)]^d \quad (k = 1, 2),
\end{equation}

and

\begin{equation}
v_0 \sum_{k=1}^2 \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^{\frac{2}{\mu}} \zeta^2 \, dx
\leq 2 \liminf_{h \to 0} \int_{B_R} (A_{k,h}S_{ij}(D(u)))(A_{k,h}D_{ij}(u))\zeta^2 \, dx.
\end{equation}

Indeed in case \( \mu > 0 \) (3.13) and (3.14) can be proved by the aid of (1.5) using a similar reasoning as in [13]), whereas in the case \( \mu = 0 \) (3.13) and (3.14) will be verified in the appendix below.

Then in (3.12) letting \( h \) tend to zero, using chain- and product rule gives

\begin{equation}
v_0 \sum_{k=1}^2 \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^{\frac{2}{\mu}} \zeta^2 \, dx
\leq 2 \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) \left( \frac{\partial u_{ij}}{\partial x_k} - \lambda_{jk} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \, dx
+ 2 \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) \frac{\partial^2 u_{ij}}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, dx
+ 2 \int_{B_R} p \left( \frac{\partial u_{ij}}{\partial x_k} - \lambda_{jk} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \, dx
+ 2 \int_{B_R} p \frac{\partial^2 u_{ij}}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, dx
+ 2 \int_{B_R} f_i \left( \frac{\partial u_{ij}}{\partial x_k} - \lambda_{jk} \right) \zeta \frac{\partial \zeta}{\partial x_k} \, dx + 2 \int_{B_R} f_i \frac{\partial^2 u_{ij}}{\partial x_k \partial x_k} \zeta^2 \, dx.
\end{equation}

where \( p \in \hat{p} \) is taken such that \( p_{B_R} = 0 \).

In order to estimate integrals \( I_1, I_2, I_3 \) and \( I_4 \) we will make extensively use of the following inequality

\begin{equation}
\mu + \|D(u)\| + \|D(u) - \lambda^+\| \geq \mu + \|\lambda^+\| \quad \text{a.e. in} \quad \Omega.
\end{equation}

1) First, observing (1.4) the estimation of integral \( I_1 \) can be easily done by the
aid of Lemma 2.1. This yields

\[ I_1 \leq \frac{c}{(R - \rho)^2} \int_{{B}_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2} |\nabla u - \lambda|^2 \, dx. \]

Then using (3.16) gives

\[ I_1 \leq \frac{c}{(R - \rho)^2} (\mu + \|\lambda^+\|)^{q-2} \int_{{B}_r} |\nabla u - \lambda|^2 \, dx. \tag{3.17} \]

2) Analogously as above, using Lemma 3.1 and then applying Young’s inequality we obtain

\[ I_2 \leq \frac{v_0}{4} \sum_{k=1}^{2} \int_{{B}_R} \mathcal{V}_\mu \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \zeta^2 \, dx \]

\[ + \int_{{B}_R} (\mu + \|\lambda^+\|)^{q-2} \int_{{B}_r} \|\nabla u - \lambda\|^2 \, dx. \tag{3.18} \]

3) To estimate integral \( I_3 \) we apply Cauchy Schwarz’s inequality and make use of the estimate (3.11). It follows

\[ I_3 \leq \frac{c}{(R - \rho)^2} \left( \int_{{B}_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{2(q-2)} |\nabla u - \lambda|^2 \, dx \right. \]

\[ + \left. R^2 \int_{{B}_R} (\mathcal{V}_\mu) \, dx \right)^{1/2} \left( \int_{{B}_R} |\nabla u - \lambda|^2 \, dx \right)^{1/2}. \]

In addition, applying Young’s inequality together with (3.16) shows that

\[ I_3 \leq \frac{c}{(R - \rho)^2} \left\{ (\mu + \|\lambda^+\|)^{q-2} \int_{{B}_r} |\nabla u - \lambda|^2 \, dx + \rho^2 \int_{{B}_r} (1 + |\mathcal{V}_\mu|^{2+\sigma/2}) \, dx \right\}. \tag{3.19} \]

