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Partial Boundary Regularity of Solutions of Nonlinear Superelliptic Systems.

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Sunto. – *Si dimostra un risultato di regolarità parziale globale per le soluzioni deboli u del problema di Dirichlet associato al sistema superellittico non lineare* $\operatorname{div} A(x,u,Du)+B(x,u,Du)=0$, *con ipotesi di crescita naturale polinomiale delle funzioni coefficienti A e B. Si applica il metodo indiretto della forma bilineare e non si fa uso di una diseguaglianza di Caccioppoli né di una diseguaglianza di Hölder al contrario.*

Summary. – *We prove global partial regularity of weak solutions u of the Dirichlet problem for the nonlinear superelliptic system* $\operatorname{div} A(x,u,Du)+B(x,u,Du)=0$, *under natural polynomial growth of the coefficient functions A and B. We employ the indirect method of the bilinear form and do not use a Caccioppoli or a reverse Hölder inequality.*

1. – Introduction.

We are interested in the regularity of the vector-valued weak solutions $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$ of the Dirichlet problem for the nonlinear system

$$\operatorname{div} A(x,u,Du)+B(x,u,Du)=0\ , \tag{1}$$

or, in components,

$$\sum_{a=1}^{n} D_a\big(A_i^a(x,u,Du)\big)+B_i(x,u,Du)=0 \quad \text{for } i=1,\dots,N\ .$$

Here Ω is a bounded open subset of $\boldsymbol{R}^n$, $n \geq 2$, $N \geq 1$, and $Du(x) \in \boldsymbol{R}^{N\times n}$ denotes the gradient of u at a.e. point $x \in \Omega$. The coefficient functions A and B are defined on the set

$$\mathfrak{Z} = \overline{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{R}^{N\times n}$$

with values in $\boldsymbol{R}^{N\times n}$ and $\boldsymbol{R}^N$ respectively.

The standard hypotheses are that $A(x,u,P)$ be uniformly superelliptic, Hölder continuous in (x,u) with some exponent $0<\alpha<1$, and of class C^1 in P; and that $A(x,u,P)$, its partial derivative $A_P(x,u,P)$ and $B(x,u,P)$ have *natural polynomial growth* with exponent $q\geq 2$ in the variable P, i.e.

$$\begin{aligned}&|A(x,u,P)|\leq c\left(1+|P|^{q-1}\right),\quad |A_P(x,u,P)|\leq c\left(1+|P|^{q-2}\right),\\&|B(x,u,P)|\leq a|P|^q+b\ .\end{aligned}\tag{2}$$

Moreover, one assumes the smallness condition

$$2a\sup_{\Omega}|u|<\gamma\ ,$$

the constants $\kappa,\gamma>0$ being determined by the superellipticity [1] of the system:

$$A_P(x,u,P)\cdot(\xi,\xi)=\sum_{i,j,\alpha,\beta}D_{P^j_\beta}A^\alpha_i(x,u,P)\,\xi^j_\alpha\xi^j_\beta\geq\left(\kappa+\gamma(q-1)|P|^{q-2}\right)|\xi|^2\ ,\tag{3}$$

valid for all $(x,u,P)\in\mathfrak{Z}$ and $\xi\in\boldsymbol{R}^{N\times n}$.

DEFINITION 1. – *We say that* $u\in W^{1,q}(\Omega,\boldsymbol{R}^N)$ *is a weak solution of system* (1) *if*

$$\int_\Omega A(x,u,Du)\cdot D\varphi\,dx=\int_\Omega B(x,u,Du)\cdot\varphi\,dx\tag{4}$$

for every $\varphi\in W^{1,q}_0(\Omega,\boldsymbol{R}^N)$, *with* $\sup_\Omega|\varphi|<\infty$ *if* $a\neq 0$. *In components,* (4) *reads as*

$$\int_\Omega\left(\sum_{i,\alpha}A^\alpha_i(x,u,Du)D_\alpha\varphi^i\right)dx=\int_\Omega\left(\sum_i B_i(x,u,Du)\varphi^i\right)dx\ .$$

The problem of regularity of a weak solution $u\in W^{1,q}(\Omega,\boldsymbol{R}^N)$ of the nonlinear superelliptic system (1) has been intensively investigated over the last 26 years. As we know (see Example II.3.2 of [5]), we can in general only expect *partial regularity* if $N>1$, i.e. Hölder continuity of the gradient Du outside of a closed set of Lebesgue measure zero. Here the *optimal Hölder exponent* of Du is α, the same as that of the coefficient function $A(x,u,P)$ in the variables (x,u).

The general method of the proof is to compare the given solution u with a solution of a linear system with constant coefficients, for which standard elliptic estimates are available. For the direct approach, this comparison is carried out on an arbitrary ball either under a Dirichlet boundary condition, or with the so-called A-harmonic approximation (which is itself procured by a contradiction

(1) *Superellipticity* is often referred to as *strong ellipticity* or simply *ellipticity*. We reserve the term *ellipticity* for the weaker *condition of Legendre-Hadamard.*

argument); for the indirect approach, it is shown that a sequence of blow-up functions $w_m \in W^{1,2}(B, \boldsymbol{R}^N)$, rescaled to the unit ball B, converges weakly to such a solution.

Partial regularity of the solutions $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$ of the nonlinear superelliptic system (1) was shown, following the direct approach, by Giaquinta and Modica [6] and by Ivert [15] for $q = 2$, and by Tan [18] for $q > 2$. Their proofs are based on a reverse Hölder inequality with increasing supports for $Du - P_0$, for any constant $P_0 \in \boldsymbol{R}^{N\times n}$. This reverse Hölder inequality in turn is derived from a Caccioppoli inequality by invoking the higher integrability theorem of Gehring, Giaquinta and Modica (see [5], Proposition V.1.1, [7], Theorem 6.6).

A direct proof of partial regularity with the optimal Hölder exponent a was given by Hamburger [9] for the system

$$\text{div}\,\tilde{A}(x, Du) + B(x, u, Du) = 0\ , \tag{5}$$

whose leading part does not depend explicitly on the variable u. This already covers the general case as soon as we know that the solution u is Lipschitz continuous on the regular set Ω_u, by simply writing (1) in the form (5) with the composite coefficient function $\tilde{A}(x, P) = A(x, u(x), P)$, which is Hölder continuous in $x \in \Omega_u$ with exponent a.

