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The Energy Density of Non Simple Materials Grade Two Thin Films via a Young Measure Approach.

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Sunto. – *Tecniche di riduzione dimensionale vengono adoperate al fine di descrivere l'energia di film sottili costituiti da materiali non semplici di grado due. Il rilassamento e la Γ convergenza conducono ad un limite definito su un opportuno spazio di misure di Young bidimensionali. La "deformazione" relativa al modello limite è consistente con la teoria di Cosserat.*

Summary. – *Dimension reduction is used to derive the energy of non simple materials grade two thin films. Relaxation and Γ convergence lead to a limit defined on a suitable space of bi-dimensional Young measures. The underlying "deformation" in the limit model takes into account the Cosserat theory.*

1. – Introduction.

In recent years a wide literature has been devoted to the study of martensitic thin films and materials of grade two because of their many applications. For a survey on thin structures in elasticity we refer to [14], [17], [24] and to the monographs of [2], [13]. (Non-simple materials of grade two were first introduced in [41], [42] and the theory developed by many authors is collected in [15]).

Dimension reduction through a Γ-convergence approach has been first used in [1] in the context of thin rods. Later Γ-convergence arguments in the $3D$-$2D$ setting have been adopted by Le Dret and Raoult in [33] and [34]. Finally we mention [7] (among the wide literature on the topic) where the $3D$-$2D$ reduction via Γ-convergence for inhomogeneous thin films with oscillating boundaries and for optimal design in the non linear elasticity framework has been studied. It is also worthwhile to mention [22] and [23] where, in the thin films analysis, the authors take into account geometric rigidity.

Analogous approaches have been used also for thin multidomains in order to derive junction conditions in the limit (for instance see [26] in the nonlinear setting, [27] in the context of linearized elasticity, [28] and [25] in the framework of non simple materials).

Very recently, in the context of thin structures a growing attention has been devoted to the bulk interfacial energies especially in order to derive the Cosserat theory for the equilibrium configuration. We refer to [6] and [40] where a convex bulk interfacial energy is added to the usual free energy of the film (i.e. an energy of the kind $k\int_{\Omega_\varepsilon} |D^2u|^2dx + \int_{\Omega_\varepsilon} W(Du)dx$ where $\Omega_\varepsilon = \omega\times]-\varepsilon;\varepsilon[$ represents the thin film, u is the displacement and Du and D^2u are its gradient and its Hessian tensor, respectively). Their analysis is also performed in order to avoid "quasiconvexification" in the limit free energy. The analogous model for wires can be found in [35] and [36]. Thin films modelled through interfacial energies were also studied in [5], in the framework of functions with bounded Hessian. Non convex bulk energies in the context of non simple materials of grade two were first studied in [39] also for inhomogeneous thin films.

In the past few years dimension reduction problem for thin domains has been approached using Young measures, in order to capture the oscillating behavior of the minimizing sequences. A nonlinear membrane model is obtained in [10] (see also [9]) considering a special class of Young measures which takes into account also the bending moments.

A different method, again based on dimension reduction via Γ-convergence and still involving Young Measures has been developed in [20] and [21] obtaining representation results for the energies of thin films and rods. This last approach we follow here in order to determine the limit energy of a thin film described through a non convex energy depending on second order derivatives of the displacement.

As in [39] we consider a $3D$ body $\Omega_\varepsilon = \omega\times]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[$ whose bulk energy is

$$\int_{\Omega_\varepsilon} W(D^2u)dx$$

First, as usual in dimensional reduction, we re-scale the energy in the transverse direction by dividing through ε, and, we extend the functionals above to the space of $3D$ Young measure in such a way to each "deformation" $\left(D^2u, D\left(\frac{1}{\varepsilon}D_3u\right)\right)$ we associate the Young measure $\mu\otimes\upsilon := \delta_{\left(D^2u, D\left(\frac{1}{\varepsilon}D_3u\right)\right)}$. Finally since one expects to get a $2D$ model for the energy, we consider the average with respect to the third variable. In this space we study Γ-convergence which leads us to a variational limit of the kind:

$$\int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0(h|d)d\sigma_{x_\alpha}(h|\xi)dx_\alpha,$$

defined on the space of $2D$ Young measures satisfying given boundary conditions (for the notations adopted here we refer to the next section, we just recall that $W_0(h|\xi) := \inf_{z\in\mathbb{R}^3} W\begin{pmatrix} h & | & \xi \\ \xi & | & z \end{pmatrix}$).

