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Fernando Cobos, Luz M. Fernández-Cabrera, Antonio Manzano, Antón Martínez

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Inclusion Indices of Quasi-Banach Spaces.

FERNANDO COBOS - LUZ M. FERNÁNDEZ-CABRERA -
ANTONIO MANZANO - ANTÓN MARTÍNEZ (*)

Sunto. – *Vengono studiati gli indici di inclusione per spazi quasi-Banach. Prima si considera il caso di spazi di funzioni su* $[0,1]$*, poi il caso degli spazi di successioni e, infine, si sviluppa un approccio astratto, usando indici definiti dalla scala degli spazi di interpolazione reale generata da una coppia quasi-Banach.*

Summary. – *We investigate inclusion indices for quasi-Banach spaces. First we consider the case of function spaces on* $[0,1]$*, then the sequence case and finally we develop an abstract approach dealing with indices defined by the real interpolation scale generated by a quasi-Banach couple.*

0. – Introduction.

Let E be a Banach space of Lebesgue-measurable functions on $[0,1]$, such that $L_\infty[0,1] \hookrightarrow E \hookrightarrow L_1[0,1]$. The inclusion indices of E are defined by $\delta_E = \sup\{p \geq 1 : E \hookrightarrow L_p[0,1]\}$ and $\gamma_E = \inf\{p \leq \infty : L_p[0,1] \hookrightarrow E\}$. Inclusion indices are an useful tool in the study of properties of embeddings between function spaces. In particular, they allow us to estimate the grade of proximity between function spaces $E \hookrightarrow F$ when $\delta_E = \gamma_E$, $\delta_F = \gamma_F$ and one has certain information on the inclusion from E into F (see [13], [9] and [11]).

If E is symmetric (that is, rearrangement invariant) and φ_E is its fundamental function, then it was proved in [13], p. 249, that

$$(0.1)\qquad \delta_E = \liminf_{t\to 0} \frac{\log t}{\log \varphi_E(t)} \qquad \text{and} \qquad \gamma_E = \limsup_{t\to 0} \frac{\log t}{\log \varphi_E(t)}.$$

Boyd indices and fundamental indices (see [2], [16], [17] and [18]) are different from inclusion indices.

It is natural to investigate the notion of inclusion indices by using the whole scale of L_p-spaces, that is $\{L_p[0,1]\}_{p>0}$, and not only the Banach part $1 \leq p \leq \infty$.

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Accordingly, we investigate in this paper such problem. The natural spaces to work with are now quasi-Banach spaces.

Besides the case of function spaces, we consider also the sequence case and abstract versions of these results using indices defined by the real interpolation scale generated by a quasi-Banach couple. In all these cases, we establish analytic formulae for computing the indices. In particular, we show that (0.1) still holds for symmetric quasi-Banach spaces. Note that the proof of (0.1) in the Banach case does not work in our setting because it is based on the well-known fact that any symmetric Banach space E lies between the Lorentz and the Marcinkiewicz spaces with fundamental function φ_E. As far as we are aware, no similar fact is known for symmetric quasi-Banach spaces (see [1] for a partial result in the p-Banach case).

Furthermore, we establish proximity results based on the equality of the indices. To use them, we need to know when the indices of a given space coincide. We pay special attention to this question, proving several criteria for equality.

The paper is organized as follows. In Section 1 we deal with function spaces on $[0,1]$. Section 2 is devoted to sequence spaces. Results of these two sections refer mainly to symmetric spaces. In Section 3 we develop the abstract approach to these results. When we specify the results to the former concrete cases, we can apply them to general spaces, not necessarily symmetric. The final Section 4 refers to a class of spaces defined by using bounded linear functionals: rank-one interpolation spaces. Because of this, we deal there with Banach spaces exclusively. We characterize equality of indices for this kind of spaces.

1. – Function spaces on $[0,1]$.

In this section we work on $[0,1]$ with the Lebesgue measure m. Let E be a quasi-Banach space of measurable functions on $[0,1]$. We say that E is a *lattice* if whenever $\|g(x)\| \leq \|f(x)\|$ a.e. with $f \in E$ and g measurable, then $g \in E$ and $\|g\|_E \leq \|f\|_E$. A quasi-Banach lattice E is called *symmetric* (or *rearrangement invariant*) if $\|\chi_{[0,1]}\|_E = 1$ and whenever $f \in E$ and g is equimeasurable with f, then $g \in E$ and $\|f\|_E = \|g\|_E$. In particular, if E is symmetric and $f \in E$, we have

$$\|f\|_E = \|f^*\|_E \,.$$

Here f^* is the non-increasing rearrangement of f

$$f^*(t) = \inf\{\sigma > 0 : \mathrm{m}\{x : |f(x)| > \sigma\} \leq t\}.$$

Any symmetric quasi-Banach space E satisfies that $L_\infty[0,1] \hookrightarrow E$, where the symbol $\hookrightarrow$ means continuous inclusion. Indeed

$$\|f\|_E = \|f^*\|_E \leq \| \|f\|_{L_\infty[0,1]} \chi_{[0,1]}\|_E = \|f\|_{L_\infty[0,1]}.$$

Moreover, it is not difficult to show that if $\{f_n\} \to f$ in E, then $\{f_n\}$ tends to f in measure.

If E is symmetric then the function $\varphi_E(t) = \|\chi_D\|_E$ where $D \subset [0,1]$ with $\mathrm{m}(D) = t$ is well-defined and it is called the *fundamental function* of E. Note that φ_E is non-decreasing so $\lim_{t \to 0} \varphi_E(t)$ exists. Using that f^* is a right-continuous function it is easy to see that

$$\lim_{t \to 0} \varphi_E(t) > 0 \qquad \text{if and only if} \qquad E = L_\infty[0,1]. \tag{1.1}$$

Let F be a quasi-Banach space of measurable functions on $[0,1]$. We define the *lower inclusion index* of F by

$$\delta_F = \sup\{0 < p < \infty : F \hookrightarrow L_p[0,1]\}.$$

If $F \not\hookrightarrow L_p[0,1]$ for any $0 < p < \infty$, then we put $\delta_F = 0$.

The *upper inclusion index* of F is defined by

$$\gamma_F = \inf\{0 < p < \infty : L_p[0,1] \hookrightarrow F\}.$$

If there is no $0 < p < \infty$ such that $L_p[0,1] \hookrightarrow F$, we put $\gamma_F = \infty$.

In the Banach case, it was shown in [13], p. 249, that the indices are related to the fundamental function. The proof is based on the well-known fact that working with symmetric Banach spaces, there are the smallest and the largest symmetric space with a given fundamental function (see [16], Thms. II.5.5 and II.5.7, or [2], Thm. II.5.13). Although there is no similar result in the quasi-Banach case, we show next that the indices can be still computed by using the fundamental function.

