The Flow Associated to Weakly Differentiable Vector Fields: Recent Results and Open Problems

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_1_25_0>
The Flow Associated to Weakly Differentiable Vector Fields: Recent Results and Open Problems.

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Sunto. – In questa nota descriviamo alcuni recenti sviluppi della teoria dei flussi associati a campi vettoriali poco regolari rispetto alle variabili spaziali, ad esempio con regolarità di tipo Sobolev o BV. Dopo aver illustrato alcune applicazioni a leggi di conservazione e equazioni della fluidodinamica, diamo una presentazione di tipo assistomatico del problema, usando un linguaggio di tipo probabilistico ispirato dalla teoria di L.C. Young. Nella parte finale discutiamo dei risultati ancora più recenti sulla regolarità del flusso stesso rispetto alle variabili spaziali.

Summary. – In this note we describe some recent developments of the theory of flows associated to vector fields with a low regularity with respect to the spatial variables, for instance with a Sobolev or BV regularity. After the illustration of some applications of this theory to conservation laws and PDE’s in fluid dynamics, we give an axiomatic presentation of the problem, based on a probabilistic approach inspired by the work of L.C. Young. In the final part we discuss very recent results on the regularity of the flow itself with respect to the spatial variables.

1. – Motivation and description of the problem.

We are interested in solving the characteristic ODE for vector fields \( b(t, x) \) having a low regularity (e.g. Sobolev or BV) with respect to the spatial variables. These fields naturally show up in many PDE’s, for instance in fluid mechanics and conservation laws, where the regularizing effects of the PDE are very mild, or absent.

A classical example is the 2 – d incompressible Euler equation, in the vorticity formulation:

\[
\begin{align*}
\frac{d}{dt} \omega_t + \nabla_x \cdot (\mathbf{v}_t \omega_t) &= 0 \\
\omega_0 &= \bar{\omega} \\
\text{with } \mathbf{v}_t &= \text{curl } \omega_t.
\end{align*}
\]

(*) Conferenza tenuta a Torino il 5 luglio 2006 in occasione del “Joint Meeting S.I.M.A.I. - S.M.A.I. - S.M.F. - U.M.I. sotto gli auspici dell’E.M.S. Mathematics and its Applications”.
Here we can canonically represent \( v_t \) as \( \nabla^+ \tilde{\phi}_t \) for some scalar potential \( \tilde{\phi}_t \); the coupling between density and velocity then becomes \( \Delta \tilde{\phi}_t = \omega_t \), with constant Dirichlet boundary conditions, to account for the fact that \( v_t \) has no normal component. The fact that the transport equation with a divergence-free field preserves (and does not improve) the \( L^p \) regularity and elliptic regularity theory give

\[
\bar{\omega} \in L^p \implies \omega_t \in L^p \implies v_t \in W^{1,p}.
\]

Therefore, since all implications are optimal, we can’t expect, for initial vorticities \( \bar{\omega} \in L^p \), a velocity better than \( W^{1,p} \).

Informally, the problem we are interested to can be described as follows: we are given a vector field

\[
b(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d
\]

and we would like to select, for \( L^d \)-a.e. \( x \) (here and in the sequel \( L^d \) stands for the Lebesgue measure), a solution \( X(t, x) \) to

\[
(\text{ODE}) \begin{cases}
\dot{\gamma}(t) = b(t, \gamma(t)) \\
\gamma(0) = x
\end{cases}
\]

in such a way that this selection is stable with respect to smooth approximations of \( b \). At the same time, we want to give some axiomatic characterization of this selection.

Notice that this is typically a weaker requirement, compared to the statement

\[
\text{for } L^d \text{-a.e. } x, \text{ there is a unique solution to the ODE.}
\]

As a matter of fact, the latter uniqueness result is open in many cases in which the former holds (e.g. \( b \in W^{1,p} \) with \( p \in (d, \infty) \)), and known only under special assumptions (e.g. Lipschitz, one-sided Lipschitz, Osgood, ...).

The first paper where this viewpoint is introduced is DiPerna-Lions [63], where the problem is solved for fields with a Sobolev regularity w.r.t. the spatial variables and absolutely continuous divergence, with a bounded density with respect to \( L^d \).