As the higher integrability theorem is considered to be rather involved, it is desirable to find a simpler partial regularity proof which avoids the use of a reverse Hölder inequality. It is also desirable to prove partial regularity with the optimal Hölder exponent a in one single step. Such proofs by the method of A-harmonic approximation were supplied for the interior by Duzaar and Grotowski [1], and for the boundary by Grotowski [8]. Indirect partial regularity proofs for (1), which do not even employ a Caccioppoli inequality, were proposed by Yan and Li [19] and by Hamburger [11].

The present paper has a twofold aim. First, with just a few minor amendments to [11], we obtain the optimal Hölder exponent a in one step (see Theorem 1). Secondly, by adapting this proof to the boundary, we show global partial regularity for the Dirichlet problem. At the same time, we generalize the characterization of the regular boundary set given in [8] (see Theorem 2).

Our indirect proof of partial regularity at the boundary employs the *method of the bilinear form*, which was introduced by Hamburger [10] in establishing convergence $w_m \to w$ in $W^{1,2}(B_r^+, \boldsymbol{R}^N)$, for $0 < r < 1$, of a sequence of blow-up functions $w_m \in W^{1,2}(B^+, \boldsymbol{R}^N)$, which is known to converge only weakly. This technique has already been applied in the interior to solutions of nonlinear superelliptic and quasimonotone systems in [11] and [12], and to minimizers of convex, quasiconvex and polyconvex variational integrals in [10], [14] and [13]. We show that the blow-up functions w_m are approximate solutions of suitable rescaled systems. This has two consequences. First, passing to the limit as $m \to \infty$ we infer that w solves a linear elliptic system with constant coefficients. Secondly, we derive the key estimate

$$\limsup_{m\to\infty}\int_{B^+} \eta G(Y_m)\cdot(Dw_m, Dw_m)\, dz \le \int_{B^+} \eta G(Y_0)\cdot(Dw, Dw)\, dz \;.$$

Here η is a cut-off function, the bilinear form $G(Y)$ depends continuously on Y, and the constant function Y_0 is the limit in L^2 of a suitable sequence of functions $\{Y_m\}$. We finally deduce from this estimate with the help of uniform positivity of $G(Y)$ that $w_m \to w$ in $W^{1,2}(B_r^+, \boldsymbol{R}^N)$. In this manner we achieve partial boundary regularity of weak solutions of the Dirichlet problem for nonlinear superelliptic systems.

For the coefficient function $A : \mathfrak{Z} \to \boldsymbol{R}^{N\times n}$ we shall assume the following hypotheses, for an exponent $q \ge 2$.

HYPOTHESIS 1. – *We suppose that $A(x,u,P)$ is of class C^1 in P, and we assume that A_P is continuous and of polynomial growth*

$$\left|A_P(x,u,P)\right| \le c\left(1 + |P|^{q-2}\right) \;.$$

HYPOTHESIS 2. – *We suppose that $A(x,u,P)$ is Hölder continuous in (x,u) uniformly with respect to P:*

$$\left|A(x,u,P) - A(y,v,P)\right| \le K(|u| + |P|)\left(|x-y| + |u-v|\right)^{a}$$

for all $(x,u,P), (y,v,P) \in \mathfrak{Z}$. Here $K(s)$ is a nondecreasing function and $0 < a < 1$.

HYPOTHESIS 3. – *We suppose that A is uniformly superelliptic*

$$A_P(x,u,P)\cdot(\xi,\xi) \ge g(x)\left(\kappa + \gamma(q-1)\left|PE(x) + F(x)\right|^{q-2}\right)\left|\xi E(x)\right|^2$$

for some $\kappa, \gamma > 0$, and all $(x,u,P) \in \mathfrak{Z}$ and $\xi \in \boldsymbol{R}^{N\times n}$; and for some continuous structure functions $g \in C(\overline{\Omega})$, $E \in C(\overline{\Omega}, \boldsymbol{R}^{n\times n})$ and $F \in C(\overline{\Omega}, \boldsymbol{R}^{N\times n})$ such that $g(x) > 0$ and $E(x)$ is invertible for all $x \in \overline{\Omega}$.

We also assume the coefficient function $B : \mathfrak{Z} \to \boldsymbol{R}^N$ has critical polynomial growth. In order to save ourselves a separate treatment of the case $B = 0$, we agree that $2aM = 0$ if $a = 0$ and $M = \infty$. Of course, condition (8) for $M = \infty$ is void.

HYPOTHESIS 4. – *We suppose that $B(x,u,P)$ is a Carathéodory function, i.e. measurable in x and continuous in (u,P). Moreover, we assume that, for con-*

stants $a, b \in [0, \infty[$ *and* $M \in [0, \infty]$,

$$|B(x,u,P)| \leq g(x)\left(a|PE(x)+F(x)|^q+b\right) \text{ and } 2aM < \gamma \tag{6}$$

for all $(x,u,P) \in \mathfrak{Z}$ *with* $|u| \leq M$.

REMARK 1. – (a) Hypotheses 3 and 4 are usually stated with the constant structure functions $g = 1$, $E = I$ and $F = 0$ as (3) and (2). This structure is, however, disturbed by the procedure of "subtracting the boundary values and straightening the boundary."

(b) We have therefore already written Hypotheses 1 to 4 in a more general form that is invariant under this procedure. To see this, suppose that $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$ is a solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x,u,Du)+B(x,u,Du)=0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \ , \end{cases}$$

with $\varphi \in C^{1,a}(\overline{\Omega}, \boldsymbol{R}^N)$. We consider a boundary chart $\psi : \overline{B^+} \to \psi(\overline{B^+}) \subset \overline{\Omega}$ such that $\psi(B^+) \subset \Omega$ and $\psi(\Gamma) \subset \partial\Omega$, which together with its inverse ψ^{-1} is of class $C^{1,a}$. Then the function $\tilde{u} = (u - \varphi) \circ \psi \in W^{1,q}(B^+, \boldsymbol{R}^N)$ solves the Dirichlet problem

$$\begin{cases} \operatorname{div} \tilde{A}(x,\tilde{u},D\tilde{u})+\tilde{B}(x,\tilde{u},D\tilde{u})=0 & \text{in } B^+ \\ \tilde{u} = 0 & \text{on } \Gamma \ . \end{cases}$$

