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The Flow Associated to Weakly Differentiable Vector Fields: Recent Results and Open Problems.

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Sunto. – *In questa nota descriviamo alcuni recenti sviluppi della teoria dei flussi associati a campi vettoriali poco regolari rispetto alle variabili spaziali, ad esempio con regolarità di tipo Sobolev o BV. Dopo aver illustrato alcune applicazioni a leggi di conservazione e equazioni della fluidodinamica, diamo una presentazione di tipo assiomatico del problema, usando un linguaggio di tipo probabilistico ispirato dalla teoria di L.C. Young. Nella parte finale discutiamo dei risultati ancora più recenti sulla regolarità del flusso stesso rispetto alle variabili spaziali.*

Summary. – *In this note we describe some recent developments of the theory of flows associated to vector fields with a low regularity with respect to the spatial variables, for instance with a Sobolev or BV regularity. After the illustration of some applications of this theory to conservation laws and PDE's in fluid dynamics, we give an axiomatic presentation of the problem, based on a probabilistic approach inspired by the work of L.C. Young. In the final part we discuss very recent results on the regularity of the flow itself with respect to the spatial variables.*

1. – Motivation and description of the problem.

We are interested in solving the characteristic ODE for vector fields $\boldsymbol{b}(t,x)$ having a low regularity (*e.g.* Sobolev or BV) with respect to the spatial variables. These fields naturally show up in many PDE's, for instance in fluid mechanics and conservation laws, where the regularizing effects of the PDE are very mild, or absent.

A classical example is the $2-d$ incompressible EULER equation, in the vorticity formulation:

$$\begin{cases} \dfrac{d}{dt}\omega_t + \nabla_x \cdot (\boldsymbol{v}_t\omega_t) = 0 \\ \omega_0 = \bar{\omega} \end{cases} \qquad \text{with} \quad \boldsymbol{v}_t := \operatorname{curl}\omega_t.$$

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Here we can canonically represent $\boldsymbol{v}_t$ as $\nabla^{\perp}\phi_t$ for some scalar potential ϕ_t; the coupling between density and velocity then becomes $\Delta\phi_t = \omega_t$, with constant Dirichlet boundary conditions, to account for the fact that $\boldsymbol{v}_t$ has no normal component. The fact that the transport equation with a divergence-free field preserves (and does not improve) the L^p regularity and elliptic regularity theory give

$$\bar{\omega} \in L^p \quad \Longrightarrow \quad \omega_t \in L^p \quad \Longrightarrow \quad \boldsymbol{v}_t \in W^{1,p}.$$

Therefore, since all implications are optimal, we can't expect, for initial vorticities $\bar{\omega}$ in L^p, a velocity better than $W^{1,p}$.

Informally, the problem we are interested to can be described as follows: we are given a vector field

$$\boldsymbol{b}(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$$

and we would like to select, for $\mathcal{L}^d$-a.e. x (here and in the sequel $\mathcal{L}^d$ stands for the Lebesgue measure), a solution $\boldsymbol{X}(t,x)$ to

$$\text{(ODE)} \qquad \begin{cases} \dot{\gamma}(t) = \boldsymbol{b}(t,\gamma(t)) \\ \gamma(0) \;=\; x \end{cases}$$

in such a way that this selection is stable with respect to smooth approximations of $\boldsymbol{b}$. At the same time, we want to give some axiomatic characterization of this selection.

Notice that this is typically a weaker requirement, compared to the statement

for $\mathcal{L}^d$-a.e. x, there is a unique solution to the ODE.

As a matter of fact, the latter uniqueness result is *open* in many cases in which the former holds (e.g. $\boldsymbol{b} \in W^{1,p}$ with $p \in (d,\infty)$), and known only under special assumptions (e.g. LIPSCHITZ, one-sided LIPSCHITZ, OSGOOD, ...).

The first paper where this viewpoint is introduced is DIPERNA-LIONS [63], where the problem is solved for fields with a Sobolev regularity w.r.t. the spatial variables and absolutely continuous divergence, with a bounded density with respect to $\mathcal{L}^d$.

More recently, there has been some progress on several aspects of this theory, mainly:

- fields with a *BV* regularity w.r.t. the spatial variables and absolutely continuous divergence bounded below (AMBROSIO, [7]);
- the axiomatic characterization of the flow;
- the regularity of the flow $\boldsymbol{X}(t,x)$ w.r.t. x;
- the validity of quantitative stability estimates.

In this lecture I am going to illustrate all these aspects, referring to the Lecture Notes [11] and [12] for a more detailed presentation of the state of the art on this subject.

2. – Two applications.

2.1 – *The Keyfitz-Kranzer system.*

Let us consider the Cauchy problem (studied in one space dimension by KEYFITZ-KRANZER [75], and more recently by BRESSAN [36])

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\boldsymbol{f}_i(|u|)u) = 0, \qquad u : \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}^n,$$

with the initial condition $u(\cdot, 0) = \bar{u} \in L^\infty$ and $\boldsymbol{f}$ smooth. Formally, $\rho = |u|$ satisfies the *scalar* equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\boldsymbol{f}_i(\rho)\rho) = 0. \tag{1}$$

If we also assume that $\bar{\rho} = |\bar{u}| \in L^\infty \cap BV_{\mathrm{loc}}$, then KRUZKHOV's theory [57] provides us with the *entropy* solution, with the same regularity.

In order to obtain a solution of (KK) we can formally decouple it, considering the scalar conservation law for ρ and a *system* of transport equations

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\boldsymbol{f}_i(\rho)\theta) = 0$$

with the initial condition $\theta(0, \cdot) = \bar{\theta}$ (here $u = \theta\rho$, $\bar{u} = \bar{\theta}|\bar{u}|$, $|\theta| = |\bar{\theta}| = 1$). Now, a *formal* solution to the system, fulfilling $|\theta| = 1$, is given by

$$\theta(t, x) := \bar{\theta}\Big([X(t, \cdot)]^{-1}(x)\Big),$$

where $X(t, x)$ is the forward flow associated to $\boldsymbol{b} = \boldsymbol{f}(\rho)$. This field is bounded, BV_{loc}, but its divergence *is not* absolutely continuous: indeed, its divergence is $-\partial_t \rho$, that is not better than a measure (due to the fact that even entropy solubions may develop discontinuities in a finite time).