THEOREM 1.1. – *Let E be a symmetric quasi-Banach space on* $[0,1]$. *Then*

$$\delta_E = \liminf_{t \to 0} \frac{\log t}{\log \varphi_E(t)}.$$

PROOF. – Assume first that $\lim_{t \to 0} \varphi_E(t) > 0$. Then

$$\liminf_{t \to 0} \frac{\log t}{\log \varphi_E(t)} = \infty.$$

On the other hand, by (1.1), $E = L_\infty[0,1]$, so $\delta_E = \infty$.

Assume now that $\lim_{t \to 0} \varphi_E(t) = 0$. If there is some $0 < p < \infty$ such that $E \hookrightarrow L_p[0,1]$, then we can find $C > 0$ so that $Ct^{1/p} \leq \varphi_E(t)$ for all $0 < t \leq 1$. Taking logarithms and using that $\varphi_E(t) \leq \|\chi_{[0,1]}\|_E = 1$, we get

$$\frac{p \log C}{\log \varphi_E(t)} + \frac{\log t}{\log \varphi_E(t)} \geq p.$$

Hence

$$\liminf_{t\to 0} \frac{\log t}{\log \varphi_E(t)} \geq p. \tag{1.2}$$

It follows that

$$\delta_E \leq \liminf_{t\to 0} \frac{\log t}{\log \varphi_E(t)}.$$

If $\liminf\limits_{t\to 0}\,[\log t/\log \varphi_E(t)] = 0$, then (1.2) implies that there is no $0 < p < \infty$ such that $E \hookrightarrow L_p[0,1]$. Hence $\delta_E = 0$ and we obtain the desired equality. Let us establish the remaining case. Take any p with $0 < p < \liminf\limits_{t\to 0}\,[\log t/\log \varphi_E(t)]$. We should show that $E \hookrightarrow L_p[0,1]$. With this aim, choose q with $p < q < \liminf\limits_{t\to 0}\,[\log t/\log \varphi_E(t)]$. There is $0 < t_0 \leq 1$ such that $\log \varphi_E(t) > \log t^{1/q}$ for any $0 < t \leq t_0$. Let $C > 0$ so that

$$t^{1/q} \leq C\varphi_E(t) \quad \text{for every} \quad 0 < t \leq 1. \tag{1.3}$$

Using (1.3), for any $f \in E$ and any $0 < t \leq 1$, we have

$$\|f\|_E = \|f^*\|_E \geq \|f^*\chi_{[0,t]}\|_E \geq f^*(t)\varphi_E(t) \geq C^{-1}t^{1/q}f^*(t).$$

Therefore

$$\|f\|_{L_p[0,1]} = \left(\int_0^1 (f^*(t))^p dt\right)^{1/p} \leq C\|f\|_E \left(\int_0^1 t^{-p/q} dt\right)^{1/p} = C\left(\frac{q}{q-p}\right)^{1/p} \|f\|_E.$$

This yields that $E \hookrightarrow L_p[0,1]$ and completes the proof. $\square$

The corresponding result for the upper index reads as follows.

THEOREM 1.2. – *Let E be a symmetric quasi-Banach space on* $[0,1]$. *Then*

$$\gamma_E = \limsup_{t\to 0} \frac{\log t}{\log \varphi_E(t)}.$$

PROOF. – Let $0 < p < \infty$ with $L_p[0,1] \hookrightarrow E$ and let $C > 0$ be such that $\varphi_E(t) \leq Ct^{1/p}$ for any $0 < t \leq 1$. Then $\lim\limits_{t\to 0} \varphi_E(t) = 0$. Moreover, taking logarithms we get

$$\frac{p \log C}{\log \varphi_E(t)} + \frac{\log t}{\log \varphi_E(t)} \leq p.$$

This implies that $\limsup\limits_{t\to 0}\,[\log t/\log \varphi_E(t)] \leq p$ and therefore

$$\limsup_{t\to 0} \frac{\log t}{\log \varphi_E(t)} \leq \gamma_E.$$

If $\limsup_{t\to 0}[\log t/\log \varphi_E(t)] = \infty$, there is no $0 < p < \infty$ such that $L_p[0,1] \hookrightarrow E$. Whence $\gamma_E = \infty$ and equality holds. To establish the remaining case, take any $0 < p, q < \infty$ with

$$\limsup_{t\to 0} \frac{\log t}{\log \varphi_E(t)} < q < p < \infty. \tag{1.4}$$

We claim that the function $g(t) = t^{-1/p}$ belongs to E. Indeed, by (1.4), there is $0 < t_0 \le 1$ such that $\varphi_E(t) < t^{1/q}$ for any $0 < t \le t_0$. Choose $C > 0$ so that

$$\varphi_E(t) \le Ct^{1/q} \quad \text{for any} \quad 0 < t \le 1. \tag{1.5}$$

Put $g_n = g\chi_{[2^{-n},1]}$. Then $g_n \in E$ because $g_n \le g(2^{-n})\chi_{[0,1]}$. Let us show that $\{g_n\}$ is a Cauchy sequence in E.

Let c be the constant in the triangle inequality of E and define ρ by the equation $(2c)^\rho = 2$. Using [3], Lemma 3.10.1 and (1.5) we have for $n < m$

$$\|g_m - g_n\|_E^\rho = \left\| \sum_{j=n}^{m-1} g\chi_{[2^{-(j+1)},2^{-j}]} \right\|_E^\rho \le 2\sum_{j=n}^{m-1} g(2^{-(j+1)})^\rho \|\chi_{[2^{-(j+1)},2^{-j}]}\|_E^\rho \le$$

$$2C^\rho \sum_{j=n}^{m-1} 2^{(j+1)\rho\left(\frac{1}{p}-\frac{1}{q}\right)} \longrightarrow 0 \text{ as } n \to \infty.$$

This yields that $\{g_n\}$ is a Cauchy sequence in the complete space E. Its limit should be the function g because $\{g_n\}$ converges pointwise to g. So $g \in E$.

Now for any $f \in L_p[0,1]$ we obtain

$$\|f\|_E = \|f^*\|_E \le \|g\|_E \sup_{0<t\le 1}\left\{t^{1/p} f^*(t)\right\} \le \|g\|_E \|f\|_{L_p[0,1]}.$$

Consequently, $L_p[0,1] \hookrightarrow E$. This implies that

$$\gamma_E \le \limsup_{t\to 0} \frac{\log t}{\log \varphi_E(t)}$$

and ends the proof. □

Next we state two immediate consequences of these formulae.