Its coefficient functions are given by

$$\begin{aligned} \tilde{A}(x,u,P)\cdot\xi &= \tilde{g}(x)A\left(\psi(x), u+\tilde{f}(x), P\tilde{E}(x)+\tilde{F}(x)\right)\cdot\left(\xi\tilde{E}(x)\right) , \\ \tilde{B}(x,u,P) &= \tilde{g}(x)B\left(\psi(x), u+\tilde{f}(x), P\tilde{E}(x)+\tilde{F}(x)\right) \end{aligned}$$

for $(x,u,P) \in \overline{B^+} \times \boldsymbol{R}^N \times \boldsymbol{R}^{N\times n}$ and $\xi \in \boldsymbol{R}^{N\times n}$, where

$$\tilde{g} = |\det D\psi| \ , \ \tilde{E} = (D\psi)^{-1} \ , \ \tilde{F} = D\varphi \circ \psi \text{ and } \tilde{f} = \varphi \circ \psi \ . \tag{7}$$

We notice that $\tilde{g}$, $\tilde{E}$ and $\tilde{F}$ are of class $C^{0,a}$, and $\tilde{f}$ is of class $C^{1,a}$ on $\overline{B^+}$. It is easy to check that if A and B satisfy Hypotheses 1 to 4 then so do $\tilde{A}$ and $\tilde{B}$, with possibly different structure constants and functions. For example, if $g = 1$, $E = I$ and $F = 0$ then $\tilde{g}$, $\tilde{E}$ and $\tilde{F}$ are given by (7).

(c) Including the factor $q - 1$ in Hypothesis 3 yields the smallness condition in (6), and the uniformly strict monotonicity at zero

$$\left(A(x,u,P)-A(x,u,0)\right)\cdot P \geq \kappa|P|^2+\gamma|P|^q$$

in their familiar form (for $g = 1$, $E = I$ and $F = 0$).

We first state a classical result concerning partial regularity in the interior for nonlinear superelliptic systems (see Theorems VI.2.1 and VI.2.2 of [5], [6], [15], [19], [18], [9], [11], [1]). For an extension to nonlinear quasimonotone systems, we refer to [3] and [12].

THEOREM 1. – *Let A and B satisfy Hypotheses* 1 *to* 4, *with exponent* $q \geq 2$. *Let* $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$, *with*

$$\sup_{\Omega} |u| \leq M \ , \tag{8}$$

be a weak solution of the system

$$\operatorname{div} A(x,u,Du) + B(x,u,Du) = 0 \ .$$

Then there exists an open set $\Omega_u \subset \Omega$, *whose complement has Lebesgue measure zero, such that the gradient Du is locally Hölder continuous in* Ω_u, *with the exponent* $0 < a < 1$ *of Hypothesis* 2*:*

$$u \in C^{1,a}(\Omega_u, \boldsymbol{R}^N) \ \textit{and} \ \mathcal{L}^n(\Omega \setminus \Omega_u) = 0 \ .$$

Moreover, the regular set is characterized by

$$\Omega_u = \Big\{ x_0 \in \Omega : \sup_{k \in \boldsymbol{N}} \big(|u_{x_0,r_k}| + |Du_{x_0,r_k}| \big) < \infty \quad \textit{and}$$

$$\liminf_{k \to \infty} \fint_{B_{r_k}(x_0)} |Du - Du_{x_0,r_k}|^q \, dx = 0 \ \textit{for some sequence} \ r_k \searrow 0 \Big\} \ .$$

We now turn to the Dirichlet problem for nonlinear superelliptic systems. Our main result is contained in the following

THEOREM 2. – *Let A and B satisfy Hypotheses* 1 *to* 4, *with exponent* $q \geq 2$. *Let* Ω *be of class* $C^{1,a}$, *and let* $\varphi \in C^{1,a}(\overline{\Omega}, \boldsymbol{R}^N)$. *Let* $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$, *with*

$$\sup_{\Omega} |u - \varphi| \leq M \ ,$$

be a weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x,u,Du) + B(x,u,Du) = 0 & \textit{in } \Omega \\ u = \varphi & \textit{on } \partial\Omega \ . \end{cases}$$

Then there exists a relatively open set $\Omega_u \subset \overline{\Omega}$, *whose complement has Lebesgue measure zero, such that the gradient Du is locally Hölder continuous* *in Ω_u, with the exponent $0 < a < 1$ of Hypothesis 2:*

$$u \in C^{1,a}(\Omega_u, \boldsymbol{R}^N) \text{ and } \mathcal{L}^n(\overline{\Omega} \setminus \Omega_u) = 0 .$$

Moreover, the regular set is characterized by

$$\Omega_u = \Bigg\{ x_0 \in \overline{\Omega} : \sup_{k \in \boldsymbol{N}} \left(|u_{x_0,r_k}| + |Du_{x_0,r_k}| \right) < \infty \quad \text{and}$$

$$\liminf_{k \to \infty} \fint_{\Omega_{r_k}(x_0)} |Du - Du_{x_0,r_k}|^q \, dx = 0 \text{ for some sequence } r_k \searrow 0 \Bigg\} .$$

Theorem 2 improves the characterization of the regular boundary set $\partial\Omega \cap \Omega_u$ given by Grotowski [8]. In the case $q = 2$, he showed that if $x_0 \in \partial\Omega$ and

$$\liminf_{r \searrow 0} \fint_{\Omega_r(x_0)} |Du(x) - D\varphi(x_0) - \xi \otimes \upsilon(x_0)|^2 \, dx = 0$$

for some $\xi \in \boldsymbol{R}^N$ then $x_0 \in \Omega_u$. Here υ is a unit normal field to $\partial\Omega$, and we know a posteriori that $\xi = D_\upsilon(u - \varphi)(x_0)$. It is not difficult to see that a point $x_0 \in \partial\Omega$ satisfying Grotowski's criterion lies in the characterization of Ω_u as stated by Theorem 2.

The example in [4] illustrates that singularities may occur at the boundary, in spite of smooth boundary data. Example 1.1 in [8] shows that the Hölder exponent a is indeed optimal. For bounds on the Hausdorff dimension of the singular set $\overline{\Omega} \setminus \Omega_u$ in certain cases, we refer to the interesting recent papers [16], [17] and [2].