However, we can use the PDE

$$\frac{d}{dt}\rho + \nabla \cdot \big(\rho \boldsymbol{f}(\rho)\big) = 0,$$

to view $\boldsymbol{b} = \boldsymbol{f}(\rho)$ as a part of the divergence-free, autonomous and BV_{loc} space-time field $\boldsymbol{B} = (\rho, \boldsymbol{f}(\rho))$. Using the flow associated to $\boldsymbol{B}$ we can recover, via a reparameterization, a flow associated to $\boldsymbol{b}$.

This is the "Lagrangian" strategy used in AMBROSIO-DE LELLIS [8] to prove existence of solutions to (KK) when $\rho^{-1} \in L^\infty$. Later on, using a more "Eulerian" strategy, this has been improved in AMBROSIO-BOUCHUT-DE LELLIS. [13], removing the assumption $\rho^{-1} \in L^\infty$.

2.2 – *The semi-geostrophic equation.*

The semigeostrophic equations are a simplifies model of the atmosphere/ocean flows [54], described by the system of transport equations

$$
\text{(SGE)} \qquad \begin{cases} \dfrac{d}{dt}\partial_2 p + \boldsymbol{u}\cdot\nabla\partial_2 p = -\boldsymbol{u}_2 + \partial_1 p \\ \dfrac{d}{dt}\partial_1 p + \boldsymbol{u}\cdot\nabla\partial_1 p = -\boldsymbol{u}_1 - \partial_2 p \\ \dfrac{d}{dt}\partial_3 p + \boldsymbol{u}\cdot\nabla\partial_3 p = 0. \end{cases}
$$

Here $\boldsymbol{u}$, the velocity, is a divergence-free field, p is the pressure and $\rho := -\partial_3 p$ represents the density of the fluid. We consider the problem in $[0,T]\times\Omega$, with Ω bounded and convex. Initial conditions are given on the pressure and a no-flux condition through $\partial\Omega$ is imposed for all times.

Introducing the modified pressure $P_t(x) := p_t(x) + (x_1^2 + x_2^2)/2$, (SGE) can be written in a more compact form as

$$
(2) \qquad \frac{d}{dt}\nabla P + \boldsymbol{u}\cdot\nabla^2 P = J(\nabla P - x) \quad \text{with} \quad J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Existence (and uniqueness) of solutions are still open for this problem. In [23] and [55], existence results have been obtained in the so-called dual coordinates, where we replace the physical variable x by $X = \nabla P_t(x)$. Under this change of variables, *and assuming P_t to be convex*, the system becomes

$$
(3) \qquad \frac{d}{dt}a_t + D_x\cdot(\boldsymbol{U}_t a_t) = 0 \quad \text{with} \quad \boldsymbol{U}_t(X) := J\big(X - \nabla P_t^*(X)\big)
$$

with $a_t := (\nabla P_t)_\#(L_\Omega)$ (here we denote by L_Ω the restriction of L^d to Ω). Indeed, for any test function φ we can use the fact that $\boldsymbol{u}$ is divergence-free to obtain:

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}\varphi\,da_t &= \int_{\mathbb{R}^d}\nabla\varphi(\nabla P_t)\cdot\frac{d}{dt}\nabla P_t\,dx \\
&= \int_{\mathbb{R}^d}\nabla\varphi(\nabla P_t)\cdot J(\nabla P_t - x)\,dx + \int_{\mathbb{R}^d}\nabla\varphi(\nabla P_t)\nabla^2 P_t\cdot\boldsymbol{u}\,dx \\
&= \int_{\mathbb{R}^d}\nabla\varphi\cdot J(X - \nabla P_t^*)\,da_t + \int_{\mathbb{R}^d}\nabla(\varphi\circ\nabla P_t)\cdot\boldsymbol{u}\,dx \\
&= \int_{\mathbb{R}^d}\nabla\varphi\cdot\boldsymbol{U}_t\,da_t.
\end{aligned}
$$

Existence of a solution to (3) can be obtained by a suitable time discretization scheme. Now the question is: can we go back to the original physical variables ? An important step forward has been achieved by CULLEN and FELDMAN in [56], with the concept of *Lagrangian* solution of (SGE).

Taking into account that the vector field $\boldsymbol{U}_t(X) = J(X - \nabla P_t^*(X))$ is BV, bounded and divergence-free, there is a well defined, stable and measure preserving flow $\boldsymbol{X}(t, X) = \boldsymbol{X}_t(X)$ relative to $\boldsymbol{U}$. This flow can be carried back to the physical space with the transformation

$$F_t(x) := \nabla P_t^* \circ \boldsymbol{X}_t \circ \nabla P_0(x),$$

thus defining maps F_t preserving L_Ω^d.

Using the stability theorem can also show that $Z_t(x) := \nabla P_t(F_t(x))$ solve, in the distributions sense, the Lagrangian form of (2), i.e.

$$\frac{d}{dt} Z_t(x) = J(Z_t - F_t) \tag{4}$$

This provides us with a sort of weak solution of (2), and it is still an open problem how the Eulerian form could be recovered.

2.3 – *Some open problems.*

One of the open problems is to extend the theory to BV fields $\boldsymbol{b}$ of *bounded compression*, namely such that the continuity equation starting from $\bar{\rho} = 1$ has a bounded solution ρ. This requirement is weaker than boundedness of the negative part of the divergence, and it is related to a compactness conjecture recently made by BRESSAN. Indeed, examples of vector fields with bounded compression and unbounded divergence are provided exactly by the Keyfitz-Kranzer system, as the solution ρ of (1) is bounded, by the maximum principle in the context of entropy solutions to scalar conservation laws. This problem is discussed is discussed in AMBROSIO-DE LELLIS-MALY [18], and solved for SBV velocity fields (namely, those fields $\boldsymbol{b}$ whose distributional derivative $D\boldsymbol{b}$ has *no* Cantor part, i.e. has only an absolutely continuous part and a part concentrated on hypersurfaces). Surprisingly, this finer regularity is not known even for entropy solutions of scalar conservation laws, except for the 1-dimensional case, where it is proved (using the Hopf-Lax formula) by AMBROSIO-DE LELLIS in [9].