COROLLARY 1.3. – *Let E be a symmetric quasi-Banach space on* $[0,1]$. *Then*

$$\delta_E = \gamma_E \quad \textit{if and only if} \quad \lim_{t\to 0} \frac{\log t}{\log \varphi_E(t)} \quad \textit{exists.}$$

COROLLARY 1.4. – *Let E be a symmetric quasi-Banach space on* $[0,1]$. *Assume that there is $0 < p < \infty$ such that for any $0 < \varepsilon < 1/p$, there are positive* *constants c_ε, C_ε so that*

$$c_\varepsilon\, t^{\frac{1}{p}+\varepsilon} \le \varphi_E(t) \le C_\varepsilon\, t^{\frac{1}{p}-\varepsilon} \quad \textit{for any} \quad 0<t\le 1. \tag{1.6}$$

Then $\delta_E = \gamma_E = p$.

The following result shows that if φ_E has regular variation at 0, then the indices are equal.

COROLLARY 1.5. – *Let E be a symmetric quasi-Banach space on $[0,1]$. If $\lim_{t\to 0}[\varphi_E(2t)/\varphi_E(t)]$ exists, then $\delta_E = \gamma_E$.*

PROOF. – Since $\varphi_E(t) \le \varphi_E(2t)$, we have

$$\lim_{t\to 0}\frac{\varphi_E(2t)}{\varphi_E(t)} = 2^\beta \quad \text{for some} \quad 0\le\beta<\infty.$$

Assume $0<\beta<\infty$ and take any $0<\varepsilon<\beta$. There is $0<t_0<1/2$ such that

$$2^{-(\beta+\varepsilon)}\varphi_E(2t) \le \varphi_E(t) \le 2^{-(\beta-\varepsilon)}\varphi_E(2t) \quad \text{for all} \quad 0<t\le t_0.$$

Given any $k\in\mathbb{N}$, if $2^{-(k+1)}t_0 \le t \le 2^{-k}t_0$, we get

$$2^{-(\beta+\varepsilon)(k+1)}\varphi_E(2^{k+1}t) \le \varphi_E(t) \le 2^{-(\beta-\varepsilon)(k+1)}\varphi_E(2^{k+1}t).$$

Since

$$t^{\beta+\varepsilon}(2t_0)^{-(\beta+\varepsilon)}\varphi_E(t_0) \le t^{\beta+\varepsilon}(t2^{k+1})^{-(\beta+\varepsilon)}\varphi_E(2^{k+1}t) = 2^{-(\beta+\varepsilon)(k+1)}\varphi_E(2^{k+1}t)$$

and

$$2^{-(\beta-\varepsilon)(k+1)}\varphi_E(2^{k+1}t) \le t^{\beta-\varepsilon}t_0^{-(\beta-\varepsilon)}\varphi_E(2t_0),$$

it follows that

$$t^{\beta+\varepsilon}[(2t_0)^{-(\beta+\varepsilon)}\varphi_E(t_0)] \le \varphi_E(t) \le t^{\beta-\varepsilon}[t_0^{-(\beta-\varepsilon)}\varphi_E(2t_0)].$$

Hence, we can find $C_\varepsilon>0$ such that

$$\frac{1}{C_\varepsilon}t^{\beta+\varepsilon} \le \varphi_E(t) \le C_\varepsilon t^{\beta-\varepsilon} \quad \text{for any} \quad 0<t\le 1.$$

This implies that $\delta_E = \gamma_E = 1/\beta$.

If $\beta=0$, a similar argument yields $\delta_E=\gamma_E=\infty$. □

We end this section with the extension of [9], Thm. 2.3, to the quasi-Banach case. Recall that a bounded linear operator $T\in\mathcal{L}(X,Y)$ from the quasi-Banach lattice X into the quasi-Banach space Y is said to be *disjointly strictly singular* if the restriction of T to any infinite-dimensional subspace generated by a sequence of disjoint vectors in X is not an isomorphism (see [12]). For example, if $0 < q < p < \infty$, the inclusion operator $L_p[0,1] \hookrightarrow L_q[0,1]$ is disjointly strictly singular.

We let $\mathcal{M}$ denote the space of all measurable functions on $[0,1]$ which are finite almost everywhere, endowed with the topology of convergence in measure.

THEOREM 1.6. – *Let E be a quasi-Banach function lattice and let F be a quasi-Banach function space with $L_\infty[0,1] \hookrightarrow E \hookrightarrow F \hookrightarrow \mathcal{M}$. Assume that $\delta_E = \gamma_E$ and $\delta_F = \gamma_F$. If the inclusion operator $E \hookrightarrow F$ is not disjointly strictly singular, then either:*

(i) $L_\infty[0,1] \subseteq E \subseteq F \subseteq \bigcap_{q<\infty} L_q[0,1]$ *or*

(ii) $\bigcup_{q>0} L_q[0,1] \subseteq E \subseteq F \subseteq \mathcal{M}$ *or*

(iii) $\bigcup_{q>p} L_q[0,1] \subseteq E \subseteq F \subseteq \bigcap_{q<p} L_q[0,1]$ *for some* $0 < p < \infty$.

PROOF. – Since $E \hookrightarrow F$, we have $\delta_F = \gamma_F \le \delta_E = \gamma_E$. The inequality cannot be strict, because if $\gamma_F < \delta_E$ then we could find $0 < p, q < \infty$ such that $\gamma_F < q < p < \delta_E$. Hence

$$E \hookrightarrow L_p[0,1] \hookrightarrow L_q[0,1] \hookrightarrow F$$

and the inclusion $E \hookrightarrow F$ would be disjointly strictly singular. Therefore, $\delta_F = \gamma_F = \delta_E = \gamma_E$.

Let $\delta_E = p$. If $p = \infty$ we get (i). If $p = 0$ we obtain (ii), and if $0 < p < \infty$ we derive (iii). $\square$

2. – Sequence spaces.

We now work on $\mathbb{N} = \{1, 2, \cdots\}$ with $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$. The non-increasing rearrangement of a sequence $\mu = \{\mu_n\} \in \ell_\infty$ is the sequence $\mu^* = \{s_n(\mu)\}$ defined by

$$s_n(\mu) = \inf\{\|\mu - \tau\|_{\ell_\infty} : \tau = \{\tau_m\} \in \ell_\infty, \ \mathrm{card}\{m \in \mathbb{N} : \tau_m \neq 0\} < n\}$$

(see [20]). Here $\mathrm{card}A$ designates the cardinality of the set A. Clearly, if $\xi \in c_0$ and $\{\xi_n^*\}$ is the rearrangement of the elements of $\{\xi_n\}$ by magnitude of the absolute values, $|\xi_1^*| \ge |\xi_2^*| \ge \cdots$, we have $s_n(\xi) = |\xi_n^*|$.

Given any subset $D \subseteq \mathbb{N}$, we put $e_D = \{\tau_n\}$ where $\tau_n = 1$ if $n \in D$ and $\tau_n = 0$ if $n \notin D$.