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2. – A decay estimate for the excess.

In what follows, all constants c may depend on the data including the coefficient functions A and B, and on the solution u itself. The Landau symbol $o(1)$ stands for any quantity for which $\lim_{m \to \infty} o(1) = 0$. We write $B_r(x_0) = \{x \in \boldsymbol{R}^n : |x - x_0| < r\}$, $B_r = B_r(0)$, and $B = B_1$ for the unit ball (we also used the symbol B in (1)); we write $B_r^+(x_0) = \{x \in B_r(x_0) : x^n > 0\}$, $B_r^+ = B_r^+(0)$, and $B^+ = B_1^+$ for the corresponding half-balls; and we write $\Gamma = \{x \in B : x^n = 0\}$ for the equatorial hyperplane of the unit ball. For $x_0 \in \overline{\Omega}$ and $r > 0$, we define the intersection $\Omega_r(x_0) = B_r(x_0) \cap \Omega$. The mean of a function f on $\Omega_r(x_0)$ is denoted by

$$f_{x_0,r} = \fint_{\Omega_r(x_0)} f\,dx = \frac{1}{\mathcal{L}^n\big(\Omega_r(x_0)\big)} \int_{\Omega_r(x_0)} f\,dx \ .$$

We assume Hypotheses 1 to 4 with $q \geq 2$, and we let $u \in W^{1,q}(\Omega, \boldsymbol{R}^N)$, subject to (8), be a weak solution of system (1). We define the excess of Du on $\Omega_r(x_0)$:

$$U(x_0, r) = \fint_{\Omega_r(x_0)} \left(|Du - Du_{x_0,r}|^2 + |Du - Du_{x_0,r}|^q\right) dx \ .$$

In view of Remark 1(b), the conclusions of Theorems 1 and 2 follow in a routine way from the next two propositions (see [5], pp. 197–199, [7], pp. 349–352, [9], Section 5, [8], Section 3.6).

PROPOSITION 1. – *Let $L > 0$ and $\tau \in\ ]0,1[$ be given. Then there exist positive constants $c_1(L)$, $H(L,\tau)$ and $\varepsilon(L,\tau)$ such that if*

$$B_r(x_0) \subset \Omega \ , \ |u_{x_0,r}| \leq L \ , \ |Du_{x_0,r}| \leq L \ \textit{and} \ U(x_0,r) \leq \varepsilon$$

then

$$U(x_0, \tau r) \leq c_1 \tau^2 U(x_0, r) + H r^{2a} \ .$$

PROPOSITION 2. – *Let $\Omega = B^+$, and let $u = 0$ on Γ. Let $L > 0$ and $\tau \in\ ]0,1[$ be given. Then there exist positive constants $c_1(L)$, $H(L,\tau)$ and $\varepsilon(L,\tau)$ such that if*

$$B_r^+(x_0) \subset B^+ \ \textit{with} \ x_0 \in \Gamma \ , \ |Du_{x_0,r}| \leq L \ \textit{and} \ U(x_0,r) \leq \varepsilon$$

then

$$U(x_0, \tau r) \leq c_1 \tau^2 U(x_0, r) + H r^{2a} \ .$$

We provide a proof of Proposition 2. The proof of Proposition 1 is very similar, and we shall list the necessary modifications at the end of the paper.

PROOF. – We will determine the constant c_1 later on. If the proposition were not true then there would exist a sequence of half-balls $B_{r_m}^+(x_m) \subset B^+$ with $x_m \in \Gamma$ such that, setting (2)

$$u_m = 0 \ , \ P_m = Du_{x_m,r_m} \ , \ \lambda_m^2 = U(x_m, r_m) \ , \tag{9}$$

we have

$$|u_m| \leq L \ , \ |P_m| \leq L \ , \ \lambda_m \searrow 0 \ , \tag{10}$$

(2) u_m will be nontrivial in the proof of Proposition 1.

but

$$U(x_m, \tau r_m) > c_1 \tau^2 \lambda_m^2 + m r_m^{2a} \ . \tag{11}$$

Since (11) implies $\lambda_m > 0$, we can define the rescaled functions

$$w_m(z) = \frac{u(x_m + r_m z) - u_m - r_m P_m \cdot z}{r_m \lambda_m}$$

for $z \in B^+$. We notice that

$$Dw_m(z) = \frac{Du(x_m + r_m z) - P_m}{\lambda_m} \ , \tag{12}$$

$$\int_\Gamma w_m \, d\mathcal{H}^{n-1} = 0 \ , \ (Dw_m)_{0,1} = 0 \ . \tag{13}$$

Then (9) and (11) become

$$\fint_{B^+} |Dw_m|^2 \, dz + \lambda_m^{q-2} \fint_{B^+} |Dw_m|^q \, dz = 1 \, , \tag{14}$$

$$\begin{aligned} c_1 \tau^2 + m \lambda_m^{-2} r_m^{2a} < & \fint_{B_\tau^+} \left| Dw_m - (Dw_m)_{0,\tau} \right|^2 dz \\ & + \lambda_m^{q-2} \fint_{B_\tau^+} \left| Dw_m - (Dw_m)_{0,\tau} \right|^q dz \ . \end{aligned} \tag{15}$$

From (14), (13) and Lemma 1 we immediately have

$$\|w_m\|_{W^{1,2}(B^+)} \le c \, , \quad \lambda_m^{(q-2)/q} \|w_m\|_{W^{1,q}(B^+)} \le c \ . \tag{16}$$

We infer from (15), (14) and (10) that

$$\lambda_m^{-1} r_m^a \searrow 0 \text{ and } r_m \searrow 0 \ . \tag{17}$$

By assumption (8) on u, and (10) and (17), we also note that

$$|u_m| \le M \, , \quad \sup_{B^+} |u_m + r_m P_m \cdot z + r_m \lambda_m w_m| \le M \ , \tag{18}$$

$$\sup_{B^+} |r_m \lambda_m w_m| \le 2M + o(1) \ . \tag{19}$$

We denote the rescaled quantity $(x, u, sDu + (1-s)P_m) \in \mathfrak{Z}$ for $0 \le s \le 1$:

$$Z_m(s, z) = \left(x_m + r_m z, \ u_m + r_m P_m \cdot z + r_m \lambda_m w_m, \ P_m + s \lambda_m Dw_m\right) \ .$$