4) In order to estimate \( I_4 \) we first apply Hölder’s and Young’s inequality and then making use of (3.10) (with \( \varepsilon > 0 \) arbitrarily chosen). Hence, together with (3.16) one obtains

\[ I_4 \leq \frac{c}{R - \rho} \left( \int_{{B}_R} |\mathcal{V}_\mu|^q \, dx \right)^{1/q} \left( \sum_{k=1}^{2} \int_{{B}_R} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^q \, dx \right)^{1/q} \]

\[ \leq \frac{v_0}{4} \sum_{k=1}^{2} \int_{{B}_R} \mathcal{V}_\mu \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \zeta^2 \, dx \]
+ \frac{c \rho^2}{(R - \rho)^2} \left( \int_{B_R} |p|^{q'} \, dx \right)^{2/q'} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(2-q)/q} \tag{9}

\leq \frac{v_0}{4} \sum_{k=1}^{2} \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \frac{\zeta^2}{x} \, dx 

+ \frac{c \rho^2}{(R - \rho)^2} \left\{ \frac{\xi^{2-q}}{4} (\mu + \|\lambda^+\|)^{(2-q)/q} \int_{B_R} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(2-q)/q}

\times \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(q-2)/q}

\right\}.

Thus, choosing \( \varepsilon := v_0 \frac{(R - \rho)^2}{4c \rho^2} (\mu + \|\lambda^+\|)^{(2-q)/q} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(q-2)/q} \)

one arrives at

\begin{align*}
I_4 & \leq \frac{v_0}{4} \sum_{k=1}^{2} \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \frac{\zeta^2}{x} \, dx 

& \quad + \frac{c \xi^{2-q}}{(R - \rho)^2/(q-1)} (\mu + \|\lambda^+\|)^{(q-2)/q} \int_{B_R} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(q-2)/q}

\times \int_{B_R} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(q-2)/q}

& \quad + \frac{c \xi^2}{(R - \rho)^2} \int_{B_R} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(2-q)/q}

& \quad + \frac{v_0}{4} \sum_{k=1}^{2} \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx.
\end{align*}

\begin{equation}
(3.20)
\end{equation}

\(^{(\ast)}\) Observing \( \nabla D u = 0 \) almost everywhere on \( \{ y \in \Omega | D(u(y)) = 0 \} \) by Hölder's inequality,

\begin{align*}
\int_{B_R} \left\| \frac{\partial}{\partial x_1} D(u) \right\| \, dx &= \int_{B_R} (\mu + \|D(u)\|^{q/2})^{q/2} \frac{V^2}{\xi^q} \left\| \frac{\partial}{\partial x_1} D(u) \right\| \, dx 

& \leq \left( \int_{B_R} V^2 \left\| \frac{\partial}{\partial x_1} D(u) \right\|^q \, dx \right)^{q/2} \left( \int_{B_R} (\mu + \|D(u)\|^q \, dx \right)^{(2-q)/2}.
\end{align*}
5) Using Cauchy Schwarz’s inequality and Young’s inequality we get

\[ I_5 \leq \frac{c r^2}{(R - \rho)} \int_{B_r} |\nabla u - \lambda|^2 \, dx + c \int_{B_r} (1 + |\tilde{f}|^{(2+\sigma)/2}) \, dx. \]  

6) Finally, by the aid of Hölder’s inequality and Young’s inequality it follows that

\[ I_6 \leq c \int_{B_r} (1 + |\Delta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2}) \, dx. \]  