Another open problem is the regularity of solutions: this is obviously related to the regularity of the flow, that we will discuss in the final part of this lecture.

3. – An axiomatic characterization of the flow.

In this section we will present an axiomatic characterization of the flow that can be used, in many specific situations, to obtain existence, uniqueness and stability properties of the flow itself.

DEFINITION 1 [Flow associated to $\boldsymbol{b}$]. – *We say that $\boldsymbol{X}$ is a flow associated to $\boldsymbol{b}$ if*

- *$\boldsymbol{X}(\cdot, x)$ is an absolutely continuous solution of $\gamma'(t) = \boldsymbol{b}(t, \gamma(t))$ in $[0, T]$, starting from x, for L^d-a.e. x; in other words, the integral identity*

$$\boldsymbol{X}(t,x) = x + \int_0^t \boldsymbol{b}(\tau, \boldsymbol{X}(\tau, x))\, d\tau \qquad \forall t \in [0, T]$$

  holds for L^d-a.e. $x \in \mathbb{R}^d$;
- *$\boldsymbol{X}(t,\cdot)_\# L^d \leq C L^d$ for some constant C independent of t, i.e.*

$$\int_{\mathbb{R}^d} \varphi(\boldsymbol{X}(t,x))\, dx \leq C \int_{\mathbb{R}^d} \varphi(x)\, dx \qquad \forall \varphi \in C_c(\mathbb{R}^d),\ \varphi \geq 0. \tag{5}$$

This is slightly different from the DIPERNA-LIONS axiomatization: first, only *upper* bounds on the measure produced by the flow are imposed; second, the semigroup property

$$\boldsymbol{X}(s-t, \boldsymbol{X}(t,x)) = \boldsymbol{X}(s,x) \qquad L^d\text{-a.e., for all } 0 \leq\ t \leq\ s \leq\ T$$

is not an axiom, but a theorem (see Remark 31 of [12] for a detailed proof). In this connection, let us notice that at least the absolute continuity of $\boldsymbol{X}(t,\cdot)_\# L^d$ is needed to make the notion of flow *invariant* with respect to modifications of $\boldsymbol{b}$ in L^{d+1}-negligible sets. Indeed, assume that $\{\boldsymbol{b} \neq \boldsymbol{b}'\}$ is L^{d+1}-negligible and that $\boldsymbol{X}$ satifies the first condition in the definition of flow, relative to $\boldsymbol{b}$; then, if $\boldsymbol{X}(t,\cdot)_\# L^d$ is absolutely continuous for L^1-a.e. t, by Fubini's theorem we obtain that

$$\boldsymbol{X}(t,\cdot)_\# L^d \Big(\{y :\ \boldsymbol{b}(t,y) \neq \tilde{\boldsymbol{b}}(t,y)\} \Big) = 0$$

for L^1-a.e. t, hence

$$L^{d+1}\big(\{(t,x) :\ \boldsymbol{b}(t, \boldsymbol{X}(t,x)) \neq \boldsymbol{b}'(t, \boldsymbol{X}(t,x))\}\big) = 0.$$

Again Fubini's theorem gives that

$$L^1\big(\{t :\ \boldsymbol{b}(t, \boldsymbol{X}(t,x)) \neq \boldsymbol{b}'(t, \boldsymbol{X}(t,x))\}\big) = 0$$

for L^d-a.e. x. Hence, for any such x, the first condition in the definition of the flow is satified with $\boldsymbol{b}$ whenever it is satified with $\boldsymbol{b}'$.

The difference between the two approaches in [63] and [7] can be better explained recalling the following two well-known facts:

- If $\boldsymbol{Y}(t, s, x)$ is the characteristic at t starting from x at time s (so that $\boldsymbol{X}(t, x) = \boldsymbol{Y}(t, 0, x)$), then $\boldsymbol{Y}(t, \cdot, \cdot)$ formally solve the system of *transport equations*

$$\frac{d}{ds}\boldsymbol{Y}(t, s, x) + \boldsymbol{b}(s, x) \cdot \nabla_x \boldsymbol{Y}(t, s, x) = 0 \qquad \text{in } (0, T) \times \mathbb{R}^d.$$

- The density $\mu(t) = \boldsymbol{X}(t, \cdot)_{\#} L^d$ produced by *any* choice of characteristics of the ODE solves the *continuity equation*

$$\frac{d}{dt}\mu(t) + \nabla_x \cdot \big(\boldsymbol{b}(t, \cdot)\mu(t)\big) = 0 \qquad \text{in } \ \mathrm{D}'\ ((0, \mathrm{T}) \times \mathbb{R}^{\mathrm{d}}).$$

So, while in the DiPerna-Lions [63] approach one uses the well-posedness of the transport problem to show uniqueness of characteristics, in the approach developed in [7] only the continuity equation is involved, choosing suitable families of characteristics to produce different solutions of the continuity equation.

3.1 – *The probabilistic viewpoint.*

Inspired by L.C.Young theory [98], we consider measures in the space of continuous maps, and invextigate the uniqueness and stability of the flow at this level. In the context of fluid mechanics, this is also related to the study of generalized geodesics in the space of measure-preserving maps, extensively studied by Brenier and (see [27], [28], [29], [30]) in connection with weak solutions to the incompressible Euler equation

$$\frac{d}{dt}\boldsymbol{v} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{v} = 0.$$

Let us introduce the following notation: we shall denote by Γ the space $C([0, T]; \mathbb{R}^d)$ and by $e_t : \Gamma \to \mathbb{R}^d$ are the evaluation maps,

$$e_t(\gamma) := \gamma(t) \qquad t \in [0, T].$$

The following result (see Theorem 8.2.1 in [15], and also [11]), shows that there exists a close link between solutions of the continuity equation and measures in Γ concentrated on solutions of the ODE. See also [71] for the stochastic counterpart of this result.