It follows from (16) and (10) by compactness of the Sobolev imbeddings $W^{1,2}(B^+) \hookrightarrow L^2(B^+)$ and $W^{1,q}(B^+) \hookrightarrow L^q(B^+)$, and of the trace $W^{1,2}(B^+) \to$

$L^2(\partial B^+)$ that, on passing to a subsequence and relabelling, we have

$$
(20)\qquad
\begin{array}{ll}
Dw_m \rightharpoonup Dw & \text{weakly in } L^2(B^+,\boldsymbol{R}^{N\times n})\ ,\\
w_m \to w & \text{in } L^2(B^+,\boldsymbol{R}^{N})\ ,\\
w_m \to w & \text{in } L^2(\partial B^+,\boldsymbol{R}^{N})\ ,\\
\lambda_m Dw_m \to 0 & \text{in } L^2(B^+,\boldsymbol{R}^{N\times n})\ ;\\
\lambda_m^{(q-2)/q} Dw_m \rightharpoonup 0 & \text{weakly in } L^q(B^+,\boldsymbol{R}^{N\times n})\ ,\\
\lambda_m^{(q-2)/q} w_m \to 0 & \text{in } L^q(B^+,\boldsymbol{R}^{N}) \text{ (for } q>2)\ ;\\
(x_m,u_m,P_m) \to Z_0=(x_0,u_0,P_0) & \text{in } \mathfrak{Z}\ ,\\
Z_m \to Z_0 & \text{in } L^2([0,1]\times B^+,\mathfrak{Z})\ .
\end{array}
$$

Since w_m is linear on Γ, we deduce that also w is linear on Γ. In details,

$$w_m = -\lambda_m^{-1} P_m \cdot z \text{ and } w = S\cdot z \text{ on } \Gamma\ ,$$

where

$$(21)\qquad -\lambda_m^{-1} P_m \pi \to S \text{ in } \boldsymbol{R}^{N\times n}$$

for the orthogonal projection $\pi(z^1,\ldots,z^n) = (z^1,\ldots,z^{n-1},0)$ of $\boldsymbol{R}^n$ onto $\boldsymbol{R}^{n-1}\times\{0\}$.

Now suppose that we can show that $w\in W^{1,2}(B^+,\boldsymbol{R}^N)$ is a weak solution of the following linear system with constant coefficients:

$$(22)\qquad \operatorname{div}\ (A_P(Z_0)\cdot Dw) = 0\ .$$

Then the function $W = w - S\cdot z \in W^{1,2}(B^+,\boldsymbol{R}^N)$ solves the Dirichlet problem

$$\begin{cases} \operatorname{div}\ (A_P(Z_0)\cdot DW) = 0 & \text{in } B^+\\ W = 0 & \text{on } \Gamma\ . \end{cases}$$

We infer from Hypothesis 1 and (10) that

$$|A_P(Z_0)| \le c\ ,$$

and from Hypothesis 3 that (22) is uniformly superelliptic:

$$A_P(Z_0)\cdot(\xi,\xi) \ge g(x_0)\kappa\lambda^2|\xi|^2 \text{ for all } \xi\in\boldsymbol{R}^{N\times n}\ .$$

Here we have used the estimate

$$(23)\qquad |\xi E(x)| \ge \lambda|\xi|$$

for some $\lambda>0$, and all $x\in\overline{B^+}$ and $\xi\in\boldsymbol{R}^{N\times n}$. Hence, from the relevant regularity theory applied to the function $W = w - S\cdot z$ (see [7], Theorems 10.5, 10.7(10.29)) we conclude that w is smooth on $B^+ \cup \Gamma$ and

$$\fint_{B_\tau^+} \left|Dw - Dw_{0,\tau}\right|^2 dz \le c_2 \tau^2 \fint_{B^+} \left|Dw - Dw_{0,1}\right|^2 dz \;, \tag{24}$$

where by (20), (13) and (14)

$$Dw_{0,1} = 0 \text{ and } \fint_{B^+} |Dw|^2 \, dz \le \liminf_{m\to\infty} \fint_{B^+} |Dw_m|^2 \, dz \le 1 \;. \tag{25}$$

On the other hand, if we also know that

$$\begin{aligned} &Dw_m \to Dw && \text{in } L^2\left(B_\tau^+, \boldsymbol{R}^{N\times n}\right) \;, \\ &\lambda_m^{(q-2)/q} Dw_m \to 0 && \text{in } L^q\left(B_\tau^+, \boldsymbol{R}^{N\times n}\right) \text{ (for } q > 2) \end{aligned} \tag{26}$$

then it would follow from (15) that

$$c_1 \tau^2 \le \fint_{B_\tau^+} \left|Dw - Dw_{0,\tau}\right|^2 dz \;.$$

If we now choose $c_1 = 2c_2$, we obtain a contradiction to (24) and (25). This proves the proposition. □

LEMMA 1. – *(Poincaré boundary inequality) There exists a positive constant* $c(n, N, q)$ *such that for every* $w \in W^{1,q}\left(B^+, \boldsymbol{R}^N\right)$ *with* $\int_\Gamma w \, d\mathcal{H}^{n-1} = 0$ *we have*

$$\int_{B^+} |w|^q \, dz \le c \int_{B^+} |Dw|^q \, dz \;.$$

PROOF. – If the lemma were not true then there would exist a sequence of functions $w_m \in W^{1,q}\left(B^+, \boldsymbol{R}^N\right)$ such that

$$\int_\Gamma w_m \, d\mathcal{H}^{n-1} = 0 \;, \int_{B^+} |w_m|^q \, dz = 1 \text{ and } \int_{B^+} |Dw_m|^q \, dz < \frac{1}{m} \;.$$

Hence $\{w_m\}$ is bounded in $W^{1,q}\left(B^+, \boldsymbol{R}^N\right)$, and by compactness of the relevant operators, there exists a relabelled subsequence such that

$$\begin{aligned} &Dw_m \rightharpoonup Dw && \text{weakly in } L^q\left(B^+, \boldsymbol{R}^{N\times n}\right) \;, \\ &w_m \to w && \text{in } L^q\left(B^+, \boldsymbol{R}^N\right) \;, \\ &w_m \to w && \text{in } L^q\left(\partial B^+, \boldsymbol{R}^N\right) \;. \end{aligned}$$

We deduce, using the weak sequential lower semicontinuity of the norm, that

$$\int_\Gamma w \, d\mathcal{H}^{n-1} = 0 \;, \int_{B^+} |w|^q \, dz = 1 \text{ and } \int_{B^+} |Dw|^q \, dz = 0 \;.$$

These equations cannot be simultaneously satisfied. □

The remainder of this work is devoted to showing (22) and (26), which are the assertions of Lemmas 3 and 5 respectively.