Now, inserting the estimates (3.17)-(3.22) into (3.15) and taking into account the estimate

\[
\int_{B_r} |\tilde{f}|^2 \, dx \left( \int_{B_r} (\mu + \|D(u)\|^q) \, dx \right)^{(2-q)/q} \\
\leq 2 \left( \int_{B_r} (\mu + \|D(u)\|^{(2-q)\lambda_+^q/2}) \, dx \right)^{(\sigma-2)/(\sigma+2)} \left( \int_{B_r} |\tilde{f}|^{(\sigma+2)/\sigma} \, dx \right)^{4/(\sigma+2)} \\
\leq c \int_{B_r} \left( 1 + \|D(u)\|^{(2-q)\lambda_+^q/2} + |\tilde{f}|^{(\sigma+2)/\sigma} \right) \, dx,
\]

which easily follows by the aid of Hölder’s and Young’s inequality, implies

\[
\sum_{k=1}^{2} \int_{B_r} V_{\mu}^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx \\
\leq \frac{c r^{2(2-q)/q}}{(R - \rho)^{2(2-q)/4}} \left( \mu + \|\lambda^+\|^{q'(q-2)} \left( \int_{B_r} (\mu + \|D(u)\|^q) \, dx \right)^{\frac{2-q}{q'(q-2)}} \right) \\
\times \int_{B_r} \|D(u) - \lambda^+\|^2 \, dx \\
+ \frac{c r^{2/(q-1)}}{(R - \rho)^{2/(q-1)}} \int_{B_r} \left( 1 + \|D(u)\|^{(2-q)\lambda_+^q/2} + \|\Delta u\|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \right) \, dx \\
+ \frac{1}{2} \sum_{k=1}^{2} \int_{B_r} V_{\mu}^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx,
\]

where \(c = \text{const} > 0\) depends on \(c_0/v_0\) and \(q\) only. Now the assertion (3.2) immediately follows from the last estimate together with Lemma 2.2. \(\square\)

Using a similar reasoning which let to (3.2) we have the following alternative Caccioppoli inequality
THEOREM 3.2. – Let \( u \in [W^{1,q}(\Omega)]^2 \) be a weak solution to the equations (1.1), (1.2). Let all assumptions of the Theorem be fulfilled. Then for each \( x_0 \in \Omega, 0 < r < \text{dist} (x_0, \partial \Omega) \) and \( \lambda \in \mathbb{R}^2 \times \mathbb{R}^2 \) there holds

\[
\sum_{k=1}^{2} \int_{B_r} V^2_{\mu} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx \\
\leq c \left\{ \int_{B_r} |\nabla u - \lambda|^2 \, dx + \int_{B_r} (\mu + \|D(u)\|^2)^{q/2} \, dx \right\}^{q/2} \\
+ c \left( \int_{B_r} |f|^2 \, dx \right)^{q/2} + c \int_{B_r} \left( 1 + |D(u)|^{(2+\sigma)/\sigma} + |f|^{(2+\sigma)^2} \right) \, dx,
\]

(3.23)

where \( c = \text{const} > 0 \) depending on \( c_0/v_0 \) and \( q \) only.

PROOF. – Let \( x_0 \in \Omega \) and \( 0 < r < \frac{1}{2} \text{dist} (x_0, \partial \Omega) \) be fixed. Let \( \zeta \in C_c^\infty (B_r) \) be a cut-off function for the ball \( B_r \), i.e. \( 0 \leq \zeta \leq 1 \) in \( B_r, \zeta \equiv 1 \) on \( B_{r/2} \), such that

\[
\left| \frac{\partial \zeta}{\partial x_i} \right| \leq \frac{c}{r}, \quad \left| \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \right| \leq \frac{c}{r^2} \quad \text{in} \quad B_r \quad (i,j = 1,2)
\]

(\( c = \text{const} \) independent of \( r \)). As in the proof of Theorem 3.1 we insert the admissible test function \( \varphi_j = \Delta_k, h (\zeta^2 (\Delta_k, h u_j) - \lambda_{jk}, h) \) \((0 < h < r, j = 1,2)\) into (3.8) applying the transformation formula of the Lebesgue integral and passing to the limit \( h \to 0 \) (cf. also the appendix below). This yields