A quasi-Banach lattice of bounded sequences E is called *symmetric* (or *re-arrangement invariant*) if $\|e_{\{1\}}\|_E = 1$ and whenever $\xi \in E$ and $\mu \in \ell_\infty$ with $\xi^* = \mu^*$, then $\mu \in E$ and $\|\xi\|_E = \|\mu\|_E$.

If E is symmetric, we have $\mathfrak{f} \subseteq E$ where $\mathfrak{f}$ is the set of all sequences $\xi = \{\xi_n\}$ which have a finite number of coordinates $\xi_n \neq 0$. Moreover $E \hookrightarrow \ell_\infty$ (see [20], Prop. 13.2.4).

The fundamental function of the symmetric sequence space E is defined by

$$\varphi_E(n) = \|e_{\{1,\cdots,n\}}\|_E.$$

The function φ_E is non-decreasing with $\varphi_E(1) = 1$. Moreover,

$$\lim_{n\to\infty} \varphi_E(n) = c < \infty \quad \text{implies} \quad E = c_0 \text{ or } E = \ell_\infty.$$

Indeed, let $\xi = \{\xi_n\} \in c_0$ and write $\eta_n = \xi e_{\{1,\cdots,n\}}$. Then $\{\eta_n\}$ is a Cauchy sequence in E because for $m > n$

$$\|\eta_m - \eta_n\|_E \leq \max\{|\xi_j| : n+1 \leq j \leq m\}\varphi_E(m-n) \leq$$

$$c \max\{|\xi_j| : n+1 \leq j \leq m\} \longrightarrow 0 \text{ as } n \to \infty.$$

It follows that $\xi \in E$. So $c_0 \hookrightarrow E \hookrightarrow \ell_\infty$. Now [20], Thm. 13.1.8 yields that $E = c_0$ or $E = \ell_\infty$.

Let F be a quasi-Banach space of sequences. The *lower inclusion index* of F is defined by

$$\delta_F = \sup\{0 < p < \infty : \ell_p \hookrightarrow F\}.$$

If there is no $0 < p < \infty$ such that $\ell_p \hookrightarrow F$, we put $\delta_F = 0$.

We define the *upper inclusion index* of F by

$$\gamma_F = \inf\{0 < p < \infty : F \hookrightarrow \ell_p\}.$$

If $F \not\hookrightarrow \ell_p$ for any $0 < p < \infty$, then we write $\gamma_F = \infty$.

The techniques used in the proof of Theorems 1.1 and 1.2 may be modified to derive the following analytic formulae for computing the indices of a symmetric sequence space E in terms of its fundamental function φ_E.

THEOREM 2.1. – *Let E be a symmetric quasi-Banach sequence space. Then*

(a) $\delta_E = \liminf_{n\to\infty} [\log n / \log \varphi_E(n)]$

(b) $\gamma_E = \limsup_{n\to\infty} [\log n / \log \varphi_E(n)]$.

In order to extend [9], Thm. 2.7. to the full range of parameters, we recall that a bounded linear operator $T \in \mathcal{L}(X,Y)$ between two quasi-Banach spaces X and Y is called *strictly singular* if it fails to be an isomorphism on any infinite dimensional subspace (see [14] and [20]). Comparing this notion with the concept of disjoint strict singularity mentioned in the previous section, it is clear that any strictly singular operator is disjointly strictly singular. On the other hand, for $0 < q < p < \infty$, the inclusion $L_p[0,1] \hookrightarrow L_q[0,1]$ is a disjointly strictly singular operator which is not strictly singular.

THEOREM 2.2. – *Let E and F be quasi-Banach spaces of sequences with $\mathfrak{f} \subseteq E \hookrightarrow F \hookrightarrow \ell_\infty$. Assume that $\delta_E = \gamma_E$ and $\delta_F = \gamma_F$. If the inclusion operator $E \hookrightarrow F$ is not strictly singular, then either:*

(i) $\mathfrak{f} \subseteq E \subseteq F \subseteq \bigcap_{q>0} \ell_q$ *or*

(ii) $\bigcup_{q<\infty} \ell_q \subseteq E \subseteq F \subseteq \ell_\infty$ *or*

(iii) $\bigcup_{q<p} \ell_q \subseteq E \subseteq F \subseteq \bigcap_{q>p} \ell_q$ *for some* $1 < p < \infty$.

PROOF. – The argument is similar to the one in the proof of Theorem 1.6 but using now that for $0 < p < q < \infty$ the inclusion $\ell_p \hookrightarrow \ell_q$ is strictly singular (see, for example, [15], Thm. 5.3). □

If E is a symmetric quasi-Banach space, it follows from Theorem 2.1 that

$$\delta_E = \gamma_E \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \quad \text{exists.}$$

In particular, if $\varphi_E(n)$ "behaves like" $n^{1/p}$ for some $0 < p < \infty$, in the sense that for any $0 < \varepsilon < 1/p$ there are positive constants c_ε, C_ε such that

$$c_\varepsilon n^{\frac{1}{p}-\varepsilon} \le \varphi_E(n) \le C_\varepsilon n^{\frac{1}{p}+\varepsilon} \quad \text{for any} \quad n \in \mathbb{N},$$

then $\delta_E = \gamma_E = p$. This is the case when φ_E has regular variation at infinity, that is to say, when $\lim_{n\to\infty} [\varphi_E(2n)/\varphi_E(n)]$ exists.

3. – Inclusion indices relative to an interpolation scale.

In this section we develop an abstract approach to the results on function spaces and sequence spaces. We study inclusion indices defined by the real interpolation scale generated by a quasi-Banach couple. In the Banach case, this question was considered in [11].

Let $\bar{X} = (X_0, X_1)$ be a *quasi-Banach couple*, that is, two quasi-Banach spaces X_0, X_1 which are continuously embedded in some Hausdorff topological vector space. The Peetre's *K-functional* and *J-functional* are defined by

$$K(t,x) = K(t,x;X_0,X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_j \in X_j\},\ t > 0,\ x \in X_0 + X_1,$$

and

$$J(t,x) = J(t,x;X_0,X_1) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\}, \quad t > 0, \quad x \in X_0 \cap X_1.$$

A quasi-Banach space X is said to be an *intermediate space* with respect to the couple $\bar{X}$ if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$. Following [4], we put

$$\psi_X(t) = \psi_X(t; X_0, X_1) = \sup\{K(t, x; X_0, X_1) : x \in X, \|x\|_X = 1\},$$
$$\rho_X(t) = \rho_X(t; X_0, X_1) = \inf\{J(t, x; X_0, X_1) : x \in X_0 \cap X_1, \|x\|_X = 1\}.$$

Variants of the functions ψ_X and ρ_X have been used in many papers. We refer, for instance, to [8], [19] or [22]. See also [7], [9] and [10] for other properties of these functions.