More recently, there has been some progress on several aspects of this theory, mainly:

- fields with a \( BV \) regularity w.r.t. the spatial variables and absolutely continuous divergence bounded below (Ambrosio, [7]);
- the axiomatic characterization of the flow;
- the regularity of the flow \( X(t, x) \) w.r.t. \( x \);
- the validity of quantitative stability estimates.

In this lecture I am going to illustrate all these aspects, referring to the Lecture Notes [11] and [12] for a more detailed presentation of the state of the art on this subject.
2. – Two applications.

2.1 – The Keyfitz-Kranzer system.

Let us consider the Cauchy problem (studied in one space dimension by Keyfitz-Kranzer [75], and more recently by Bressan [36])

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (f_i(u)u) = 0, \quad u : \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}^n,$$

with the initial condition $u(\cdot, 0) = \bar{u} \in L^\infty$ and $f$ smooth. Formally, $\rho = |u|$ satisfies the scalar equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (f_i(\rho)\rho) = 0. \quad (1)$$

If we also assume that $\bar{\rho} = |\bar{u}| \in L^\infty \cap BV_{loc}$, then Kruzhkov’s theory [57] provides us with the entropy solution, with the same regularity.

In order to obtain a solution of (KK) we can formally decouple it, considering the scalar conservation law for $\rho$ and a system of transport equations

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (f_i(\rho)\theta) = 0,$$

with the initial condition $\theta(0, \cdot) = \bar{\theta}$ (here $u = \theta \rho$, $\bar{u} = \bar{\theta}|\bar{u}|$, $|\theta| = |\bar{\theta}| = 1$). Now, a formal solution to the system, fulfilling $|\theta| = 1$, is given by

$$\theta(t, x) := \bar{\theta}\left([X(t, \cdot)]^{-1}(x)\right),$$

where $X(t, x)$ is the forward flow associated to $b = f(\rho)$. This field is bounded, $BV_{loc}$, but its divergence is not absolutely continuous: indeed, its divergence is $-\partial_t \rho$, that is not better than a measure (due to the fact that even entropy solutions may develop discontinuities in a finite time).

However, we can use the PDE

$$\frac{d}{dt} \rho + \nabla \cdot (\rho f(\rho)) = 0,$$

to view $b = f(\rho)$ as a part of the divergence-free, autonomous and $BV_{loc}$ space-time field $B = (\rho, f(\rho))$. Using the flow associated to $B$ we can recover, via a reparameterization, a flow associated to $b$.

This is the “Lagrangian” strategy used in Ambrosio-De Lellis [8] to prove existence of solutions to (KK) when $\rho^{-1} \in L^\infty$. Later on, using a more “Eulerian” strategy, this has been improved in Ambrosio-Bouchut-De Lellis, [13], removing the assumption $\rho^{-1} \in L^\infty$. 
2.2 – The semi-geostrophic equation.

The semigeostrophic equations are a simplifies model of the atmosphere/ocean flows [54], described by the system of transport equations

\[
\begin{align*}
\frac{d}{dt}\partial_2 p + \mathbf{u} \cdot \nabla \partial_2 p &= -u_2 + \partial_1 p \\
\frac{d}{dt}\partial_1 p + \mathbf{u} \cdot \nabla \partial_1 p &= -u_1 - \partial_2 p \\
\frac{d}{dt}\partial_3 p + \mathbf{u} \cdot \nabla \partial_3 p &= 0.
\end{align*}
\]

(SGE)

Here \( \mathbf{u} \), the velocity, is a divergence-free field, \( p \) is the pressure and \( \rho := -\partial_3 p \) represents the density of the fluid. We consider the problem in \([0, T] \times \Omega \), with \( \Omega \) bounded and convex. Initial conditions are given on the pressure and a no-flux condition through \( \partial \Omega \) is imposed for all times.

