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A Note on Strong Lie Derived Length of Group Algebras

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Sunto. – Per un’algebra gruppale KG di un gruppo non-abeliano G su di un campo K di caratteristica positiva p si studia la lunghezza derivata forte di Lie dell’algebra di Lie associata.

Summary. – For a group algebra KG of a non-abelian group G over a field K of positive characteristic p we study the strong Lie derived length of the associated Lie algebra.

1. – Introduction.

Let G be a group and let KG be the group algebra of G over a field K. We consider the Lie algebra associated with KG by setting \([x, y] := xy - yx\) for every \(x, y \in KG\). We define by induction \(\delta^{(0)}(KG) := KG\) and \(\delta^{(n+1)}(KG)\) the associative ideal generated by \([\delta^{(n)}(KG), \delta^{(n)}(KG)]\), where this symbol denotes the additive subgroup generated by all the Lie commutators \([a, b]\) with \(a, b \in \delta^{(n)}(KG)\). KG is strongly Lie solvable if there exists an integer \(m\) such that \(\delta^{(m)}(KG) = 0\) and the minimal \(m\) with this property is called the strong Lie derived length of KG. Such an \(m\) is usually denoted by \(dl^L(KG)\).

If \(K\) has characteristic \(p > 0\) and \(G\) is non-abelian, it is well-known that the group algebra \(KG\) is strongly Lie solvable if, and only if, the commutator subgroup \(G'\) of \(G\) is a finite \(p\)-group (see Theorem V.5.1 of [5]).

If \(t(G')\) denotes the nilpotency index of the augmentation ideal \(\Delta(G')\) of \(KG'\), as an immediate consequence of Lemma 2.1 of [6] and of Lemma 2.2 of [4] we have the following elementary bounds

\[
\lceil \log_2(t(G') + 1) \rceil \leq dl^L(KG) \leq \lceil \log_2(2t(G')) \rceil,
\]

where \(\lceil r \rceil\) denotes the upper integral part of a real number \(r\).

Very little it is known about the strong Lie derived length of group algebras. The most remarkable works in this area are the papers by C. Baginski [1], M. Sahai [4] and A. Shalev [6], [7].
With the extra assumption that \( G \) is nilpotent, the evaluation of \( dl^L(KG) \) is more accurate. Denote by \( KG^{(1)} := KG \) and \( KG^{(m+1)} \) the associative ideal generated by \( [KG^{(m)}, KG] \); we denote by \( cl^L(KG) \) the minimal integer \( n \) such that \( KG^{(n+1)} = 0 \), the strong Lie nilpotency class of \( KG \). An easy induction allows to verify that \( \delta^{(m)}(KG) \subseteq KG^{2^m} \) for every non-negative integer \( m \). So we have

\[
(1) \quad [\log_2(t(G') + 1)] \leq dl^L(KG) \leq [\log_2(cl^L(KG) + 1)].
\]

An easy induction shows that if \( G \) is any group with \( G/G^{p} \) nilpotent for some prime \( p \), then also \( G/G^{mp} \) is nilpotent for all \( n \geq 1 \). In particular, if the condition \( \gamma_2(G) \leq G^{p} \) is satisfied, \( G \) is nilpotent, provided that \( G' \) is a finite \( p \)-group. Under the same assumptions, by Theorem 3.1 of [2], \( cl^L(KG) = t(G') \) and by (1) we obtain

\[
(2) \quad dl^L(KG) = [\log_2(t(G') + 1)].
\]

This result extends Corollary 4 of [1], which deals with finite \( p \)-groups whose commutator subgroup is cyclic.

We define by induction \( \delta^{(0)}(KG) := KG \) and \( \delta^{(n+1)}(KG) := [\delta^{(n)}(KG), \delta^{(n)}(KG)] \). Recall that \( KG \) is Lie solvable if there exists an integer \( n \) such that \( \delta^{(n)}(KG) = 0 \) and the minimal \( n \) with this property is called the Lie derived length of \( KG \). Such an \( n \) is usually denoted by \( dl_L(KG) \). Clearly \( \delta^{(n)}(KG) \subseteq \delta^{(m)}(KG) \) for all non-negative integer \( n \). Thus a strongly Lie solvable group algebra \( KG \) is Lie solvable and \( dl_L(KG) \leq dl^L(KG) \). But equality does not always hold. In fact, let \( G \) be a 2-group of maximal class of order \( 2^m \) with \( n \geq 5 \) and let \( K \) be a field of characteristic 2. Then \( G \) contains an abelian subgroup of index 2 and, by Theorem 1 of [3], \( dl_L(KG) \leq 3 \), whereas \( dl^L(KG) = n - 1 > 3 \) since \( G' \) is cyclic of order \( 2^n - 2 \).

We remark that, obviously, we obtain with precision the strong Lie derived length of a group algebra \( KG \) also when the Lie derived length reaches its upper bound \( [\log_2(2t(G'))] \). For example, when the group \( G \) is a semidirect product of an elementary abelian \( p \)-group by an automorphism of prime order \( q \), where \( p \neq q \) and \( q > 2 \) (Theorem C(1) of [6]).