Subsequently, we mostly work with *ordered couples*, that is, quasi-Banach couples (A_0, A_1) with $A_0 \hookrightarrow A_1$.

For $0 < \theta < 1$ and $0 < q \le \infty$, the *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all elements $a \in A_1$ having a finite quasi-norm

$$\|a\|_{\theta,q} = \begin{cases} \Big(\int\limits_0^\infty \big(t^{-\theta} K(t,a)\big)^q \, dt/t \Big)^{1/q} & \text{if } 0 < q < \infty \\ \sup\limits_{t>0} \{t^{-\theta} K(t,a)\} & \text{if } q = \infty \end{cases}$$

(see [3] and [23]).

Since $A_0 \hookrightarrow A_1$, for $0 < \mu < \theta < 1$ it holds

$$(A_0, A_1)_{\mu,p} \hookrightarrow (A_0, A_1)_{\theta,q} \quad \text{for any} \quad 0 < p, q \le \infty \tag{3.1}$$

(see [3], Thm. 3.4.1).

The equivalence theorem is still valid for quasi-Banach couples (see [3], Thm. 3.11.3). As a consequence, given a quasi-Banach space A with $A_0 \hookrightarrow A \hookrightarrow A_1$ if there is a constant $C > 0$ such that

$$\|a\|_A \le C t^{-\theta} J(t, a) \quad \text{for any} \quad a \in A_0 \quad \text{and} \quad t \ge 1,$$

it follows that

$$(A_0, A_1)_{\theta,q} \hookrightarrow A \quad \text{for some} \quad q \le 1. \tag{3.2}$$

DEFINITION 3.1. – *Let A be a quasi-Banach space with $A_0 \hookrightarrow A \hookrightarrow A_1$. The inclusion indices of A relative to the scale $\{(A_0, A_1)_{\theta,1}\}$ are defined by*

$$\bar{\delta}_A = \sup\{0 < \theta < 1 : \bar{A}_{\theta,1} \hookrightarrow A\}\,, \quad \bar{\gamma}_A = \inf\{0 < \theta < 1 : A \hookrightarrow \bar{A}_{\theta,1}\}.$$

If there is no $0 < \theta < 1$ such that $\bar{A}_{\theta,1} \hookrightarrow A$ [respectively, $A \hookrightarrow \bar{A}_{\theta,1}$], then we let $\bar{\delta}_A = 0$ [respectively, $\bar{\gamma}_A = 1$].

REMARK 3.2. – We have taken the value 1 for the second parameter of the real interpolation scale, but we can equally choose any other value because, by (3.1), it has no influence on the definition of the indices.

REMARK 3.3. – In the Banach case, definition of indices given in [11] is slightly different from Definition 3.1. There, one works with the complex interpolation scale and the supremum and the infimum of the set $\{1/(1-\theta)\}$.

Next we establish analytic formulae to calculate the indices. The proofs are patterned on those of [11] for the Banach case.

THEOREM 3.4. – *Let A be a quasi-Banach space with $A_0 \hookrightarrow A \hookrightarrow A_1$. Then*

$$\bar{\delta}_A = \liminf_{t\to\infty} \frac{\log \rho_A(t)}{\log t}.$$

PROOF. – We may assume, multiplying by a constant the norms of A_0 and A_1 if necessary, that $\rho_A(1) = 1$. Hence $\rho_A(t) \geq 1$ for each $t \geq 1$ because the function ρ_A is non-decreasing. Moreover, since $\rho_A(t)/t$ is non-increasing, we have $\rho_A(t)/t \leq 1$ for each $t \geq 1$. It follows that

$$0 \leq \liminf_{t\to\infty} \frac{\log \rho_A(t)}{\log t} \leq 1.$$

Suppose that $\bar{A}_{\theta,1} \hookrightarrow A$ for some $0 < \theta < 1$. Then there is a constant $C > 0$ such that $\|a\|_A \leq Ct^{-\theta} J(t,a)$ for all $a \in A_0$ and $t > 0$. Hence $t^\theta \leq C\rho_A(t)$. Taking logarithms and lower limits, we get $\theta \leq \liminf_{t\to\infty} [\log \rho_A(t)/\log t]$. This yields that

$$\bar{\delta}_A \leq \liminf_{t\to\infty} \frac{\log \rho_A(t)}{\log t}.$$

If $\liminf_{t\to\infty} [\log \rho_A(t)/\log t] = 0$, the desired equality holds. To establish the remaining case, suppose that

$$0 < \theta < \mu < \liminf_{t\to\infty} \frac{\log \rho_A(t)}{\log t} \tag{3.3}$$

and let us show that $\bar{A}_{\theta,1} \hookrightarrow A$. By (3.3), there exists $t_0 > 1$ such that $t^\mu < \rho_A(t)$ for any $t > t_0$. Find $C > 0$ such that $1/\rho_A(t) \leq Ct^{-\mu}$ for all $t \geq 1$. Given any $a \in A_0$ and $t \geq 1$, we obtain

$$\|a\|_A \leq \frac{J(t,a)}{\rho_A(t)} \leq Ct^{-\mu} J(t,a).$$

Then (3.1) and (3.2) imply that $\bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\mu,q} \hookrightarrow A$ and finishes the proof. □

THEOREM 3.5. – *Let A be a quasi-Banach space with $A_0 \hookrightarrow A \hookrightarrow A_1$. Then*

$$\bar{\gamma}_A = \limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t}.$$

PROOF. – We may assume, multiplying by a constant the norms of A_0 and A_1 if necessary, that $\psi_A(1) = 1$. Then $\psi_A(t) \geq 1$ and $\psi_A(t)/t \leq 1$ for any $t \geq 1$. So

$$0 \leq \limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t} \leq 1.$$

Let $0 < \theta < 1$ with $A \hookrightarrow \bar{A}_{\theta,1}$. Since $\bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\theta,\infty}$, we can find $C > 0$ such that $t^{-\theta}K(t,a) \leq C\|a\|_A$ for any $t > 0$ and $a \in A$. It follows that $\psi_A(t) \leq Ct^\theta$ and so $\limsup_{t\to\infty} [\log \psi_A(t)/\log t] \leq \theta$. This implies that

$$\limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t} \leq \bar{\gamma}_A.$$

If $\limsup_{t\to\infty} [\log \psi_A(t)/\log t] = 1$, we are done. In order to prove the remaining case, take any μ, θ with

$$\limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t} < \mu < \theta < 1.$$

There is $t_0 > 1$ so that $\psi_A(t) < t^\mu$ for every $t \geq t_0$. Let $M > 0$ such that $\psi_A(t) \leq Mt^\mu$ for any $t \geq 1$. Using (3.1) and the embedding $A_0 \hookrightarrow A_1$, we get

$$\begin{aligned}\|a\|_{\theta,1} &\leq M_1\|a\|_{\mu,\infty} = M_1 \sup_{0<t<\infty} \{t^{-\mu}K(t,a)\} \\ &\leq M_2 \sup_{t\geq 1}\{t^{-\mu}K(t,a)\} \leq M_2 M \sup_{t\geq 1}\{K(t,a)/\psi_A(t)\} \leq M_2 M\|a\|_A.\end{aligned}$$

Consequently, $A \hookrightarrow \bar{A}_{\theta,1}$. This implies that

$$\bar{\gamma}_A \leq \limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t}$$

and completes the proof. □

Theorems 3.4 and 3.5 can be used to complement the results of the previous sections. We illustrate it with an example.