Introducing the modified pressure \( P_t(x) := \rho_t(x) + (x_1^2 + x_2^2)/2 \), (SGE) can be written in a more compact form as

\[
\frac{d}{dt} \nabla P + \mathbf{u} \cdot \nabla^2 P = J(\nabla P - x) \quad \text{with} \quad J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Existence (and uniqueness) of solutions are still open for this problem. In [23] and [55], existence results have been obtained in the so-called dual coordinates, where we replace the physical variable \( x \) by \( X = \nabla P_t(x) \). Under this change of variables, and assuming \( P_t \) to be convex, the system becomes

\[
\frac{d}{dt} a_t + D_x \cdot (U_t(a_t)) = 0 \quad \text{with} \quad U_t(X) := J(X - \nabla P_t^*(X))
\]

with \( a_t := (\nabla P_t)_{\#}(L_\Omega) \) (here we denote by \( L_\Omega \) the restriction of \( L^d \) to \( \Omega \)). Indeed, for any test function \( \varphi \) we can use the fact that \( \mathbf{u} \) is divergence-free to obtain:

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \varphi \, da_t = \int_{\mathbb{R}^d} \nabla \varphi(\nabla P_t) \cdot \frac{d}{dt} \nabla P_t \, dx
\]

\[
= \int_{\mathbb{R}^d} \nabla \varphi(\nabla P_t) \cdot J(\nabla P_t - x) \, dx + \int_{\mathbb{R}^d} \nabla \varphi(\nabla P_t)\nabla^2 P_t \cdot \mathbf{u} \, dx
\]

\[
= \int_{\mathbb{R}^d} \nabla \varphi \cdot J(X - \nabla P_t^*) \, da_t + \int_{\mathbb{R}^d} \nabla(\varphi \circ \nabla P_t) \cdot \mathbf{u} \, dx
\]

\[
= \int_{\mathbb{R}^d} \nabla \varphi \cdot U_t \, da_t.
\]
Existence of a solution to (3) can be obtained by a suitable time discretization scheme. Now the question is: can we go back to the original physical variables? An important step forward has been achieved by Cullen and Feldman in [56], with the concept of Lagrangian solution of (SGE).

Taking into account that the vector field \( U_t(X) = J(X - \nabla P_t(X)) \) is BV, bounded and divergence-free, there is a well defined, stable and measure preserving flow \( X(t, X) = X_t(X) \) relative to \( U \). This flow can be carried back to the physical space with the transformation

\[
F_t(x) := \nabla P_t^c \circ X_t \circ \nabla P_0(x),
\]

thus defining maps \( F_t \) preserving \( L^d_\Omega \).

Using the stability theorem can also show that \( Z_t(x) := \nabla P_t(F_t(x)) \) solve, in the distributions sense, the Lagrangian form of (2), i.e.

\[
\frac{d}{dt} Z_t(x) = J(Z_t - F_t)
\]

This provides us with a sort of weak solution of (2), and it is still an open problem how the Eulerian form could be recovered.

2.3 – Some open problems.

One of the open problems is to extend the theory to BV fields \( b \) of bounded compression, namely such that the continuity equation starting from \( \bar{\rho} = 1 \) has a bounded solution \( \rho \). This requirement is weaker than boundedness of the negative part of the divergence, and it is related to a compactness conjecture recently made by Bressan. Indeed, examples of vector fields with bounded compression and unbounded divergence are provided exactly by the Keyfitz-Kranzer system, as the solution \( \rho \) of (1) is bounded, by the maximum principle in the context of entropy solutions to scalar conservation laws. This problem is discussed in Ambrosio-De Lellis-Maly [18], and solved for SBV velocity fields (namely, those fields \( b \) whose distributional derivative \( Db \) has no Cantor part, i.e. has only an absolutely continuous part and a part concentrated on hypersurfaces). Surprisingly, this finer regularity is not known even for entropy solutions of scalar conservation laws, except for the 1-dimensional case, where it is proved (using the Hopf-Lax formula) by Ambrosio-De Lellis in [9].

Another open problem is the regularity of solutions: this is obviously related to the regularity of the flow, that we will discuss in the final part of this lecture.
3. – An axiomatic characterization of the flow.

In this section we will present an axiomatic characterization of the flow that can be used, in many specific situations, to obtain existence, uniqueness and stability properties of the flow itself.