\[
v_0 \sum_{k=1}^{2} \int_{B_r} V^2_{\mu} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \zeta^2 \, dx \\
\leq 2 \int_{B_r} (S_{ij}(D(u)) - S_{ij}(0)) \left( \frac{\partial u_j}{\partial x_k} - \lambda_{jk} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \, dx \\
+ 2 \int_{B_r} (S_{ij}(D(u)) - S_{ij}(0)) \frac{\partial^2 u_j}{\partial x_k \partial x_i} \zeta \frac{\partial \zeta}{\partial x_k} \, dx \\
+ 2 \int_{B_r} p \left( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \, dx \\
+ 2 \int_{B_r} p \frac{\partial^2 u_i}{\partial x_k \partial x_i} \zeta \frac{\partial \zeta}{\partial x_k} \, dx \\
+ 2 \int_{B_r} \tilde{f}_i \left( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \right) \zeta \frac{\partial \zeta}{\partial x_k} \, dx + 2 \int_{B_R} \tilde{f}_i \frac{\partial^2 u_i}{\partial x_k \partial x_i} \zeta^2 \, dx \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

(3.24)
1) Observing (1.4) and applying Hölder’s inequality we get

\[ I_1 \leq \frac{c}{r^2} \int_{B_r} \left( \mu + \|D(u)\|^q \right) \left( \int_{B_r} |\nabla u - \lambda|^q \, dx \right)^{1/q} \left( \int_{B_r} |\nabla u - \lambda|^q \, dx \right)^{1/q} \]

(3.25)

\[ \leq c \left\{ \int_{B_r} |\nabla u - \lambda|^2 \, dx + \int_{B_r} (\mu + \|D(u)\|^2) \, dx \right\}^{q/2}. \]

2) Similarly, making use of Hölder’s inequality (cf. footnote (3)) and Young’s inequality gives

\[ I_2 \leq \frac{c}{r^2} \left( \int_{B_r} (\mu + \|D(u)\|^q) \, dx \right)^{1/q} \left( \sum_{k=1}^2 \int_{B_r} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^q \, dx \right)^{1/q} \]

(3.26)

\[ \leq \frac{v_0}{4} \int_{B_r} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx + c \left( \int_{B_r} (\mu + \|D(u)\|^2) \, dx \right)^{q/2}. \]

3) In order to estimate \( I_3 \) we apply Hölder’s inequality and make use of (3.9) (with \( \lambda^+ = 0 \)), it follows

\[ I_3 \leq c \left( \int_{B_r} |p|^q \, dx \right)^{1/q} \left( \int_{B_r} |\nabla u - \lambda|^q \, dx \right)^{1/q} \]

\[ \leq c \left\{ \left( \int_{B_r} (\mu + \|D(u)\|^q) \, dx \right)^{1/q} + \left( \int_{B_r} |\tilde{r}|^2 \, dx \right)^{1/2} \right\} \]

\[ \times \left( \int_{B_r} |\nabla u - \lambda|^q \, dx \right)^{1/q}. \]

Then applying Young’s inequality gives

(3.27) \[ I_3 \leq c \left\{ \int_{B_r} |\nabla u - \lambda|^2 \, dx + \int_{B_r} (\mu + \|D(u)\|^2) \, dx \right\}^{q/2} + c \left( \int_{B_r} |\tilde{r}|^2 \, dx \right)^{q/2}. \]

4) Applying again Hölder’s and Young’s inequality we obtain

\[ I_4 \leq \frac{v_0}{4} \sum_{k=1}^2 \int_{B_r} \left\| \frac{\partial}{\partial x_k} D(u) \right\|^2 \, dx \]

\[ + c \left( \int_{B_r} |p|^q \, dx \right)^{2/q} \left( \int_{B_r} (\mu + \|D(u)\|^2) \, dx \right)^{(2-q)/q}. \]
Then estimating the right of the latter inequality from above by (3.9) (with \( \lambda^+ = 0 \)) and then applying Hölder’s and Young’s inequality gives

\[
I_4 \leq \frac{t_0}{4} \sum_{k=1}^{2} \int_{B_r} V^2 \left\| \frac{\partial}{\partial x_k} D(u) \right\|_{\zeta^2}^2 \, dx
\]

(3.28)

\[
+ c \left( \int_{B_r} (\mu + \|D(u)\|)^2 \, dx \right)^{q/2} + c \left( \int_{B_r} |\tilde{f}|^2 \, dx \right)^{q'/2}
\]