Theorem 2 [Probabilistic representation]. – *Let $\boldsymbol{b} \in L^\infty$ and let $\boldsymbol{\eta} \in P(\mathbb{R}^d \times \Gamma)$ be concentrated on the pairs (x, γ) solutions to the ODE. Then $\mu(t) := (e_t)_{\#}\boldsymbol{\eta}$ solve the continuity equation*

$$\frac{d}{dt}\mu(t) + \nabla \cdot \big(\boldsymbol{b}(t, \cdot)\mu(t)\big) = 0 \qquad \textit{in } \ D'\ ((0, T) \times \mathbb{R}^d).$$

Conversely, any solution of the continuity equation above can be represented as $(e_t)_\# \eta$ *for a suitable* $\boldsymbol{\eta}$.

This result allows several constructions (e.g. localizing with respect to initial and/or final position) that are not easy to perform at the "bottom" level of the continuity equation.

3.2 – *Existence, uniqueness and stability properties.*

In this section we illustrate the close relation between the well-posedness of the continuity equation and the existence and uniqueness of the flow. Very recently, these links have been further investigated by FIGALLI in [71] who found the stochastic counterpart of these results: well-posedness of the Fokker-Planck equation (corresponding, in the deterministic case, to the continuity equation) is tightly related to a kind of "generic uniqueness" of martingale solutions to the corresponding SDE (corresponding, in the deterministic case, to the ODE).

THEOREM 3. – *If the continuity equation has a unique solution in* $L^\infty(L^\infty)$ *for any initial datum* $\bar\mu \in L^\infty$*, then the flow exists and is uniquey determined up to* L^d*-negligible sets.*

SKETCH OF PROOF. Existence of solutions to the continuity equation implies, by the probabilistic representation, existence of a "generalized flow", namely a family of measures $\{\boldsymbol{\eta}_x\}$, $x \in \mathbb{R}^d$, satisfying:

(a) $\boldsymbol{\eta}_x$ is concentrated on absolutely continuous solutions to the ODE starting from x;

(b) $\int_{\mathbb{R}^d} \int_\Gamma \varphi(e_t(\gamma))\, d\boldsymbol{\eta}_x(\gamma)\, d\bar\mu(x) \le C \int_{\mathbb{R}^d} \varphi(y)\, dy$ for any nonnegative $\varphi \in C_c(\mathbb{R}^d)$.

The problem is to show that $\boldsymbol{\eta}_x$ are Dirac masses for $\bar\mu$-a.e. x. This result can be achieved using the well-posedness of the continuity equation, thanks to the following criterion.

THEOREM 4. – *Let* $\boldsymbol{\eta}_x$ *be a generalized flow. If the continuity equation with any initial datum* $\bar\mu \le \upsilon \in L^\infty$ *is well posed in* $L^\infty(L^\infty)$*, then* $\boldsymbol{\eta}_x$ *is a Dirac mass for* υ*-a.e.* x.

The proof (see [7], or Lemma 15 and Theorem 18 in [11]) is achieved by a kind of dyadic decomposition of $\mathbb{R}^d$: if $\boldsymbol{\eta}_x$ happen not to be Dirac masses in a set with positive υ-measure, we find two solutions of the continuity equation with the same initial value $\bar\mu \le \upsilon$, that become orthogonal before time T. So, using

Theorem 4 and denoting $\boldsymbol{\eta}_x = \delta_{\boldsymbol{X}(\cdot,x)}$, we have recovered our deterministic flow and proved Theorem 5.

Now, let us analyze some specific cases: it turns out that the continuity equation is well-posed in $L^\infty(L^\infty)$ (and therefore the whole previous theory applies) in the case of a Sobolev or BV regularity with respect to the spatial variables, if one assumes uniform bounds on the divergence of $\boldsymbol{b}$ and $|\boldsymbol{b}|$ (under weaker assumptions one can still hope to obtain well-posedness in smaller function spaces for which still Theorem 4 applies).

The proof of the well-posedness in $L^\infty(L^\infty)$ is based on a careful regularization of the PDE

$$\frac{d}{dt}w_t + \nabla_x \cdot (\boldsymbol{b}w_t) = 0$$

that allows to show that all solutions are *renormalized*, i.e.

$$\frac{d}{dt}w_t^2 + \nabla_x \cdot (\boldsymbol{b}w_t^2) = -w_t^2 \nabla_x \cdot \boldsymbol{b}.$$

Starting from the Sobolev case studied in [63], this required several technical refinements, by LIONS [78], BOUCHUT [33], COLOMBINI-LERNER [46], [47], and finally AMBROSIO [7]. In the BV case the key idea has been introduced in [33], of smoothing the PDE faster in the directions where $\boldsymbol{b}$ is less regular, compared to those where $\boldsymbol{b}$ is more regular (in the case of a jump discontinuity, for instance, the normal and the tangential directions to the jump set, respectively). The BV regularity seems to be critical, at least for the techniques presently known.

The flow built by the previous procedure is also stable with respect to many approximations. Here I illustrate a typical result.

THEOREM 5. – *Assume that $\boldsymbol{b}_h \to \boldsymbol{b}$ in L^1_{loc}, that*

$$\sup_h \|\boldsymbol{b}_h\|_\infty < \infty,$$

and assume that $\boldsymbol{X}_h$ are flows relative to $\boldsymbol{b}_h$ satisfying, for some constant C,

$$\boldsymbol{X}_h(t,\cdot)_\# \mathscr{L}^d \le C\mathscr{L}^d \qquad \forall t \in [0,T],\ h \in \mathbb{N}.$$

Then, if the continuity equation in L^∞ with the velocity field $\boldsymbol{b}$ is well posed, we have

$$\lim_{h\to\infty} \int_{B_R} \sup_{t\in[0,T]} |\boldsymbol{X}_h(t,x) - \boldsymbol{X}(t,x)|\,dx = 0 \qquad \forall R,\ T > 0.$$

Again, the proof is achieved by looking at the convergence of the Young measures induced by $\boldsymbol{X}_h$, whose limit induces a generalized flow, see in particular [7], or Theorem 21 and Lemma 22 of [11].