**Definition 1** [Flow associated to $b$.] We say that $X$ is a flow associated to $b$ if

- $X(\cdot, x)$ is an absolutely continuous solution of $\dot{\gamma}(t) = b(t, \gamma(t))$ in $[0, T]$, starting from $x$, for $L^d$-a.e. $x$; in other words, the integral identity

$$X(t, x) = x + \int_0^t b(\tau, X(\tau, x)) \, d\tau \quad \forall t \in [0, T]$$

holds for $L^d$-a.e. $x \in \mathbb{R}^d$;
- $X(t, \cdot)_# L^d \leq CL^d$ for some constant $C$ independent of $t$, i.e.

$$(5) \quad \int_{\mathbb{R}^d} \varphi(X(t, x)) \, dx \leq C \int_{\mathbb{R}^d} \varphi(x) \, dx \quad \forall \varphi \in C_c(\mathbb{R}^d), \ \varphi \geq 0.$$ 

This is slightly different from the DiPerna-Lions axiomatization: first, only upper bounds on the measure produced by the flow are imposed; second, the semigroup property

$$X(s - t, X(t, x)) = X(s, x) \quad L^d\text{-a.e., for all } 0 \leq t \leq s \leq T$$

is not an axiom, but a theorem (see Remark 31 of [12] for a detailed proof). In this connection, let us notice that at least the absolute continuity of $X(t, \cdot)_# L^d$ is needed to make the notion of flow invariant with respect to modifications of $b$ in $L^{d+1}$-negligible sets. Indeed, assume that $\{b \neq b'\}$ is $L^{d+1}$-negligible and that $X$ satisfies the first condition in the definition of flow, relative to $b$; then, if $X(t, \cdot)_# L^d$ is absolutely continuous for $L^1$-a.e. $t$, by Fubini’s theorem we obtain that

$$X(t, \cdot)_# L^d\left(\{y : b(t, y) \neq \bar{b}(t, y)\}\right) = 0$$

for $L^1$-a.e. $t$, hence

$$L^{d+1}\left(\{(t, x) : b(t, X(t, x)) \neq b'(t, X(t, x))\}\right) = 0.$$ 

Again Fubini’s theorem gives that

$$L^1\left(\{t : b(t, X(t, x)) \neq b'(t, X(t, x))\}\right) = 0$$

for $L^d$-a.e. $x$. Hence, for any such $x$, the first condition in the definition of the flow is satified with $b$ whenever it is satified with $b'$.

The difference between the two approaches in [63] and [7] can be better ex-
plained recalling the following two well-known facts:

- If $Y(t, s, x)$ is the characteristic at $t$ starting from $x$ at time $s$ (so that $X(t, x) = Y(t, 0, x)$), then $Y(t, \cdot, \cdot)$ formally solve the system of transport equations
  
  $$ \frac{d}{ds} Y(t, s, x) + b(s, x) \cdot \nabla Y(t, s, x) = 0 \quad \text{in} \ (0, T) \times \mathbb{R}^d. $$

- The density $\mu(t) = X(t, \cdot, \cdot) L^d$ produced by any choice of characteristics of the ODE solves the continuity equation
  
  $$ \frac{d}{dt} \mu(t) + \nabla \cdot (b(t, \cdot) \mu(t)) = 0 \quad \text{in} \ D' ((0, T) \times \mathbb{R}^d). $$

So, while in the DiPerna-Lions [63] approach one uses the well-posedness of the transport problem to show uniqueness of characteristics, in the approach developed in [7] only the continuity equation is involved, choosing suitable families of characteristics to produce different solutions of the continuity equation.

3.1 – The probabilistic viewpoint.

Inspired by L.C. Young theory [98], we consider measures in the space of continuous functions, and investigate the uniqueness and stability of the flow at this level. In the context of fluid mechanics, this is also related to the study of generalized geodesics in the space of measure-preserving maps, extensively studied by Brenier and (see [27], [28], [29], [30]) in connection with weak solutions to the incompressible Euler equation

$$ \frac{d}{dt} v + \nabla \cdot (v \otimes v) + \nabla p = 0, \quad \nabla \cdot v = 0. $$

Let us introduce the following notation: we shall denote by $\Gamma$ the space $C([0, T]; \mathbb{R}^d)$ and by $e_t : \Gamma \to \mathbb{R}^d$ are the evaluation maps,

$$ e_t(\gamma) := \gamma(t) \quad t \in [0, T]. $$

The following result (see Theorem 8.2.1 in [15], and also [11]), shows that there exists a close link between solutions of the continuity equation and measures in $\Gamma$ concentrated on solutions of the ODE. See also [71] for the stochastic counterpart of this result.