Let $0 < r < \infty$. Consider the quasi-Banach spaces $A_0 = L_\infty[0,1] \hookrightarrow A_1 = L_r[0,1]$. It is well-known that interpolating this couple with parameters $(\theta, 1)$ we obtain Lorentz function spaces

$$\bar{A}_{\theta,1} = (L_\infty[0,1], L_r[0,1])_{\theta,1} = L_{s,1}[0,1]$$

(see [3] or [23]). Here $s = r/\theta$. If E is a quasi-Banach function space with $L_\infty[0,1] \hookrightarrow E \hookrightarrow L_r[0,1]$, the indices of E defined by the interpolation scale do not coincide with those considered in Section 1, but they are related. Indeed

$$\begin{aligned}\bar{\delta}_E &= \sup\{0 < \theta < 1 : (L_\infty[0,1], L_r[0,1])_{\theta,1} \hookrightarrow E\} \\ &= \sup\{r/s : L_{s,1}[0,1] \hookrightarrow E\} \\ &= \frac{r}{\inf\{0 < s < \infty : L_s[0,1] \hookrightarrow E\}} = \frac{r}{\gamma_E}.\end{aligned}$$

Similarly, $\bar{\gamma}_E = r/\delta_E$. Hence

$$\delta_E = \gamma_E \quad \text{if and only if} \quad \bar{\delta}_E = \bar{\gamma}_E.$$

Now we can characterize those function spaces, not necessarily symmetric, satisfying that $\delta_E = \gamma_E$. We work with the functions

$$\rho_E(t) = \rho_E(t; L_\infty[0,1], L_r[0,1]) \quad \text{and} \quad \psi_E(t) = \psi_E(t; L_\infty[0,1], L_r[0,1]).$$

THEOREM 3.6. – *Let $0 < r < \infty$ and E be a quasi-Banach function space with $L_\infty[0,1] \hookrightarrow E \hookrightarrow L_r[0,1]$. Then a necessary and sufficient condition for $\delta_E = \gamma_E$ is that the limits*

$$\lim_{t\to\infty} \frac{\log \rho_E(t)}{\log t} \quad \text{and} \quad \lim_{t\to\infty} \frac{\log \psi_E(t)}{\log t}$$

exist and coincide.

PROOF. – If the limits exist and are equal, using Theorems 3.4 and 3.5, we get

$$\begin{aligned} \bar{\delta}_E &= \liminf_{t\to\infty} \frac{\log \rho_E(t)}{\log t} = \lim_{t\to\infty} \frac{\log \rho_E(t)}{\log t} \\ &= \lim_{t\to\infty} \frac{\log \psi_E(t)}{\log t} = \limsup_{t\to\infty} \frac{\log \psi_E(t)}{\log t} = \bar{\gamma}_E. \end{aligned}$$

Let us show that the converse holds. According to [3], Thm. 5.2.1, we have

$$K(t,f; L_\infty[0,1], L_r[0,1]) \sim t\Big(\int_0^{t^{-r}} (f^*(s))^r ds\Big)^{1/r}.$$

Hence, there is a constant $C > 0$ such that for any $t \geq 1$,

$$K(t, \chi_{(0,t^{-r})}; L_\infty[0,1], L_r[0,1]) \geq C.$$

It follows that

$$\begin{aligned} \rho_E(t) &\leq \frac{J(t, \chi_{(0,t^{-r})}; L_\infty[0,1], L_r[0,1])}{\|\chi_{(0,t^{-r})}\|_E} \\ &= \frac{1}{\|\chi_{(0,t^{-r})}\|_E} \leq \frac{K(t, \chi_{(0,t^{-r})}; L_\infty[0,1], L_r[0,1])}{C\|\chi_{(0,t^{-r})}\|_E} \leq \frac{1}{C}\psi_E(t). \end{aligned}$$

Consequently, if $\bar{\delta}_E = \bar{\gamma}_E$, using Theorems 3.4 and 3.5, we derive

$$\limsup_{t\to\infty} \frac{\log \rho_E(t)}{\log t} \leq \limsup_{t\to\infty} \frac{\log \psi_E(t)}{\log t} = \bar{\gamma}_E = \bar{\delta}_E = \liminf_{t\to\infty} \frac{\log \rho_E(t)}{\log t}$$

and

$$\limsup_{t\to\infty} \frac{\log \psi_E(t)}{\log t} = \overline{\gamma}_E = \overline{\delta}_E = \liminf_{t\to\infty} \frac{\log \rho_E(t)}{\log t} \leq \liminf_{t\to\infty} \frac{\log \psi_E(t)}{\log t}.$$

This implies that $\lim\limits_{t\to\infty} [\log \rho_E(t)/\log t]$ and $\lim\limits_{t\to\infty} [\log \psi_E(t)/\log t]$ exist and coincide. □

For the sequence case, the corresponding result to Theorem 3.6 reads:

THEOREM 3.7. – *Let $0 < r < \infty$ and let E be a quasi-Banach sequence space with $\ell_r \hookrightarrow E \hookrightarrow \ell_\infty$. Let*

$$\rho_E(t) = \rho_E(t; \ell_r, \ell_\infty) \quad \textit{and} \quad \psi_E(t) = \psi_E(t; \ell_r, \ell_\infty).$$

Then a necessary and sufficient condition for $\delta_E = \gamma_E$ is that the limits

$$\lim_{t\to\infty} \frac{\log \rho_E(t)}{\log t} \quad \textit{and} \quad \lim_{t\to\infty} \frac{\log \psi_E(t)}{\log t}$$

exist and coincide.

The proof is similar to that of Theorem 3.6.