5) The integrals \( I_5, I_6 \) may be estimated in the same way as in Theorem 3.1. Thus

\[
I_5 + I_6 \leq c \left( \int_{B_r} |\nabla u - \lambda|^2 \, dx \right)^{q/2} + c \left( \int_{B_r} |\tilde{f}|^2 \, dx \right)^{q'/2}
\]

(3.29)

\[
+ c \int_{B_r} (1 + |\Delta u|^{(\sigma+2)/\sigma} + |\tilde{f}|^{(\sigma+2)/2}) \, dx.
\]

Now, inserting (3.25)-(3.29) into (3.24) gives (3.22). \( \square \)

4. – Proof of the Main Result.

By means of Sobolev’s embedding theorem to prove the Theorem it suffices to obtain the higher integrability of second order derivative of \( u \). This can be achieved with the help the result of Giaquinta and Modica (cf. [7]) based on Gehring’s lemma [8] after having established appropriate “reverse Hölder inequalities”. For, we consider the following two cases.

**First case:** \( \mu + \|D(u)_{B_r}\| > \left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2} \)

Making use of triangular inequality we obtain

\[
\left( \int_{B_r} (\mu + \|D(u)\|)^2 \, dx \right)^{1/2} \leq \mu + \|D(u)_{B_r}\| + \left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2}
\]

\[
\leq 2(\mu + \|D(u)_{B_r}\|).
\]

Next, from (3.2) (with \( \lambda := (\nabla u)_{B_r} \)) applying Hölder’s inequality and making use of (4.1) it follows
\[
\int_{B_{r/2}} V_{\mu}^2 \| \nabla D(u) \|^2 \, dx \\
\leq \frac{c}{r^2} (\mu + \| (D(u))_{B_r} \|^2)^{\frac{\gamma(q-2)}{q-1}} \left( \int_{B_r} (\mu + \| D(u) \|)^2 \, dx \right)^{\frac{2-q}{2(q-1)}} \\
\times \int_{B_r} | \nabla u - (\nabla u)_{B_r} |^2 \, dx \\
+ c \int_{B_r} \left( 1 + \| D(u) \| (2-q)^{\frac{\gamma(q-2)}{q-2}} + |Au|^{(\sigma+2)/\sigma} + |f|^{(\sigma+2)/2} \right) \, dx \\
\leq c (\mu + \| (D(u))_{B_r} \|)^{\gamma-2} \int_{B_r} | \nabla u - (\nabla u)_{B_r} |^2 \, dx + c \int_{B_r} \tilde{g} \, dx,
\]

where

\[
\tilde{g}(x) := 1 + \| D(u(x)) \|^{\frac{\gamma(q-2)}{2}} + |Au(x)|^{(\sigma+2)/\sigma} + |f|^{(\sigma+2)/2}, \quad x \in \Omega.
\]

To proceed we first note that

\[
\int_{B_r} | \nabla u - (\nabla u)_{B_r} |^2 \, dx \leq c r^2 \left( \int_{B_r} \| \nabla D(u) \| \, dx \right)^2,
\]

which easily follows by the aid of Poincaré’s inequality together with (3.3).

Then making use of Hölder’s inequality and (4.1) gives

\[
\int_{B_r} | \nabla u - (\nabla u)_{B_r} |^2 \, dx \\
\leq c r^2 \left( \int_{B_r} (\mu + \| D(u) \|)^{2-q}/2 V_{\mu} \| \nabla D(u) \| \, dx \right)^2 \\
\leq c r^2 \left( \int_{B_r} (\mu + \| D(u) \|) \, dx \right)^{2-q} \left( \int_{B_r} [V_{\mu} \| \nabla D(u) \|]^{2/q} \, dx \right)^q \\
\leq c r^2 (\mu + \| (D(u))_{B_r} \|)^{2-q} \left( \int_{B_r} [V_{\mu} \| \nabla D(u) \|]^{2/q} \, dx \right)^q.
\]

Thus inserting the latter inequality into (4.2) gives

\[
\int_{B_{r/2}} V_{\mu}^2 \| \nabla D(u) \|^2 \, dx \\
\leq c \left( \int_{B_r} [V_{\mu} \| \nabla D(u) \|]^{2/q} \, dx \right)^q + c \int_{B_r} \tilde{g} \, dx
\]

(4.3)
for all \( x_0 \in \Omega \) and \( 0 < r < \frac{1}{2} \text{dist}(x_0, \partial \Omega) \), where \( c = \text{const} > 0 \) depending only on \( c_0/v_0, q \).