We remark that the latter energy does not contradict more standard formulas for the energy of thin films such as (see [39])

$$\int_{\omega} \mathcal{Q}_{\mathcal{A}^2} W_0(D_a^2 u, D_a b) dx_a,$$

where appropriate "convexified" densities appear. Also the parametrized measure σ admits as underlying "deformation" the couple $(D_a^2 u, D_a b)$ (where the first component is the bi-dimensional Hessian of the displacement and the second represents the Gradient of the Cosserat vector relative to the cross section (see [13])). Furthermore, the energy provided by the Young measure approach provides more information. Its minimizers carry the description of the microstructure, since limits of oscillating sequences, (cf. [6]) and their baricenters recover the minimizers of the "standard" energy above.

On the other hand, still in the spirit of [20], one may conclude, under appropriate boundary data (see Remark 6.3) that the energy of a film constituted by a non simple material is

$$E(u, b) = \int_{\omega} W_0(D_a^2 u, D_a b) dx_a.$$

Because of the presence of terms of the kind $(D^2 u, Db)$ in the limit, we need to introduce "HG" Young measures, thus generalizing the notion of "Gradient" Young measures introduced in [31] and [32] (see also [38]) and exploiting the results stated in the framework of $\mathcal{A}$-quasiconvexity (see [19]).

Moreover, in order to perform our analysis we prove some relaxation results dealing with "deformations" of the kind $(D^2 u, Db)$ (see Section 5) already available in the Gradient case.

The paper is organized as follows. In Section 2 some preliminary results dealing with Young Measures, convexity notions and Γ-convergence are stated. Section 3 is devoted to set the problem in the framework of Young Measures. In the fourth section the asymptotic behavior of the sequences involved in the problem is studied and the main theorems are stated. In Section 5 relaxation results related to the limit functionals are proven. The relations with the classical formulas are investigated in the sixth section. Finally the proofs of the main theorems are shown in the last section.

2. – Preliminaries.

In this section we will assume that Ω is an open bounded subset of $\mathbb{R}^h$, $M^{h \times k}$ the set of $h \times k$ real matrices which will be often identified with the Euclidean space $\mathbb{R}^{hk}$, and assume that $Sym^s(\mathbb{R}^m)$ is the set of the s-tuples of completely symmetric bilinear forms on $\mathbb{R}^m$. When $s = 3$ we will adopt the symbol $Sym(\mathbb{R}^m)$ in place of $Sym^3(\mathbb{R}^m)$. We denote a generical element of $Sym^s(\mathbb{R}^m)$ by $H = (H_{jk})^i = H^i_{jk}$, where for $i = 1, , s$ $(H^i)_{jk}$ is a symmetric $M^{m\times m}$ matrix, i.e. $H^i_{jk} = H^i_{kj}$, for every i, j, k. (In the reminder of this paper m will take the values 2 or 3).

Given $H \in Sym(\mathbb{R}^3)$ we consider a triple $(h, \xi, c) \in Sym(\mathbb{R}^2)\times M^{3\times 2}\times\mathbb{R}^3$ defined by

$$\begin{cases} h^i_{jk} := \ H^i_{jk} & i = 1,2,3, \quad 1 \le j,k \le 2, \\ \xi_{ik} := \ H^i_{k3} & 1 \le i \le 3, \quad 1 \le k \le 2, \\ c_i := H^i_{33} & i = 1,2,3. \end{cases} \tag{2.1}$$

In symbols the decomposition above can be written as

$$H = \begin{pmatrix} h & \xi \\ \xi & c \end{pmatrix} \tag{2.2}$$

In (2.1) h has the same symmetry properties as H, while ξ is obtained by considering only the components H^i_{k3}, i.e. identifying H^i_{k3} with H^i_{3k} for every $i = 1, 2, 3$ and $k = 1, 2$.