REMARK 3.8. – If $0 < r \leq 1$, assumption $\ell_r \hookrightarrow E \hookrightarrow \ell_\infty$ is satisfied by any quasi-Banach sequence lattice E which is r-normed, that is,

$$\|\xi + \eta\|_E^r \leq \|\xi\|_E^r + \|\eta\|_E^r \quad \text{for all} \quad \xi, \eta \in E,$$

and such that

$$0 < \inf_{n\in\mathbb{N}} \{\|e_{\{n\}}\|_E\} \leq \sup_{n\in\mathbb{N}} \{\|e_{\{n\}}\|_E\} = M < \infty.$$

Indeed, embedding $E \hookrightarrow \ell_\infty$ is clear. As regards the other embedding, take any $\xi = \{\xi_j\} \in \ell_r$ and any $m > n$. We have

$$\Big\| \sum_{j=1}^{m} \xi_j e_{\{j\}} - \sum_{j=1}^{n} \xi_j e_{\{j\}} \Big\|_E^r \leq M^r \sum_{j=n+1}^{m} |\xi_j|^r \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Hence $\Big\{ \sum\limits_{j=1}^{n} \xi_j e_{\{j\}} \Big\}$ is a Cauchy sequence and so it should be convergent in E. This implies that $\xi \in E$.

We end this section by using the inclusion indices to estimate the grade of proximity between quasi-Banach spaces. We shall first establish some preliminary results.

LEMMA 3.9. – *A bounded linear operator $T \in \mathcal{L}(X, Y)$ between the quasi-Banach spaces X, Y is strictly singular if and only if given any infinite dimensional subspace M of X and any $\varepsilon > 0$, there is a subspace $\{0\} \neq N \subseteq M$ such that $\|T\|_{N,Y} < \varepsilon$.*

Here $\|T\|_{N,Y}$ stands for the norm of T acting from N into Y.

PROOF. – If $T \in \mathcal{L}(X, Y)$ is not strictly singular, there exists an infinite dimensional subspace $M \subseteq X$ such that $T : M \longrightarrow T(M)$ is an isomorphism. Let $T^{-1} \in \mathcal{L}(T(M), M)$ be the inverse operator and choose $0 < \varepsilon < 1/\|T^{-1}\|_{T(M),M}$. For any $0 \neq x \in M$, we have

$$\|Tx\|_F \geq \frac{1}{\|T^{-1}\|_{T(M),M}} \|x\|_M > \varepsilon \|x\|_M.$$

Conversely, if $T \in \mathcal{L}(X, Y)$ is strictly singular and M is an infinite dimensional subspace M of X, then $T : M \longrightarrow T(M)$ is not an isomorphism. It follows that either $T : M \longrightarrow Y$ is not injective or $T^{-1} : T(M) \longrightarrow M$ is not bounded. In the first case, there is $0 \neq x \in M$ such that $Tx = 0$. So, for the subspace $N = \langle x \rangle \subseteq M$, we have $N \neq \{0\}$ and $\|T\|_{N,Y} = 0$. In the second case, given any $\varepsilon > 0$, we can find $x \in M$ such that $\|x\|_M > \dfrac{1}{\varepsilon} \|Tx\|_Y$. Put $N = \langle x \rangle$. Then $\{0\} \neq N \subseteq M$ and $\|T\|_{N,Y} < \varepsilon$. □

In the Banach case, Lemma 3.9 is contained in [14], Thm. III.2.1. Using Lemma 3.9 we can easily derive an interpolation result for strictly singular operators between quasi-Banach spaces.

LEMMA 3.10. – *Let $\bar{Y} = (Y_0, Y_1)$ be a quasi-Banach couple, let Y be an intermediate space with respect to $\bar{Y}$ and let X be another quasi-Banach space. Suppose that T is a linear operator. If $T : X \longrightarrow Y_0$ is bounded, $T : X \longrightarrow Y_1$ is strictly singular and $\lim_{t \to \infty} \rho_Y(t; Y_0, Y_1) = \infty$, then $T : X \longrightarrow Y$ is strictly singular.*

PROOF. – Let M be any infinite dimensional subspace of X. Given any $\varepsilon > 0$, take $t_0 > 0$ such that $\|T\|_{X,Y_0}/\varepsilon < \rho_Y(t_0)$. Since $T : X \longrightarrow Y_1$ is strictly singular, there exists $\{0\} \neq N \subseteq M$ such that $\|T\|_{N,Y_1} \leq \rho_Y(t_0)\, \varepsilon / t_0$. Hence, for any $x \in N$, we have

$$\|Tx\|_Y \leq \frac{J(t_0, Tx)}{\rho_Y(t_0)} \leq \|x\|_N \max\left\{ \frac{\|T\|_{N,Y_0}}{\rho_Y(t_0)}, \frac{t_0 \|T\|_{N,Y_1}}{\rho_Y(t_0)} \right\} \leq \varepsilon \|x\|_N.$$

Using Lemma 3.9, the result follows. □

Previous interpolation results for strictly singular operators can be found in [5], [4], [6] and the references given there.

REMARK 3.11. – Disjointly strictly singular operators have the same interpolation property. This can be checked repeating the argument of the proof of Lemma 3.10 but using now [12], Prop. 2.1, instead of Lemma 3.9. We refer to [12] and [6] for similar results.

Now we can establish the abstract version of Theorem 2.2.

THEOREM 3.12. – *Let X, Y be quasi-Banach spaces with $A_0 \hookrightarrow X \hookrightarrow Y \hookrightarrow A_1$. Assume that $\bar{\delta}_X = \bar{\gamma}_X$, $\bar{\delta}_Y = \bar{\gamma}_Y$ and that the inclusion $X \hookrightarrow Y$ is not strictly singular. If there is $0 < \theta < 1$ with $\bar{\gamma}_X < \theta$ and the inclusion $\bar{A}_{\theta,1} \hookrightarrow A_1$ being strictly singular, then either*

(i) $A_0 \subseteq X \subseteq Y \subseteq \bigcap_{\eta>0} \bar{A}_{\eta,1}$ *or*

(ii) $\bigcup_{\eta<1} \bar{A}_{\eta,1} \subseteq X \subseteq Y \subseteq A_1$ *or*

(iii) $\bigcup_{\eta<\mu} \bar{A}_{\eta,1} \subseteq X \subseteq Y \subseteq \bigcap_{\eta>\mu} \bar{A}_{\eta,1}$ *for some* $0 < \mu < 1$.

PROOF. – We claim that $\bar{\gamma}_X \geq \bar{\delta}_Y$. Indeed, if this would not be the case, there would be $\bar{\gamma}_X < \theta < \mu < \bar{\delta}_Y$ such that the embedding $\bar{A}_{\theta,1} \hookrightarrow A_1$ is strictly singular and $X \hookrightarrow \bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\mu,1} \hookrightarrow Y$. Using the reiteration theorem (see [3], Thm. 3.11.5), we have

$$\rho_{\bar{A}_{\mu,1}}(t; \bar{A}_{\theta,1}, A_1) \geq C t^{(\mu-\theta)/(1-\theta)},$$

so $\lim_{t\to\infty} \rho_{\bar{A}_{\mu,1}}(t; \bar{A}_{\theta,1}, A_1) = \infty$. Lemma 3.10 yields that the inclusion $\bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\mu,1}$ is strictly singular and then the embedding $X \hookrightarrow Y$ would be strictly singular.