3. – Approximate solutions of rescaled systems.

In this section we show that w_m is, to order zero as $m \to \infty$, a weak solution of a rescaled system of inequalities.

LEMMA 2. – *For* $\varphi \in W_0^{1,q}(B^+, \boldsymbol{R}^N)$, *with* $\sup\limits_{B^+} |\varphi| < \infty$ *if* $a \neq 0$, *and* $\varepsilon > 0$, *we have*

$$\int\limits_{B^+}\int\limits_0^1 A_P(Z_m(s,z))\cdot(Dw_m, D\varphi)\, ds\, dz \tag{27}$$

$$\leq a(q-1)(1+\varepsilon)^2 \int\limits_{B^+}\int\limits_0^1 g\big|(P_m + s\lambda_m Dw_m)E + F\big|^{q-2} |Dw_m E|^2 r_m \lambda_m |\varphi|\, ds\, dz$$

$$+\, a o(1) \|r_m \lambda_m \varphi\|_{L^\infty}^{1-\beta} \|\varphi\|_{L^2}^{\beta} + o(1)\|\varphi\|_{W^{1,2}}\, .$$

Here the structure functions g, E *and* F *are evaluated at* $x_m + r_m z$, *and* $\beta = 2a(q-2)/(1+a)q < 1$.

PROOF. – Rescaling system (1) we find

$$\int\limits_{B^+} A(Z_m(1,z))\cdot D\varphi\, dz = r_m \int\limits_{B^+} B(Z_m(1,z))\cdot\varphi\, dz$$

for every $\varphi \in W_0^{1,q}(B^+, \boldsymbol{R}^N)$, with $\sup_{B^+} |\varphi| < \infty$ if $a \neq 0$. So it follows that

$$\int\limits_{B^+}\int\limits_0^1 A_P(Z_m(s,z))\cdot(Dw_m, D\varphi)\, ds\, dz$$

$$= \lambda_m^{-1} \int\limits_{B^+} \big(A(Z_m(1,z)) - A(Z_m(0,z))\big)\cdot D\varphi\, dz$$

$$= \lambda_m^{-1} r_m \int\limits_{B^+} B(Z_m(1,z))\cdot\varphi\, dz$$

$$- \lambda_m^{-1} \int\limits_{B^+} \big(A\big(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m \lambda_m w_m,\ P_m\big)$$

$$- A(x_m, u_m, P_m)\big)\cdot D\varphi\, dz = (\mathrm{I}) + (\mathrm{II})\, .$$

By virtue of Hypothesis 2, we estimate term (II) as follows using (10), (16),

(17) and the Hölder inequality

$$\text{(II)} \quad \le \lambda_m^{-1} K\big(|u_m| + |P_m|\big) \int_{B^+} \big(r_m + r_m|P_m| + r_m\lambda_m|w_m|\big)^a |D\varphi|\, dz$$
$$\le c\lambda_m^{-1} r_m^a \big(1 + \|w_m\|_{L^2}^a\big)\|\varphi\|_{W^{1,2}} = o(1)\|\varphi\|_{W^{1,2}} \;.$$

We next apply estimate (6) to term (I) and we use the inequality $(x+y)^q \le (1+\varepsilon)x^q + c(\varepsilon)y^q$. This gives

$$\text{(I)} \quad \le \lambda_m^{-1} r_m \int_{B^+} g\big(a\big|(P_m + \lambda_m Dw_m)E + F\big|^q + b\big)|\varphi|\, dz$$
$$\le a(1+\varepsilon)\int_{B^+} g\lambda_m^{q-2}|Dw_m E|^q r_m\lambda_m|\varphi|\, dz + c(\varepsilon)\lambda_m^{-1} r_m \|\varphi\|_{L^2} = \text{(i)} + \text{(ii)}\,.$$

In the case $q = 2$, we are already finished. In the case $q > 2$, the inequality $(x+y)^{q-2} \le (1+\varepsilon)x^{q-2} + c(\varepsilon)y^{q-2}$ yields

$$\text{(i)} \quad = a(q-1)(1+\varepsilon)\int_{B^+}\int_0^1 g|s\lambda_m Dw_m E|^{q-2}|Dw_m E|^2 r_m\lambda_m|\varphi|\, ds\, dz$$
$$\le a(q-1)(1+\varepsilon)^2\int_{B^+}\int_0^1 g\big|(P_m + s\lambda_m Dw_m)E + F\big|^{q-2}|Dw_m E|^2 r_m\lambda_m|\varphi| ds\, dz$$
$$+ ac(\varepsilon)\int_{B^+}|Dw_m|^2 r_m\lambda_m|\varphi|\, dz = \text{(a)} + \text{(b)}\;.$$

Noting that $0 < \beta < \beta q/(q-2) < 1$ we estimate

$$\text{(b)} \quad \le ac\|r_m\lambda_m\varphi\|_{L^\infty}^{1-\beta}\|Dw_m\|_{L^q}^2\|r_m\lambda_m\varphi\|_{L^{\beta q/(q-2)}}^\beta$$
$$\le ac\big(\lambda_m^{-1} r_m^a\big)^{\beta/a}\|r_m\lambda_m\varphi\|_{L^\infty}^{1-\beta}\big(\lambda_m^{(q-2)/q}\|Dw_m\|_{L^q}\big)^2\|\varphi\|_{L^2}^\beta\;.$$

By (17) and (16), we thus obtain Lemma 2. □

By choosing for φ a test function in Lemma 2 and sending $m \to \infty$, we derive

LEMMA 3. – *The function $w \in W^{1,2}(B^+, \boldsymbol{R}^N)$ is a weak solution of the linear superelliptic system with constant coefficients*

$$\operatorname{div}\,(A_P(Z_0)\cdot Dw) = 0\;,$$

satisfying the Dirichlet boundary condition $w = S\cdot z$ on Γ. In particular, we conclude that w is smooth on $B^+ \cup \Gamma$.