3.3 – *Other extensions and open problems*

• The extension of these results from Euclidean spaces to Riemannian manifolds is not difficult. More problematic seems to be the extension to *Non-Euclidean* geometries: for instance horizontal vector fields of the form

$$\boldsymbol{b}(x) = \sum_{i=1}^{m} c_i(x)\boldsymbol{X}_i(x)$$

where $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$ satisfy HÖRMANDER's condition and c_i are Sobolev functions, in the sense of FOLLAND-STEIN.

• In infinite dimensions (potentially interesting for applications to PDE's), very little appears to be known: the only available results, by BOGACHEV-WOLF [26] (see also [51], [52], [53]) require *exponential* integrability both of the divergence and of the spatial derivative of the field $\boldsymbol{b}$, w.r.t. a Gaussian measure γ.

4. – Regularity of the flow w.r.t. the spatial variables.

What can we say about the regularity of the flow $\boldsymbol{X}(t,\cdot)$? Recently, LE BRIS-LIONS [76] proved, in the Sobolev case, the following differentiability property, that we call *differentiability in measure*

$$\lim_{\varepsilon\downarrow 0} \frac{\boldsymbol{X}(t, x+\varepsilon y) - \boldsymbol{X}(t,x)}{\varepsilon} = \boldsymbol{W}(t,x,y) \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y. \tag{6}$$

Recall that a sequence (f_n) is said to be locally convergent in measure to f in a domain Ω if the measure of all sets $K \cap \{|f_n - f| > \varepsilon\}$ tends to 0 for all $\varepsilon > 0$ and $K \subset \Omega$ compact; equivalently if the truncated functions $\min\{1, |f_n - f|\}$ converge to 0 in L^1_{loc}. Notice that the dfferentiability property above is very weak for two reasons: first, an averaging with respect to the increment y is involved; second, convergence occurs only in measure. This result raises in a natural way the following questions:

- What is the nature of $\boldsymbol{W}$?
- Are there relations with other weak differentiability properties, as for instance the approximate differentiability (a concept deeply investigated in [70]) ?

- Can we infer from (6), as it happens for approximate differentiability, some kind of "Local Lipschitz" regularity of the flow ?

Before collecting a few answers to the questions above, let us remark that the proof by LE BRIS-LIONS is based on the nice observation that the difference quotients

$$\left(\boldsymbol{X}(t,x), \frac{\boldsymbol{X}(t,x+\varepsilon y)-\boldsymbol{X}(t,x)}{\varepsilon}\right)$$

are characteristics for the $2d$-dimensional velocity field

$$\boldsymbol{B}_\varepsilon(t,x,y) := \left(\boldsymbol{b}(t,x), \frac{\boldsymbol{b}(t,x+\varepsilon y)-\boldsymbol{b}(t,x)}{\varepsilon}\right).$$

Therefore, a suitable extension of the theory of renormalized solutions for the limit field

$$\boldsymbol{B}(t,x,y) := (\boldsymbol{b}(t,x), \nabla_x \boldsymbol{b}(t,x)y)$$

provides the result. Here BOUCHUT's anisotropic commutator argument [33] can be used with success because, even though $\boldsymbol{B}$ is oly measurable with respect to x, this lack of regularity occurs for the last d components, the ones corresponding to the y variable.

The following result shows that, at least in the $W^{1,p}$ case, with $p > 1$, a "local" Lipschitz porperty of the flow holds. Here "local" should not be understood in the topological sense but, rather, in the measure-theoretic sense, as in Lusin's theorem concerning measurable functions.

THEOREM 6 (A.-Lecumberry-Maniglia, [16]). – *Assume that $\boldsymbol{b} \in L^1(W^{1,p}_{\rm loc}) \cap L^\infty(L^\infty)$, with $p > 1$. Then, for any R, $\varepsilon > 0$ we can find $K \subset B_R \times [0,T]$ such that*

$$L^{d+1}([0,T]\times B_R \setminus K) < \varepsilon \quad \textit{and} \quad \boldsymbol{X}|_K \textit{ is a Lipschitz function.}$$

This result shows that the classical CAUCHY-LIPSCHITZ theory is much closer than expected to the "weak" theory: it suffices to remove sets of small measure. In particular all classical identities of the CAUCHY-LIPSCHITZ calculus hold, e.g.

$$\boldsymbol{X}(t,\cdot)_{\#}\rho L^d = \frac{\rho}{\det \nabla \boldsymbol{X}(t,x)} \circ \boldsymbol{X}(t,\cdot)^{-1} L^d.$$

In a joint work with MALY [17], I proved that, for a generic map f, the "LUSIN-LIPSCHITZ" property described above is stronger than differentiability in measure:

$$\lim_{\varepsilon\downarrow 0} \frac{f(x+\varepsilon y)-f(x)}{\varepsilon} = \boldsymbol{U}(x,y) \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y.$$

In turn, differentiability in measure is equivalent to

$$\frac{f(x+hy)-f(x)-\boldsymbol{M}(x)y}{h} \to 0 \quad \text{locally in measure as } h \downarrow 0$$

for all $y \in \mathbb{R}^n$, for some matrix-valued Borel map $\boldsymbol{M}(x)$. Finally, $\boldsymbol{M}$ is related to $\boldsymbol{U}$ by

$$\boldsymbol{M}(x)y = \boldsymbol{U}(x,y) \qquad \text{for } L^{2d}\text{-a.e. } (x,y).$$

Now, let us come back to the proof of Theorem 6: we formally know, by spatial differentiation of the ODE, that $\frac{d}{dt}|\nabla \boldsymbol{X}| \leq |\nabla \boldsymbol{b}||\nabla \boldsymbol{X}|$, so that

$$\log\left(1+|\nabla \boldsymbol{X}(t,x)|\right) \leq \int_0^t |\nabla \boldsymbol{b}(\boldsymbol{X}(s,x))|\, ds.$$