**Second case**: 
\[
\mu + \| (D(u))_{B_r} \| \leq \left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2}.
\]

First using the triangular inequality we easily get
\[
\left( \int_{B_r} (\mu + \| D(u) \|)^2 \, dx \right)^{1/2} \leq 2 \left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2}.
\]

Then (3.22) (with \( \lambda = (\nabla u)_{B_r} \)) reads
\[
\int_{B_{r/2}} V^2 \| \nabla D(u) \|^2 \, dx \leq c \left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2}
\]
\[
+ c \left( \int_{B_r} |\tilde{f}\|^2 \, dx \right)^{q/2} + c \int_{B_r} \left( 1 + |A u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \right) \, dx.
\]

As above using the Poincaré inequality and Hölder's inequality gives
\[
\left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2} \leq c r^q \left( \int_{B_r} (\mu + \| D(u) \|)^2 \, dx \right)^{2-q/2} \int_{B_r} (V \| \nabla D(u) \|)^{2/q} \, dx.
\]

Then taking into account (4.4) applying Young’s inequality we get
\[
\left( \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2} \leq c r^2 \left( \int_{B_r} (V \| \nabla D(u) \|)^{2/q} \, dx \right)^q.
\]

Now, inserting this inequality into (4.5) yields
\[
\int_{B_r} [V \| \nabla D(u) \|]^{2/q} \, dx
\]
\[
\leq c \left( \int_{B_r} [V \| \nabla D(u) \|]^{2/q} \, dx \right)^q
\]
\[
+ c(1 + (o(r; x_0)) \int_{B_r} \left( 1 + |A u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \right) \, dx,
\]

where
\[
o(r; x_0) := \left( \int_{B_r} |\tilde{f}|^2 \, dx \right)^{2/q - q/\sigma}.
\]
Let $\Omega' \subset \Omega$ be an open set with $\overline{\Omega'} \subset \Omega$. By the absolutely continuity of the Lebesgue integral there exists a number $0 < r_0 < \text{dist}(\Omega', \partial \Omega)$ such that

$$o(r; x_0) \leq 1 \quad \forall x_0 \in \Omega', \quad \forall 0 < r \leq r_0.$$ 

Then combining (4.3) and (4.6) gives

$$\int_{B_r} \left[ V_{\mu} \| \nabla D(u) \| \right]^2 dx \leq \left( \int_{B_r} \left[ V_{\mu} \| \nabla D(u) \| \right]^{2/q} dx \right)^q + c \int_{B_r} \tilde{g} dx,$$

for all $x_0 \in \Omega'$ and $0 < r \leq r_0$, where $c = \text{const} > 0$ depends only on $c_0, \nu_0, q$ and $\sigma$.

Now we are in a position to apply the following result of higher integrability due to M. Giaquinta and G. Modica (cf. [7]) based on Gehring’s lemma (cf. [8]).