PROOF. – By Hypothesis 1, there exists a continuous, bounded and concave function $\omega(t)$, with $\omega(0)=0$, such that

$$|A_P(Z)-A_P(Z_0)| \le \omega(|Z-Z_0|^2)(1+|P|^{q-2}) \tag{28}$$

for all $Z=(x,u,P)\in \mathfrak{Z}$.

For $\varphi \in C_c^\infty(B^+,\boldsymbol{R}^N)$, the right-hand side of (27) is easily seen to approach zero as $m\to\infty$. Therefore, using (28), the boundedness of $D\varphi$, the Hölder and Jensen inequalities in combination with the boundedness and concavity of $\omega(t)$, (16), (20) and (10) we obtain

$$\begin{aligned}
\int_{B^+} A_P(Z_0)\cdot(Dw_m, D\varphi)\,dz &\le \int_{B^+}\int_0^1 A_P(Z_m)\cdot(Dw_m,D\varphi)\,ds\,dz \\
&+\int_{B^+}\int_0^1 \omega(|Z_m-Z_0|^2)(1+\lambda_m^{q-2}|Dw_m|^{q-2})|Dw_m||D\varphi|\,ds\,dz \\
&\le o(1)+c\left\{ \fint_{B^+}\int_0^1 \omega(|Z_m-Z_0|^2)\,ds\,dz\right\}^{1/2}\|Dw_m\|_{L^2} \\
&+c\lambda_m^{(q-2)/q}(\lambda_m^{(q-2)/q}\|Dw_m\|_{L^q})^{q-1} \\
&\le o(1)+c\omega^{1/2}\left(\fint_{B^+}\int_0^1 |Z_m-Z_0|^2\,ds\,dz\right) = o(1)\ .
\end{aligned}$$

We conclude by (20) that

$$\int_{B^+} A_P(Z_0)\cdot(Dw,D\varphi)\,dz \le 0\ ,$$

and the result follows by replacing φ by $-\varphi$. □

4. – Convergence of the blow-up functions.

We introduce some further notation. We define the set

$$\mathfrak{Y} = \left\{(x,u,v,P,Q)\in \overline{\Omega}\times \boldsymbol{R}^N\times\boldsymbol{R}^N\times\boldsymbol{R}^{N\times n}\times\boldsymbol{R}^{N\times n} : |u|,|v|\le M\right\}\ ,$$

and, by virtue of (18), functions $Y_m\in L^q(B^+,\mathfrak{Y})$ by

$$Y_m = (x_m+r_m z,\ u_m+r_mP_m\cdot z+r_m\lambda_m w_m,\ u_m,\ P_m,\ \lambda_m Dw_m)\ .$$

By (10), (17) and (20), we notice the limit

$$Y_m\to Y_0=(x_0,u_0,u_0,P_0,0)\ \text{in}\ L^2(B^+,\mathfrak{Y})\ . \tag{29}$$

Next, we define a bilinear form $G(Y)$ on $\boldsymbol{R}^{N\times n}$ by

$$G(Y)\cdot(\xi,\xi')=\int_0^1 A_P(x,u,P+sQ)\cdot(\xi,\xi')\,ds$$

$$-\,a(q-1)(1+\varepsilon)^2 g(x)|u-v|\int_0^1 \left|(P+sQ)E(x)+F(x)\right|^{q-2} ds\ \langle \xi E(x),\xi' E(x)\rangle$$

for $Y=(x,u,v,P,Q)\in\mathfrak{Y}$ and $\xi,\xi'\in\boldsymbol{R}^{N\times n}$. By Hypotheses 1 and 3, and since $|u-v|\leq 2M$, the bilinear form $G(Y)$ depends continuously on $Y\in\mathfrak{Y}$ and satisfies the growth and positivity conditions

$$\left|G(Y)\right|\leq c\left(1+|P|^{q-2}+|Q|^{q-2}\right)\ , \tag{30}$$

$$G(Y)\cdot(\xi,\xi)\geq\gamma'\left(1+|Q|^{q-2}\right)|\xi|^2 \tag{31}$$

for some $\gamma'>0$, and all $Y=(x,u,v,P,Q)\in\mathfrak{Y}$ and $\xi\in\boldsymbol{R}^{N\times n}$.

To see (31), we note by Hypothesis 3 that

$$G(Y)\cdot(\xi,\xi)\geq\tilde{\gamma}g(x)\int_0^1\left(1+\left|(P+sQ)E(x)+F(x)\right|^{q-2}\right)ds\ \left|\xi E(x)\right|^2\ ,$$

with $\tilde{\gamma}=\min\left(\kappa,\left(\gamma-2aM(1+\varepsilon)^2\right)(q-1)\right)>0$ for sufficiently small $\varepsilon>0$. We combine this with estimate (23) and with

LEMMA 4. – *There exists a positive constant $c(q)$ depending only on $q>2$ such that for all $P,Q\in\boldsymbol{R}^{N\times n}$ we have*

$$\int_0^1|P+sQ|^{q-2}\,ds\geq c(q)\left(|P|^{q-2}+|Q|^{q-2}\right)\ .$$

PROOF. – The function

$$f(x)=\frac{1}{1+x^{q-2}}\int_0^1|1-sx|^{q-2}\,ds$$

is positive and continuous for $x\in[0,\infty[$, and it satisfies

$$\lim_{x\to\infty}f(x)=\int_0^1 s^{q-2}\,ds=\frac{1}{q-1}\ .$$

Hence $c(q)=\inf_{[0,\infty[}f>0$. This immediately implies the result. □

LEMMA 5. – *For* $0<r<1$, *we have the limits*

$$Dw_m \to Dw \text{ in } L^2\big(B_r^+, \boldsymbol{R}^{N\times n}\big)\ , \tag{32}$$

$$\lambda_m^{(q-2)/q} Dw_m \to 0 \quad \text{in } L^q\big(B_r^+, \boldsymbol{R}^{N\times n}\big)\ (\text{for } q>2)\ . \tag{33}$$