Even though the weak theory is stable w.r.t. smooth approximations, we can't extend the bound above in this form even if we integrate it in space: this is due to the fact that the concavity of the logarithm prevents the lower semicontinuity of

$$g \mapsto \int \log(1+|\nabla g|)\, dx.$$

The key idea is then to replace the gradient by a difference quotient; after an integration and a change of variable, this leads to the estimate

$$\int_{B_R} \log\left(1 + r^{-d}\int_{B_r(x)} \frac{|\boldsymbol{X}(t,y)-\boldsymbol{X}(t,x)|}{r}\, dy\right) \leq C(d)\int_0^t \int_{B_{R+Ct}} |\nabla \boldsymbol{b}|^{\sharp}\, ds dy. \tag{7}$$

These integral estimates on the difference quotients of $\boldsymbol{X}$ give, in particular, the Lusin-Lipschitz property. The quantity $|\nabla f|^{\sharp}$ in the previous inequality is the maximal function of $|\nabla f|$, defined by

$$|\nabla f|^{\sharp}(x) := \sup_{r\in(0,1)} \frac{1}{\omega_d r^d}\int_{B_r(x)} |\nabla f|(y)\, dy.$$

In the proof of (7) we use the classical maximal inequality (valid at Lebesgue points)

$$|f(x)-f(y)| \leq C(d)\Big[|\nabla f|^{\sharp}(x) + |\nabla f|^{\sharp}(y)\Big]|x-y|$$

and the fact that $|\nabla f| \in L^p$, $p>1$, implies $|\nabla f|^{\sharp} \in L^p$. This inequality holds even for BV functions, but $|\nabla f|^{\sharp}$ fails to be integrable in general, even when $f \in W^{1,1}$. For this reason we presently don't know whether these results can be extended up to the $W^{1,1}$ or the BV case.

This approach has been further developed by Crippa-De Lellis [50], that obtained:

- A quantitative Lusin-Lipschitz property;
- A quantitative stability (and uniqueness) property, of the form

$$\|\boldsymbol{X}^1(t,x) - \boldsymbol{X}^2(t,x)\|_{L^1(B_R)} \leq C \frac{1}{\log\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1(B_{R+Vt})}}.$$

Furthermore, the constant C in the bounds above depends only on the compressibility constant of the flow (as defined by (5)) and not on pointwise bounds on the divergence of $\boldsymbol{b}$. Therefore these estimates provide a new construction of the flow "by completion", an approach completely independent of the theory of renormalized solutions.

REFERENCES

[1] M. AIZENMAN, *On vector fields as generators of flows: a counterexample to Nelson's conjecture*, Ann. Math., **107** (1978), 287-296.

[2] G. ALBERTI, *Rank-one properties for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A, **123** (1993), 239-274.

[3] G. ALBERTI - L. AMBROSIO, *A geometric approach to monotone functions in $\mathbb{R}^n$*, Math. Z., **230** (1999), 259-316.

[4] G. ALBERTI - S.MÜLLER, *A new approach to variational problems with multiple scales*, Comm. Pure Appl. Math., **54** (2001), 761-825.

[5] F. J. ALMGREN, *The theory of varifolds - A variational calculus in the large,* Princeton University Press, 1972.

[6] L. AMBROSIO - N. FUSCO - D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, 2000.

[7] L. AMBROSIO, *Transport equation and Cauchy problem for BV vector fields*, Inventiones Mathematicae, **158** (2004), 227-260.

[8] L. AMBROSIO - C. DE LELLIS, *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions*, International Mathematical Research Notices, **41** (2003), 2205-2220.

[9] L. AMBROSIO - C. DE LELLIS, *A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton-Jacobi equations*, Journal of Hyperbolic Differential Equations, **1** (4) (2004), 813-826.

[10] L. AMBROSIO, *Lecture notes on transport equation and Cauchy problem for BV vector fields and applications*, Preprint, 2004 (available at http://cvgmt.sns.it).

[11] L. AMBROSIO, *Lecture notes on transport equation and Cauchy problem for non-smooth vector fields and applications*, Preprint, 2005 (available at http://cvgmt.sns.it).

[12] L. AMBROSIO - G. CRIPPA, *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields*, Preprint, 2006 (available at http://cvgmt.sns.it).

[13] L. AMBROSIO - F. BOUCHUT - C. DE LELLIS, *Well-posedness for a class of hyperbolic systems of conservation laws in several space dimensions*, Comm. PDE, **29** (2004), 1635-1651.