**Theorem 2** [Probabilistic representation]. – Let $b \in L^\infty$ and let $\eta \in P(\mathbb{R}^d \times \Gamma)$ be concentrated on the pairs $(x, \gamma)$ solutions to the ODE. Then $\mu(t) := (e_t)_{\#} \eta$ solve the continuity equation

$$ \frac{d}{dt} \mu(t) + \nabla \cdot (b(t, \cdot) \mu(t)) = 0 \quad \text{in} \ D' ((0, T) \times \mathbb{R}^d). $$
Conversely, any solution of the continuity equation above can be represented as \((e_t)_{\tilde{\mu}}\eta\) for a suitable \(\eta\).

This result allows several constructions (e.g. localizing with respect to initial and/or final position) that are not easy to perform at the “bottom” level of the continuity equation.

3.2 – Existence, uniqueness and stability properties.

In this section we illustrate the close relation between the well-posedness of the continuity equation and the existence and uniqueness of the flow. Very recently, these links have been further investigated by Figalli in [71] who found the stochastic counterpart of these results: well-posedness of the Fokker-Planck equation (corresponding, in the deterministic case, to the continuity equation) is tightly related to a kind of “generic uniqueness” of martingale solutions to the corresponding SDE (corresponding, in the deterministic case, to the ODE).

**Theorem 3.** – If the continuity equation has a unique solution in \(L^\infty(L^\infty)\) for any initial datum \(\tilde{\mu} \in L^\infty\), then the flow exists and is uniquely determined up to \(L^d\)-negligible sets.

**Sketch of proof.** Existence of solutions to the continuity equation implies, by the probabilistic representation, existence of a “generalized flow”, namely a family of measures \(\{\eta_x\}, x \in \mathbb{R}^d\), satisfying:

(a) \(\eta_x\) is concentrated on absolutely continuous solutions to the ODE starting from \(x\);
(b) \(\int_{\mathbb{R}^d} \int_\Gamma \varphi(e_t(y)) d\eta_x(y) d\tilde{\mu}(x) \leq C \int_{\mathbb{R}^d} \varphi(y) dy\) for any nonnegative \(\varphi \in C_c(\mathbb{R}^d)\).

The problem is to show that \(\eta_x\) are Dirac masses for \(\tilde{\mu}\)-a.e. \(x\). This result can be achieved using the well-posedness of the continuity equation, thanks to the following criterion.

**Theorem 4.** – Let \(\eta_x\) be a generalized flow. If the continuity equation with any initial datum \(\tilde{\mu} \leq v \in L^\infty\) is well posed in \(L^\infty(L^\infty)\), then \(\eta_x\) is a Dirac mass for \(v\)-a.e. \(x\).

The proof (see [7], or Lemma 15 and Theorem 18 in [11]) is achieved by a kind of dyadic decomposition of \(\mathbb{R}^d\): if \(\eta_x\) happen not to be Dirac masses in a set with positive \(v\)-measure, we find two solutions of the continuity equation with the same initial value \(\tilde{\mu} \leq v\), that become orthogonal before time \(T\). So, using
Theorem 4 and denoting $\eta_x = \delta_{X(.,x)}$, we have recovered our deterministic flow and proved Theorem 5.

Now, let us analyze some specific cases: it turns out that the continuity equation is well-posed in $L^\infty(L^\infty)$ (and therefore the whole previous theory applies) in the case of a Sobolev or $BV$ regularity with respect to the spatial variables, if one assumes uniform bounds on the divergence of $b$ and $|b|$ (under weaker assumptions one can still hope to obtain well-posedness in smaller function spaces for which still Theorem 4 applies).

The proof of the well-posedness in $L^\infty(L^\infty)$ is based on a careful regularization of the PDE

$$\frac{d}{dt} w_t + \nabla_x \cdot (bw_t) = 0$$

that allows to show that all solutions are renormalized, i.e.

$$\frac{d}{dt} w_t^2 + \nabla_x \cdot (bw_t^2) = -w_t^2 \nabla_x \cdot b.$$

Starting from the Sobolev case studied in [63], this required several technical refinements, by Lions [78], Bouchut [33], Colombini-Lerner [46], [47], and finally Ambrosio [7]. In the $BV$ case the key idea has been introduced in [33], of smoothing the PDE faster in the directions where $b$ is less regular, compared to those where $b$ is more regular (in the case of a jump discontinuity, for instance, the normal and the tangential directions to the jump set, respectively). The $BV$ regularity seems to be critical, at least for the techniques presently known.