Therefore $\bar{\delta}_X = \bar{\gamma}_X \geq \bar{\delta}_Y = \bar{\gamma}_Y$. Since the converse inequality follows from the inclusion $X \hookrightarrow Y$, we obtain that $\bar{\delta}_X = \bar{\gamma}_X = \bar{\delta}_Y = \bar{\gamma}_Y$. This implies the result. □

REMARK 3.13. – Working with quasi-Banach lattices, one can easily check that Theorem 3.12 is still valid if we replace strict singularity by disjoint strict singularity. The proof is the same but using now Remark 3.11. Writing down the outcome we get an abstract version of Theorem 1.6.

4. – Rank-one interpolation spaces and indices.

In this final section we work with certain intermediate spaces defined by using bounded linear functionals, so we deal with Banach spaces exclusively.

Given a Banach couple $\bar{B} = (B_0, B_1)$, we write $T : \bar{B} \longrightarrow \bar{B}$ to mean that T is a linear operator from $B_0 + B_1$ into $B_0 + B_1$ such that the restriction to each B_j is a bounded operator from B_j into itself, $j = 0, 1$. An intermediate Banach space B with respect to $\bar{B}$ is said to be an *interpolation space* if, for any $T : \bar{B} \longrightarrow \bar{B}$, the restriction of T to B defines a bounded operator from B into B. It is a consequence of the closed graph theorem that there is a positive constant $C = C(B, \bar{B})$ such that

$$\|T\|_{B,B} \leq C \max\left\{\|T\|_{B_0,B_0}, \|T\|_{B_1,B_1}\right\} \quad \text{for all} \quad T : \bar{B} \longrightarrow \bar{B}. \tag{4.1}$$

We say that an intermediate Banach space B with respect to $\bar{B}$ is a *rank-one interpolation space*, or a partly interpolation space, if (4.1) holds for all operators T of the special form

$$Tx = f(x)y \quad \text{where} \quad f \in (B_0 + B_1)^* \quad \text{and} \quad y \in B_0 \cap B_1.$$

This class of spaces have been considered by a number of authors. See, for example, [8], [22] or [4]. It turns out that B is a rank-one interpolation space if and only if there is a positive constant $C = C(B, \bar{B})$ such that

$$\psi_B(t; \bar{B}) \leq C\rho_B(t; \bar{B}), \quad \text{for all} \quad t > 0 \tag{4.2}$$

(see [8] and [22]).

Next we combine (4.2) with the results of the previous section.

COROLLARY 4.1. – *Let B_0, B_1 be Banach spaces with $B_0 \hookrightarrow B_1$ and let B be a rank-one interpolation space with respect to $\bar{B} = (B_0, B_1)$. If any of the limits*

$$\lim_{t\to\infty} \frac{\log \rho_B(t)}{\log t}, \quad \lim_{t\to\infty} \frac{\log \psi_B(t)}{\log t}$$

exists, then $\bar{\delta}_B = \bar{\gamma}_B$.

PROOF. – By (4.2),

$$\frac{\log \psi_B(t)}{\log t} \leq \frac{\log C}{\log t} + \frac{\log \rho_B(t)}{\log t}.$$

Hence, if $\lim_{t\to\infty} [\log \rho_B(t)/\log t]$ exists, we have

$$\begin{aligned} \bar{\gamma}_B = \limsup_{t\to\infty} \frac{\log \psi_B(t)}{\log t} &\leq \limsup_{t\to\infty} \frac{\log \rho_B(t)}{\log t} \\ &= \liminf_{t\to\infty} \frac{\log \rho_B(t)}{\log t} = \bar{\delta}_B. \end{aligned}$$

Since always $\bar{\delta}_B \leq \bar{\gamma}_B$, equality follows. The other case is analogous. □

As a consequence, we get the following complements of Theorems 3.6 and 3.7.

COROLLARY 4.2. – *Let E be a Banach function space with $L_\infty[0,1] \hookrightarrow E \hookrightarrow L_1[0,1]$. If E is a rank-one interpolation space with respect to $(L_\infty[0,1], L_1[0,1])$,* *then the following conditions are equivalent.*

(i) $\delta_E = \gamma_E$.

(ii) $\lim\limits_{t\to\infty} \dfrac{\log \rho_E(t; L_\infty[0,1], L_1[0,1])}{\log t}$ *exists.*

(iii) $\lim\limits_{t\to\infty} \dfrac{\log \psi_E(t; L_\infty[0,1], L_1[0,1])}{\log t}$ *exists.*

PROOF. – By Corollary 4.1, (ii) implies (i), and (iii) implies (i). On the other hand, Theorem 3.6 shows that (i) implies (ii) and (iii). □

For the sequence case, the results reads.

COROLLARY 4.3. – *Let E be a Banach sequence space with $\ell_1 \hookrightarrow E \hookrightarrow \ell_\infty$. If E is a rank-one interpolation space with respect to (ℓ_1, ℓ_∞), then the following conditions are equivalent.*

(i) $\delta_E = \gamma_E$.

(ii) $\lim\limits_{t\to\infty} \dfrac{\log\ \rho_E(t; \ell_1, \ell_\infty)}{\log t}$ *exists.*

(iii) $\lim\limits_{t\to\infty} \dfrac{\log\ \psi_E(t; \ell_1, \ell_\infty)}{\log t}$ *exists.*

The proof is similar to that of Corollary 4.2.

We finish the paper by recalling that a function [respectively, sequence] Banach space E is a rank-one interpolation space with respect to the couple $(L_\infty[0,1], L_1[0,1])$ [respectively, (ℓ_1, ℓ_∞)] if E lies between the Lorentz and the Marcinkiewicz spaces with the same fundamental function (see [8] and [21]).

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Fernando Cobos: Departamento de Análisis Matemático Facultad de Matemáticas
Universidad Complutense de Madrid 28040 Madrid. Spain
e-mail: cobos@mat.ucm.es

Luz M. Fernández-Cabrera: Departamento de Matemática Aplicada Escuela de Estadística
Universidad Complutense de Madrid 28040 Madrid. Spain
e-mail: luz_fernandez-c@mat.ucm.es

Antonio Manzano: Departamento de Matemáticas y Computación Escuela Politécnica Superior
Universidad de Burgos 09001 Burgos. Spain
e-mail: amanzano@ubu.es

Antón Martínez: Departamento de Matemática Aplicada I E.T.S. Ingenieros Industriales
Universidad de Vigo 36200 Vigo. Spain
e-mail: antonmar@uvigo.es

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