[14] L. AMBROSIO - G. CRIPPA - S. MANIGLIA, *Traces and fine properties of a BD class of vector fields and applications*, Ann. Sci. Toulouse, **XIV** (4) (2005), 527-561.
[15] L. AMBROSIO - N. GIGLI - G. SAVARÉ, *Gradient flows in metric spaces and in the Wasserstein space of probability measures*, Lectures in Mathematics, ETH Zurich, Birkhäuser, 2005.
[16] L. AMBROSIO - M. LECUMBERRY - S. MANIGLIA, *Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow.* Rendiconti del Seminario Fisico Matematico di Padova, **114** (2005), 29-50.
[17] L. AMBROSIO - J. MALÝ, *Very weak notions of differentiability*, Preprint, 2005 (available at http://cvgmt.sns.it).
[18] L. AMBROSIO - C. DE LELLIS - J. MALÝ, *On the chain rule for the divergence of BV like vector fields: applications, partial results, open problems*, Preprint, 2005 (available at http://cvgmt. sns. it).
[19] L. AMBROSIO - S. LISINI - G. SAVARÉ, *Stability of flows associated to gradient vector fields and convergence of iterated transport maps*, Preprint, 2005 (available at http:// cvgmt. sns. it).
[20] E. J. BALDER, *New fundamentals of Young measure convergence*, CRC Res. Notes in Math. **411**, 2001.
[21] V. BANGERT, *Minimal measures and minimizing closed normal one-currents*, Geom. funct. anal., **9** (1999), 413-427.
[22] J. BALL - R. JAMES, *Fine phase mixtures as minimizers of energy*, Arch. Rat. Mech. Anal., **100** (1987), 13-52.
[23] J.-D. BENAMOU - Y. BRENIER, *Weak solutions for the semigeostrophic equation formulated as a couples Monge-Ampere transport problem*, SIAM J. Appl. Math., **58** (1998), 1450-1461.
[24] P. BERNARD - B. BUFFONI, *Optimal mass transportation and Mather theory*, Preprint, 2004.
[25] M. BERNOT - V. CASELLES - J. M. MOREL, *Traffic plans*, Preprint, 2004.
[26] V. BOGACHEV - E. M. WOLF, *Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions*, J. Funct. Anal., **167** (1999), 1-68.
[27] Y. BRENIER, *The least action principle and the related concept of generalized flows for incompressible perfect fluids*, J. Amer. Mat. Soc., **2** (1989), 225-255.
[28] Y. BRENIER, *The dual least action problem for an ideal, incompressible fluid*, Arch. Rational Mech. Anal., **122** (1993), 323-351.
[29] Y. BRENIER, *A homogenized model for vortex sheets*, Arch. Rational Mech. Anal., **138** (1997), 319-353.
[30] Y. BRENIER, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations*, Comm. Pure Appl. Math., **52** (1999), 411-452.
[31] F. BOUCHUT - F. JAMES, *One dimensional transport equation with discontinuous coefficients*, Nonlinear Analysis, **32** (1998), 891-933.
[32] F. BOUCHUT - F. GOLSE - M. PULVIRENTI, *Kinetic equations and asymptotic theory*, Series in Appl. Math., Gauthiers-Villars, 2000.
[33] F. BOUCHUT, *Renormalized solutions to the Vlasov equation with coefficients of bounded variation*, Arch. Rational Mech. Anal., **157** (2001), 75-90.
[34] F. BOUCHUT - F. JAMES - S. MANCINI, *Uniqueness and weak stability for multidimensional transport equations with one-sided Lipschitz coefficients*, Annali Scuola Normale Superiore, Ser. 5, **4** (2005), 1-25.
[35] F. BOUCHUT - G. CRIPPA, *On the relations between uniqueness for the Cauchy problem and existence of smooth approximations for linear transport equations*, Preprint, 2006 (available at http://cvgmt.sns.it).

[36] A. BRESSAN, *An ill posed Cauchy problem for a hyperbolic system in two space dimensions*, Rend. Sem. Mat. Univ. Padova, **110** (2003), 103-117.
[37] H. BREZIS, *Analyse fonctionnelle. Théorie et applications*, Masson, Paris, 1983.
[38] L. A. CAFFARELLI, *Some regularity properties of solutions of Monge Ampère equation*, Comm. Pure Appl. Math., **44** (1991), 965-969.
[39] L. A. CAFFARELLI, *Boundary regularity of maps with convex potentials*, Comm. Pure Appl. Math., **45** (1992), 1141-1151.
[40] L. A. CAFFARELLI, *The regularity of mappings with a convex potential*, J. Amer. Math. Soc., **5** (1992), 99-104.
[41] L. A. CAFFARELLI, *Boundary regularity of maps with convex potentials*, Ann. of Math., **144** (1996), 453-496.
[42] I. CAPUZZO DOLCETTA - B. PERTHAME, *On some analogy between different approaches to first order PDE's with nonsmooth coefficients*, Adv. Math. Sci Appl., **6** (1996), 689-703.
[43] A. CELLINA, *On uniqueness almost everywhere for monotonic differential inclusions*, Nonlinear Analysis, TMA, **25** (1995), 899-903.
[44] A. CELLINA - M. VORNICESCU, *On gradient flows*, Journal of Differential Equations, **145** (1998), 489-501.
[45] F. COLOMBINI - G. CRIPPA - J. RAUCH, *A note on two dimensional transport with bounded divergence*, Accepted by Comm. Partial Differential Equations, 2005.
[46] F. COLOMBINI - N. LERNER, *Uniqueness of continuous solutions for BV vector fields*, Duke Math. J., **111** (2002), 357-384.
[47] F. COLOMBINI - N. LERNER, *Uniqueness of L^∞ solutions for a class of conormal BV vector fields*, Contemp. Math. **368** (2005), 133-156.
[48] F. COLOMBINI - T. LUO - J. RAUCH, *Neraly Lipschitzean diverge free transport propagates neither continuity nor BV regularity*, Commun. Math. Sci., **2** (2004), 207-212.
[49] G. CRIPPA - C. DE LELLIS, *Oscillatory solutions to transport equations*, Preprint, 2005 (available at http://cvgmt. sns. it).
[50] G. CRIPPA - C. DE LELLIS, *Estimates for transport equations and regularity of the DiPerna-Lions flow*, Preprint, 2006.
[51] A. B. CRUZEIRO, *Équations différentielles ordinaires: non explosion et mesures quasi-invariantes*, J. Funct. Anal., **54** (1983), 193-205.
[52] A. B. CRUZEIRO, *Équations différentielles sur l'espace de Wiener et formules de Cameron-Martin non linéaires*, J. Funct. Anal., **54** (1983), 206-227.
[53] A. B. CRUZEIRO, *Unicité de solutions d'équations différentielles sur l'espace de Wiener*, J. Funct. Anal., **58** (1984), 335-347.
[54] M. CULLEN, *On the accuracy of the semi-geostrophic approximation*, Quart. J. Roy. Metereol. Soc., **126** (2000), 1099-1115.
[55] M. CULLEN - W. GANGBO, *A variational approach for the 2-dimensional semi-geostrophic shallow water equations*, Arch. Rational Mech. Anal., **156** (2001), 241-273.
[56] M. CULLEN - M. FELDMAN, *Lagrangian solutions of semigeostrophic equations in physical space*, Preprint, 2003.
[57] C. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, Springer Verlag, 2000.
[58] C. DE LELLIS, *Blow-up of the BV norm in the multidimensional Keyfitz and Kranzer system*, Duke Math. J., **127** (2004), 313-339.
[59] L. DE PASCALE - M. S. GELLI - L. GRANIERI, *Minimal measures, one-dimensional currents and the Monge-Kantorovich problem*, Preprint, 2004 (available at http://cvgmt.sns.it).