The flow built by the previous procedure is also stable with respect to many approximations. Here I illustrate a typical result.

**Theorem 5.** Assume that $b_h \to b$ in $L^1_{loc}$, that

$$\sup_h \|b_h\|_\infty < \infty,$$

and assume that $X_h$ are flows relative to $b_h$ satisfying, for some constant $C$,

$$X_h(t, \cdot)_{\#} L^d \leq CL^d \quad \forall t \in [0, T], \; h \in \mathbb{N}.$$  

Then, if the continuity equation in $L^\infty$ with the velocity field $b$ is well posed, we have

$$\lim_{h \to \infty} \int_{R^d} \sup_{t \in [0, T]} |X_h(t, x) - X(t, x)| \, dx = 0 \quad \forall R, \; T > 0.$$  

Again, the proof is achieved by looking at the convergence of the Young
measures induced by $X_h$, whose limit induces a generalized flow, see in particular [7], or Theorem 21 and Lemma 22 of [11].

3.3 – Other extensions and open problems

- The extension of these results from Euclidean spaces to Riemannian manifolds is not difficult. More problematic seems to be the extension to Non-Euclidean geometries: for instance horizontal vector fields of the form

$$b(x) = \sum_{i=1}^{m} c_i(x)X_i(x)$$

where $(X_1, \ldots, X_m)$ satisfy Hörmander’s condition and $c_i$ are Sobolev functions, in the sense of Folland-Stein.

- In infinite dimensions (potentially interesting for applications to PDE’s), very little appears to be known: the only available results, by Bogachev-Wolf [26] (see also [51], [52], [53]) require exponential integrability both of the divergence and of the spatial derivative of the field $b$, w.r.t. a Gaussian measure $\gamma$.

4. – Regularity of the flow w.r.t. the spatial variables.

What can we say about the regularity of the flow $X(t, \cdot)$? Recently, Le Bris-Lions [76] proved, in the Sobolev case, the following differentiability property, that we call differentiability in measure

$$\lim_{\varepsilon \downarrow 0} \frac{X(t, x + \varepsilon y) - X(t, x)}{\varepsilon} = W(t, x, y) \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y. \quad (6)$$

Recall that a sequence $(f_n)$ is said to be locally convergent in measure to $f$ in a domain $\Omega$ if the measure of all sets $K \cap \{|f_n - f| > \varepsilon\}$ tends to 0 for all $\varepsilon > 0$ and $K \subset \Omega$ compact; equivalently if the truncated functions $\min\{1, |f_n - f|\}$ converge to 0 in $L^1_{\text{loc}}$. Notice that the differentiability property above is very weak for two reasons: first, an averaging with respect to the increment $y$ is involved; second, convergence occurs only in measure. This result raises in a natural way the following questions:

- What is the nature of $W$ ?
- Are there relations with other weak differentiability properties, as for instance the approximate differentiability (a concept deeply investigated in [70]) ?
• Can we infer from (6), as it happens for approximate differentiability, some kind of “Local Lipschitz” regularity of the flow?

Before collecting a few answers to the questions above, let us remark that the proof by Le Bris-Lions is based on the nice observation that the difference quotients

$$\left( X(t, x), \frac{X(t, x + \varepsilon y) - X(t, x)}{\varepsilon} \right)$$

are characteristics for the $2d$-dimensional velocity field

$$B_x(t, x, y) := \left( \frac{b(t, x)}{\varepsilon}, \frac{b(t, x + \varepsilon y) - b(t, x)}{\varepsilon} \right).$$

Therefore, a suitable extension of the theory of renormalized solutions for the limit field

$$B(t, x, y) := (b(t, x), \nabla_x b(t, x) y)$$

provides the result. Here Bouchut’s anisotropic commutator argument [33] can be used with success because, even though $B$ is only measurable with respect to $x$, this lack of regularity occurs for the last $d$ components, the ones corresponding to the $y$ variable.

The following result shows that, at least in the $W^{1,p}$ case, with $p > 1$, a “local” Lipschitz property of the flow holds. Here “local” should not be understood in the topological sense but, rather, in the measure-theoretic sense, as in Lusin’s theorem concerning measurable functions.