**Lemma 4.1.** Let $F \in L^t_\text{loc}(\Omega)$ and $G \in L^d_\text{loc}(\Omega) \ (1 < t < d < +\infty)$ be given non-negative functions. Suppose there are constants $K_0 \geq 1$ and $\nu_0 > 0$ such that

$$\int_{B_r/2(x_0)} F^t dx \leq K_0 \left( \int_{B_r(x_0)} F^t dx \right) + \int_{B_r(x_0)} G^t dx$$

for each $x_0 \in \Omega, 0 < r < \min\{\nu_0, \text{dist}(x_0, \partial \Omega)\}$. Then there exists $t < \tau_0 \leq d$, such that

$$F \in L^\tau_\text{loc}(\Omega) \quad \forall \tau \in [1, \tau_0[.$$

\[\square\]

**Proof of Theorem 1.1 continued.** Applying Lemma 4.1 with

$$F := \left[ \sum_{k=1}^2 V_{\mu_k} \left\| \frac{\partial}{\partial x_k} D(u) \right\| \right]^{2/q}, \quad G := \tilde{g}^{1/q},$$

$$t := q, \quad d := \frac{2q\sigma}{\sigma + 2}$$

gives

$$\sum_{k=1}^2 V_{\mu_k} \left\| \frac{\partial}{\partial x_k} D(u) \right\| \in L^{\tau}(\Omega') \quad \text{for some} \ 2 < \tau < \frac{4\sigma}{\sigma + 2}.$$ 

Then by an analogous reasoning which led to (3.7) we obtain

$$V_{\mu_i} D(u) \in [W^{1, \tau}(\Omega')]^d,$$

and hence applying Sobolev’s embedding theorem it follows that $D(u)$ is bounded
on $\Omega'$. Using (3.3) we obtain
\[
\left| \sum_{k,l=1}^{2} \frac{\partial^2 u_i}{\partial x_k \partial x_l} \right| \leq 2 \left( \mu + \max_{\Omega} \| D(u) \| \right)^{(2-q)/2} V_{\mu} \left( \sum_{k=1}^{2} \left| \frac{\partial}{\partial x_k} D(u) \right| \right).
\]

Thus making use of Sobolev’s embedding theorem from (4.10) it follows
\[(4.11) \quad u |_{\partial \Omega} \in [C^{1, (r-2)/r}(\Omega')]^2 .\]

This completes the proof of the theorem. \(\Box\)

5. – Appendix.

This appendix is devoted to the proof of (3.13) and (3.14) for the special case $\mu = 0$.

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded open set. Let $N \geq 1$ denote an integer. For $v : \Omega \rightarrow \mathbb{R}^N$ Lebesgue measurable we define
\[
V(v)(x) := \begin{cases} |v(x)|^{(q-2)/2} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0, \end{cases}
\]
x $\in \Omega$, and given $\lambda > 0$ we set,
\[
V_{\lambda,k}(v)(x) := \begin{cases} (|v(x)| + |v(x + \lambda e_k)|)^{(q-2)/2} & \text{if } |v(x)| + |v(x + \lambda e_k)| > 0 \\ 0 & \text{if } |v(x)| + |v(x + \lambda e_k)| = 0, \end{cases}
\]
x $\in \Omega$, dist $(x, \partial \Omega) \geq \lambda$ ($k = 1, \ldots, d$).

**Lemma A.1** Let $v \in [L^q(\Omega)]^N$. Suppose for every $G \subset \subset \Omega$ there exists $K_G > 0$ such that
\[(A.1) \quad \int_G (V_{\lambda,k}(v))^2 |A_{\lambda,k}v|^2 \, dx \leq K_G \lambda^2 \quad \forall 0 < \lambda < \text{dist} (G, \partial \Omega) \quad (k = 1, \ldots, d) .
\]
Then $v \in [W^{1,q}_{\text{loc}}(\Omega)]^N$,
\[(A.2) \quad V(v) \frac{\partial v}{\partial x_k} \in [L^2_{\text{loc}}(\Omega)]^N \quad (k = 1, \ldots, d)
\]
and for every $G \subset \subset \Omega$,
\[(A.3) \quad \int_G (V(v))^2 \left( \frac{\partial v}{\partial x_k} \right)^2 \, dx \leq 2 \liminf_{\lambda \to 0} \int_G (V_{\lambda,k}(v))^2 \left( \frac{1}{\lambda} A_{\lambda,k}v \right)^2 \, dx \leq K_G \quad (k = 1, \ldots, d).\]
PROOF. – Let $G \subset \subset \Omega$ be fixed. As in [11] (cf. also [12], [13]) from (A.1) one easily deduces that $v \in [W^{1,q}(G)]^N$ and

$$\int_G \left( t + 2|v|^{q-2} \left| \frac{\partial v}{\partial x_k} \right|^2 \right) dx \leq \liminf_{\lambda \to 0} \int_G (V_{\lambda,k}(v))^2 \left| \frac{1}{\lambda} A_{\lambda,k} v \right|^2 dx \leq K_G$$

for every $t > 0 (k = 1, \ldots, d)$.