[60] N. DEPAUW, *Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan*, C. R. Math. Sci. Acad. Paris, **337** (2003), 249-252.
[61] N. G. DE BRUIJN, *On almost additive functions*, Colloq. Math. **15** (1966), 59-63.
[62] R. J. DIPERNA, *Measure-valued solutions to conservation laws*. Arch. Rational Mech. Anal., 88 (1985), 223-270.
[63] R. J. DI PERNA - P. L. LIONS, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., **98** (1989), 511-547.
[64] R. J. DI PERNA - P. L. LIONS, *On the Cauchy problem for the Boltzmann equation: global existence and weak stability*, Ann. of Math., **130** (1989), 312-366.
[65] L. C. EVANS - R. F. GARIEPY, *Lecture notes on measure theory and fine properties of functions,* CRC Press, 1992.
[66] L. C. EVANS, *Partial Differential Equations*, Graduate studies in Mathematics, **19** (1998), American Mathematical Society.
[67] L. C. EVANS, *Partial Differential Equations and Monge-Kantorovich Mass Transfer*, Current Developments in Mathematics, 1997, 65-126.
[68] L. C. EVANS - W. GANGBO, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Memoirs AMS, **653**, 1999.
[69] L. C. EVANS - W. GANGBO - O. SAVIN, *Nonlinear heat flows and diffeomorphisms*, Preprint, 2004.
[70] H. FEDERER, *Geometric measure theory*, Springer, 1969.
[71] A. FIGALLI, *Existence and uniqueness of martingale solutions for SDE with rough or degenerate coefficients*, Preprint, 2006, available at http://cvgmt. sns. it.
[72] M. HAURAY, *On Liouville transport equation with potential in* $BV_{\rm loc}$, Comm. in PDE, **29** (2004), 207-217.
[73] M. HAURAY, *On two-dimensional Hamiltonian transport equations with* $L^p_{\rm loc}$ *coefficients*, Ann. IHP Nonlinear Anal. Non Linéaire, **20** (2003), 625-644.
[74] W. B. JURKAT, *On Cauchy's functional equation*, Proc. Amer. Math. Soc., **16** (1965), 683-686.
[75] B. L. KEYFITZ - H. C. KRANZER, *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rational Mech. Anal. **1980**, *72*, 219-241.
[76] C. LE BRIS - P. L. LIONS, *Renormalized solutions of some transport equations with partially* $W^{1,1}$ *velocities and applications*, Annali di Matematica, **183** (2004), 97-130.
[77] N. LERNER, *Transport equations with partially BV velocities*, Preprint, 2004.
[78] P. L. LIONS, *Sur les équations différentielles ordinaires et les équations de transport*, C. R. Acad. Sci. Paris Sér. I, **326** (1998), 833-838.
[79] P. L. LIONS, *Mathematical topics in fluid mechanics, Vol. I: incompressible models*, Oxford Lecture Series in Mathematics and its applications, **3** (1996), Oxford University Press.
[80] P. L. LIONS, *Mathematical topics in fluid mechanics, Vol. II: compressible models*, Oxford Lecture Series in Mathematics and its applications, **10** (1998), Oxford University Press.
[81] J. LOTT - C. VILLANI, *Weak curvature conditions and Poincaré inequalities*, Preprint, 2005.
[82] S. MANIGLIA, *Probabilistic representation and uniqueness results for measure-valued solutions of transport equations*, Preprint, 2005.
[83] J. N. MATHER, *Minimal measures*, Comment. Math. Helv., **64** (1989), 375-394.
[84] J. N. MATHER, *Action minimizing invariant measures for positive definite Lagrangian systems*, Math. Z., **207** (1991), 169-207.
[85] E. Y. PANOV, *On strong precompactness of bounded sets of measure-valued solutions of a first order quasilinear equation*, Math. Sb., **186** (1995), 729-740.

[86] G. PETROVA - B. POPOV, *Linear transport equation with discontinuous coefficients*, Comm. PDE, **24** (1999), 1849-1873.

[87] F. POUPAUD - M. RASCLE, *Measure solutions to the liner multidimensional transport equation with non-smooth coefficients*, Comm. PDE, **22** (1997), 337-358.

[88] A. PRATELLI, *Equivalence between some definitions for the optimal transport problem and for the transport density on manifolds*, preprint, 2003, to appear on Ann. Mat. Pura Appl (available at http://cvgmt. sns. it).

[89] S. K. SMIRNOV, *Decomposition of solenoidal vector charges into elementary solenoids and the structure of normal one-dimensional currents*, St. Petersburg Math. J., **5** (1994), 841-867.

[90] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[91] L. TARTAR, *Compensated compactness and applications to partial differential equations*, Research Notes in Mathematics, Nonlinear Analysis and Mechanics, ed. R. J. Knops, vol. 4, Pitman Press, New York, 1979, 136-211.

[92] R. TEMAM, *Problémes mathématiques en plasticité*, Gauthier-Villars, Paris, 1983.

[93] J. I. E. URBAS, *Global Hölder estimates for equations of Monge-Ampère type*, Invent. Math., **91** (1988), 1-29.

[94] J. I. E. URBAS, *Regularity of generalized solutions of Monge-Ampère equations*, Math. Z., **197** (1988), 365-393.

[95] A. VASSEUR, *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal., **160** (2001), 181-193.

[96] C. VILLANI, *Topics in mass transportation*, Graduate Studies in Mathematics, **58** (2004), American Mathematical Society.

[97] C. VILLANI, *Optimal transport: old and new*, Lecture Notes of the 2005 Saint-Flour Summer school.

[98] L. C. YOUNG, *Lectures on the calculus of variations and optimal control theory,* Saunders, 1969.

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