PROOF. – We fix $0<r<1$, and we let $\eta\in C_c^\infty(B)$ be a cut-off function with $0\le\eta\le 1$, and $\eta=1$ on B_r. In the proof we shall make use of the fact that, by Lemma 3, the function w and its gradient Dw are bounded on $\operatorname{supp}\eta\cap B^+$. We now insert $\varphi=\eta\big(w_m-w+\big(\lambda_m^{-1}P_m\pi+S\big)\cdot z\big)\in W_0^{1,q}\big(B^+,\boldsymbol{R}^N\big)$ in (27), for which by (21)

$$\begin{aligned}\varphi &= \eta(w_m-w)+o(1)\ ,\\ D\varphi &= \eta(Dw_m-Dw)+(w_m-w)\otimes D\eta+o(1)\ .\end{aligned}$$

By Hypothesis 1, (10), (16), (17), (19) and (20), this easily yields

$$\begin{aligned}&\int_{B^+}\eta G(Y_m)\cdot(Dw_m,Dw_m)\,dz\\ &=\int_{B^+}\int_0^1 \eta A_P(Z_m)\cdot(Dw_m,Dw_m)\,ds\,dz\\ &\quad -a(q-1)(1+\varepsilon)^2\int_{B^+}\int_0^1 g\eta|r_mP_m\cdot z+r_m\lambda_m w_m|\\ &\qquad\qquad \times\big|(P_m+s\lambda_m Dw_m)E+F\big|^{q-2}|Dw_mE|^2\,ds\,dz\\ &\le\int_{B^+}\int_0^1 \eta A_P(Z_m)\cdot(Dw_m,Dw)\,ds\,dz+o(1)\ ,\end{aligned}$$

where g, E and F are evaluated at x_m+r_mz. Similar to the proof of Lemma 3, we have

$$\int_{B^+}\int_0^1 \eta A_P(Z_m)\cdot(Dw_m,Dw)\,ds\,dz\le\int_{B^+}\eta A_P(Z_0)\cdot(Dw_m,Dw)\,dz+o(1)\ .$$

By (20), we thus arrive at the key estimate

$$\begin{aligned}\limsup_{m\to\infty}\int_{B^+}\eta G(Y_m)\cdot(Dw_m,Dw_m)\,dz &\le\int_{B^+}\eta A_P(Z_0)\cdot(Dw,Dw)\,dz\\ &=\int_{B^+}\eta G(Y_0)\cdot(Dw,Dw)\,dz\ .\end{aligned} \tag{34}$$

According to (31) and the definitions of η and Y_m, we have

(35)
$$\gamma' \int_{B_r^+} (1+\lambda_m^{q-2}|Dw_m|^{q-2})|Dw_m - Dw|^2\,dz$$
$$\le \int_{B^+} \eta G(Y_m)\cdot(Dw_m - Dw, Dw_m - Dw)\,dz$$
$$\le \int_{B^+} \eta G(Y_m)\cdot(Dw_m, Dw_m)\,dz + \int_{B^+} \eta G(Y_0)\cdot(Dw, Dw)\,dz$$
$$- \int_{B^+} \eta\big(G(Y_0)\cdot(Dw_m, Dw) + G(Y_0)\cdot(Dw, Dw_m)\big)\,dz$$
$$+ \; c\int_{B^+} \big|G(Y_m) - G(Y_0)\big|(1+|Dw_m|)\,dz\ .$$

Since G is continuous and satisfies (30), there exists a continuous, bounded and concave function $\omega(t)$, with $\omega(0) = 0$, such that

$$\big|G(Y) - G(Y_0)\big| \le \omega(|Y - Y_0|^2)\big(1 + |P|^{q-2} + |Q|^{q-2}\big)$$

for all $Y = (x, u, v, P, Q) \in \mathfrak{Y}$. So we estimate the last term of (35) as

$$\int_{B^+} |G(Y_m) - G(Y_0)|(1+|Dw_m|)\,dz$$
$$\le c\int_{B^+} \omega(|Y_m - Y_0|^2)\big(1 + |Dw_m| + \lambda_m^{q-2}|Dw_m|^{q-1}\big)\,dz$$
$$\le c\omega^{1/2}\left(\fint_{B^+} |Y_m - Y_0|^2\,dz\right)\big(1 + \|Dw_m\|_{L^2}\big) + c\lambda_m^{(q-2)/q}\big(\lambda_m^{(q-2)/q}\|Dw_m\|_{L^q}\big)^{q-1} = o(1)\ .$$

Here we have once more applied (10), the Hölder and Jensen inequalities, the boundedness and concavity of $\omega(t)$, (29) and (16).

We now infer from (35), using (34) for the first and (20) for the third term, that

$$\gamma' \limsup_{m\to\infty} \int_{B_r^+} (1+\lambda_m^{q-2}|Dw_m|^{q-2})|Dw_m - Dw|^2\,dz$$
$$\le \limsup_{m\to\infty} \int_{B^+} \eta G(Y_m)\cdot(Dw_m, Dw_m)\,dz + \int_{B^+} \eta G(Y_0)\cdot(Dw, Dw)\,dz$$
$$- \lim_{m\to\infty} \int_{B^+} \eta\big(G(Y_0)\cdot(Dw_m, Dw) + G(Y_0)\cdot(Dw, Dw_m)\big)\,dz$$
$$\le (1 + 1 - 2)\int_{B^+} \eta G(Y_0)\cdot(Dw, Dw)\,dz = 0\ ,$$

and we conclude that

$$\lim_{m\to\infty}\int_{B_r^+}|Dw_m - Dw|^2\,dz = 0\ ,\quad \lim_{m\to\infty}\lambda_m^{q-2}\int_{B_r^+}|Dw_m|^{q-2}|Dw_m - Dw|^2\,dz = 0\ .$$

The last equation implies

$$\lim_{m\to\infty}\lambda_m^{q-2}\int_{B_r^+}|Dw_m|^q\,dz = 0\ ,$$

and we have shown that (32) and (33) hold. □

The proof of Proposition 1 is the same as the previous proof except for the following four major modifications (see also [11]):

(a) We replace every half-ball by its corresponding ball.

(b) We set $u_m = u_{x_m,r_m}$. Then the first equation of (13) is no longer valid, but instead $(w_m)_{0,1}= 0$, so we deduce (16) from (14) by the Poincaré inequality.

(c) We pass from (22) directly to (24).

(d) We set $\varphi = \eta(w_m - w)\in W_0^{1,q}(B,\boldsymbol{R}^N)$ in the proof of Lemma 5.

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