For $t > 0$ we set \( w_{k,t} := (t + 2|v|^{(q-2)/2} \frac{\partial v}{\partial x_k} ) (k = 1, \ldots, d) \). Then (A4) implies that \( \{ w_{k,t} \mid t > 0 \} \) is bounded in \([L^2(G)]^N\). By virtue of reflexivity there exists $w_k \in [L^2(G)]^N$ and a decreasing sequence $t_1 > t_2 > \cdots > t_m \to 0$ such that

$$w_{k,t_m} \to w_k \text{ weakly in } [L^2(G)]^N \text{ as } m \to +\infty.$$ 

In addition taking into account (A4) by the aid of Banach-Steinhaus’ theorem we get

$$\int_G |w_k|^2 dx \leq \liminf_{\lambda \to 0} \int_G (V_{\lambda,k}(v))^2 \left| \frac{1}{\lambda} A_{\lambda,k} v \right|^2.$$ 

Next for each $\varepsilon > 0$ define

$$G_\varepsilon := \{ y \in G \mid |v(y)| > \varepsilon \}.$$ 

Using Lebesgue’s theorem it is easily seen that

$$w_{k,t} \to 2|v|^{(q-2)/2} \frac{\partial v}{\partial x_k} \text{ in } [L^q(G_\varepsilon)]^N \text{ as } t \to 0.$$ 

Thus by (A5),

$$w_k = 2^{(q-2)/2}|v|^{(q-2)/2} \frac{\partial v}{\partial x_k} \text{ a.e. in } \{ y \in G \mid v(y) \neq 0 \}.$$ 

On the other hand, observing

$$\frac{\partial v}{\partial x_k} = 0 \text{ a.e. in } \{ y \in G \mid v(y) = 0 \}$$

making use of (A6) and (A7) it follows

$$\int_G (V(v))^2 \left| \frac{\partial v}{\partial x_k} \right|^2 dx = 2^{2-q} \int_{G \cap \{ v \neq 0 \}} |w_k|^2 dx \leq 2^{2-q} \liminf_{\lambda \to 0} \int_G (V_{\lambda,k}(v))^2 \left| \frac{1}{\lambda} A_{\lambda,k} v \right|^2 dx.$$ 

Whence (A2) and (A3).
Proof of (3.13) and (3.14). Based on (3.15) and the fact that $f \in u \cdot \nabla u \in [L^q_{\text{loc}}(\Omega)]^d$ as in [11] we infer

$$
\int_{B_r} (A_{k,k} S_{ij}(D(u))(A_{k,k} D_{ij}(u))) \, dx \leq K_r \lambda^2,
$$

for all $0 < \lambda < \frac{1}{2} \text{dist} (B_r, \partial \Omega)$ (here $K_r > 0$ is independent of $\lambda$).

By the aid of (1.5) with the notation introduced above setting $v := D(u)$ one finds

$$
v_0 |V_{\lambda,k}(v)A_{\lambda,k} v|^2 \leq (A_{\lambda,k} S_{ij}(D(u))(A_{\lambda,k} D_{ij}(u)) \quad \text{a.e. in } B_r.
$$

Then integrating both sides of (A.9) over $B_r$ using (A.8) yields

$$
v_0 \int_{B_r} (V_{\lambda,k}(v))^2 |A_{\lambda,k} v|^2 \, dx \leq K_r \lambda^2.
$$

Thus, the assumptions of Lemma A.1 are satisfied for $v := D(u)$. Now both (3.13) and (3.14) are an immediate consequence of Lemma A.1.

REFERENCES


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