If the group \( G \) is as above, but with \( q = 2 \), A. Shalev in [6] proved that the Lie derived length of \( KG \) is \( [\log_2(3t(G')/2)] \).

The aim of the sequel is to prove that this is also the strong Lie derived length of \( KG \).

**Theorem.** — Let \( K \) be a field of characteristic \( p > 2 \) and let \( G = E \rtimes \langle a \rangle \) be a split extension of an elementary abelian \( p \)-group \( E \) by an automorphism \( a \) of order 2. Then

\[
dl^L(KG) = [\log_2(3t(G')/2)].
\]
2. – Proof of the Theorem.

**Lemma.** – Let $K$ be a field of characteristic $p > 2$ and let $G = E \rtimes \langle a \rangle$ be a split extension of an elementary abelian $p$-group $E$ by an automorphism $a$ that acts on $E$ by inversion. The following holds:

1. $[\mathcal{A}(G')^mKG, \mathcal{A}(G')^mKG] \subseteq \mathcal{A}(G')^{2m}KG$ provided $m$ is odd;
2. $[\mathcal{A}(G')^mKG, \mathcal{A}(G')^mKG] \subseteq \mathcal{A}(G')^{2m+1}KG$ provided $m$ is even.

**Proof.** – The first statement is trivial. In order to prove the other one, we preliminarily observe that

\[(3) \quad [\mathcal{A}(G')^2, KG] \subseteq \mathcal{A}(G')^3KG.\]

Now let $m$ be an even integer. If $m = 2$, using (3) we have

\[
[\mathcal{A}(G')^2KG, \mathcal{A}(G')^2KG] \subseteq [\mathcal{A}(G')^2, \mathcal{A}(G')^2KG]KG + [\mathcal{A}(G')^2, \mathcal{A}(G')^2KG]KG
\]

\[
\subseteq \mathcal{A}(G')^4[KG, KG]KG + \mathcal{A}(G')^2[KG, \mathcal{A}(G')^2]KG \subseteq \mathcal{A}(G')^5KG.
\]

Finally, let $m > 2$. Then

\[
[\mathcal{A}(G')^mKG, \mathcal{A}(G')^mKG] \subseteq [\mathcal{A}(G')^m, \mathcal{A}(G')^mKG]KG + [\mathcal{A}(G')^2, \mathcal{A}(G')^mKG]KG
\]

\[
\subseteq \mathcal{A}(G')^4[\mathcal{A}(G')^{m-2}, \mathcal{A}(G')^mKG]KG + \mathcal{A}(G')^2[\mathcal{A}(G')^{m-2}, \mathcal{A}(G')^2]KG
\]

\[
\subseteq \mathcal{A}(G')^4\mathcal{A}(G')^{2m-3}KG + \mathcal{A}(G')^{m}[\mathcal{A}(G')^2, \mathcal{A}(G')^{m-2}KG]
\]

\[
\subseteq \mathcal{A}(G')^{2m+1}KG,
\]

by combining (3) and by the induction hypothesis. \(\square\)

**Proof of the Theorem.** – As just remarked in [6] by A. Shalev, we may assume that $C_E(a) = 1$ and so $G' = E$.

Let $n$ be a non-negative integer. In the sequel we put $s_0 := 1$ and

\[
s_n := \begin{cases} \frac{(2^{n+2} - 1)}{3} & \text{provided } n \text{ is even,} \\ \frac{(2^{n+2} - 2)}{3} & \text{provided } n \text{ is odd.} \end{cases}
\]
First we proceed by induction to prove that

\[ \forall n \in \mathbb{N}_0 \quad \delta^{(n+1)}(KG) \subseteq A(G')^s KG. \]

Since \( G' = \gamma_3(G) \), it follows that

\[ \delta^{(2)}(KG) = A(G')^2 KG. \]

Let \( n > 2 \). By induction and by the Lemma, we obtain

\[ \delta^{(n+1)}(KG) = [\delta^{(n)}(KG), \delta^{(n)}(KG)]KG \subseteq [A(G')^{s_{n-1}} KG, A(G')^{s_{n-1}} KG]KG \subseteq A(G')^{s_n} KG. \]

Now, let \( d := d(G)(KG) \). Then, by (4), \( A(G')^{s_{d-2}} \neq 0 \) and so we have \( s_{d-2} < t(G') \).

If \( d \) is even, then \( d < \log_2(3t(G')/2 + 1/2) + 1 \) and so \( d \leq \lceil \log_2(3t(G')/2) \rceil \), since \( t(G') \) is odd.

If \( d \) is odd, since \( 2^{d-1} < \lceil 3t(G')/2 \rceil \), it follows that \( 2^{d-1} < 3t(G')/2 + 1/2 \) and, as above, \( d \leq \lceil \log_2(3t(G')/2) \rceil \).

Finally, by Theorem C(2) of [6], the result follows.

\[ \square \]

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