**Theorem 6** (A. Lecumberry-Maniglia, [16]). – Assume that $b \in L^1(W^{1,p}_{\text{loc}}) \cap L^\infty(L^\infty)$, with $p > 1$. Then, for any $R$, $\varepsilon > 0$ we can find $K \subset B_R \times [0, T]$ such that

$$L^{d+1}([0, T] \times B_R \setminus K) < \varepsilon \quad \text{and} \quad X|_K \text{ is a Lipschitz function}.$$

This result shows that the classical Cauchy-Lipschitz theory is much closer than expected to the “weak” theory: it suffices to remove sets of small measure. In particular all classical identities of the Cauchy-Lipschitz calculus hold, e.g.

$$X(t, \cdot) \#\rho L^d = \frac{\rho}{\det \nabla X(t, x)} \circ X(t, \cdot)^{-1} L^d.$$

In a joint work with Maly [17], I proved that, for a generic map $f$, the “Lusin-Lipschitz” property described above is stronger than differentiability in measure:

$$\lim_{\varepsilon \to 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon} = U(x, y) \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y.$$
In turn, differentiability in measure is equivalent to
\[
\frac{f(x + hy) - f(x) - M(x)y}{h} \to 0 \quad \text{locally in measure as } h \downarrow 0
\]
for all \( y \in \mathbb{R}^n \), for some matrix-valued Borel map \( M(x) \). Finally, \( M \) is related to \( U \) by
\[
M(x)y = U(x, y) \quad \text{for } L^{2d}\text{-a.e. } (x, y).
\]

Now, let us come back to the proof of Theorem 6: we formally know, by spatial differentiation of the ODE, that \( \frac{d}{dt} |\nabla X| \leq |\nabla b| |\nabla X| \), so that
\[
\log (1 + |\nabla X(t, x)|) \leq \int_0^t |\nabla b(X(s, x))| \, ds.
\]

Even though the weak theory is stable w.r.t. smooth approximations, we can’t extend the bound above in this form even if we integrate it in space: this is due to the fact that the concavity of the logarithm prevents the lower semicontinuity of
\[
g \mapsto \int \log (1 + |\nabla g|) \, dx.
\]

The key idea is then to replace the gradient by a difference quotient; after an integration and a change of variable, this leads to the estimate
\[
\int_{B_r} \log \left( 1 + r^{-d} \int_{B_{r(x)}} |X(t, y) - X(t, x)| \, dy \right) \leq C(d) \int_0^t \int_{B_{r(x)}} |\nabla b|^2 \, dsdy.
\]

These integral estimates on the difference quotients of \( X \) give, in particular, the LUSIN-LIPSCHITZ property. The quantity \( |\nabla f|^2 \) in the previous inequality is the maximal function of \( |\nabla f| \), defined by
\[
|\nabla f|^2(x) := \sup_{r \in (0,1)} \frac{1}{|O_d r|} \int_{B_r(x)} |\nabla f|(y) \, dy.
\]

In the proof of (7) we use the classical maximal inequality (valid at Lebesgue points)
\[
|f(x) - f(y)| \leq C(d) \left[ |\nabla f|^2(x) + |\nabla f|^2(y) \right] |x - y|
\]
and the fact that \( |\nabla f| \in L^p \), \( p > 1 \), implies \( |\nabla f|^2 \in L^p \). This inequality holds even for \( BV \) functions, but \( |\nabla f|^2 \) fails to be integrable in general, even when \( f \in W^{1,1} \). For this reason we presently don’t know whether these results can be extended up to the \( W^{1,1} \) or the \( BV \) case.

This approach has been further developed by CRIPIA-DE LELLIS [50], that
obtained:

- A quantitative Lusin-Lipschitz property;
- A quantitative stability (and uniqueness) property, of the form

$$\|X^1(t, x) - X^2(t, x)\|_{L^1(B_r)} \leq C \frac{1}{\log \|b_1 - b_2\|_{L^1(B_{2r}, \mu)}}.$$

Furthermore, the constant $C$ in the bounds above depends only on the compressibility constant of the flow (as defined by (5)) and not on pointwise bounds on the divergence of $b$. Therefore these estimates provide a new construction of the flow “by completion”, an approach completely independent of the theory of renormalized solutions.

REFERENCES


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Pervenuta in Redazione
il 30 ottobre 2006