Let E^d be a d-dimensional real vector space, by $C^\infty_{\mathrm{per}}(E^d)$ we denote the space of all regular function defined on E^d and with unit period. $\mathcal{M}(E^d)$ is the space of $\mathbb{R}$-valued Borel measures on E^d which can be viewed as the dual of the separable Banach space $C_0(E^d)$ under the duality

$$\langle \mu, \varphi\rangle = \int_{E^d} \varphi d\mu.$$

Furthermore recall that a mapping $\mu : \Omega \to \mathcal{M}(E^d)$ is said to be weakly $*$ measurable whenever the function $x \mapsto \langle \mu(x), \varphi\rangle$ is measurable for every $\varphi \in C_0(E^d)$. The space $L^\infty_w(\Omega; \mathcal{M}(E^d))$ consists of all weakly $*$ measurable mappings $\mu : \Omega \to \mathcal{M}(E^d)$ which are essentially bounded. This space will be endowed with the weak $*$ topology induced by the duality with $L^1(\Omega, C_0(E^d))$. Therefore a sequence μ^n is weakly $*$ converging to a limit μ if and only if

$$\int_\Omega \langle \mu^n_x, \varphi\rangle\, g(x)dx \to \int_\Omega \langle \mu_x, \varphi\rangle\, g(x)dx \qquad \forall\varphi \in C_0(E^d), \forall g \in L^1(\Omega). \tag{2.3}$$

A Young measure on Ω with target space E^d is an element υ of $L^\infty_w(\Omega, \mathcal{M}(E^d))$ such that $\upsilon_x := \upsilon(x)$ is a probability measure for almost every $x \in \Omega$.

A Young measure $\mu \in L^\infty_w(\Omega; \mathcal{M}(E^d))$ is said to be generated by the sequence of measurable functions $\{u^n\}$ if

$$\delta_{u^n(\cdot)} \to \mu \text{ weakly } * \text{ in } L^\infty_w(\Omega; \mathcal{M}(E^d)).$$

Every Young measure is generated by some sequence of measurable functions, see [38].

The center of mass of a Young measure $\mu \in L^\infty_w(\Omega; \mathcal{M}(E^d))$ is the function

$$x \mapsto \langle \mu_x, \mathrm{id} \rangle = \int_{E^d} \lambda d\mu_x(\lambda) \in L^\infty(\Omega; E^d).$$

It is easy to see that if $\mu^n \to \mu$ weakly $*$ in $L^\infty_w(\Omega; \mathcal{M}(E^d))$ then $\langle \mu^n_x, \mathrm{id} \rangle \to \langle \mu_x, \mathrm{id} \rangle$ weakly $*$ in $L^\infty(\Omega; E^d)$.

The following result is known as Fundamental Theorem on Young Measures (see [11]).

THEOREM 2.1. – *Let $D \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{z_n\}$ be sequence of measurable functions, $z_n : D \to E^d$. Then there exists a subsequence $\{z_{n_k}\}$ and a weak $*$ measurable map $\upsilon : D \to \mathcal{M}(E^d)$ such that the following hold:*

i) $\upsilon_x \geq 0, \|\upsilon_x\|_{\mathcal{M}(E^d)} = \int_{E^d} d\upsilon_x \leq 1$ *for a.e.* $x \in D$;

ii) *one has* (i') $\|\upsilon_x\|_{\mathcal{M}(E^d)} = 1$ *for a.e.* $x \in D$ *if and only if*

$$\lim_{M \to \infty} \sup_k \mathcal{L}^N(\{|z_{n_k} \geq M|\}) = 0;$$

iii) *if* $K \subset E^d$ *is a compact subset and dist*$(z_{n_k}; K) \to 0$ *in measure, then*

$$\mathrm{supp}\, \upsilon_x \subset K \text{ for a.e. } x \in D;$$

iv) *if* (i') *holds, then in* (iii) *one may replace "if" with "if and only if";*

v) *if* $f : \Omega \times E^d \to \mathbb{R}$ *is a normal integrand, bounded from below, then*

$$\liminf_{k \to \infty} \int_\Omega f(x, z_{n_k}(x)) dx \geq \int_\Omega \overline{f}(x) dx$$

where $\overline{f}(x) := \langle \upsilon_x, f(x, \cdot) \rangle = \int_{E^d} f(x, \lambda) d\upsilon_x(\lambda)$;

vi) *if* (i') *holds and if* $f : \Omega \times E^d \to \mathbb{R}$ *is Carathéodory and bounded from below, then*

$$\lim_{k \to \infty} \int_\Omega f(x, z_{n_k}(x)) dx = \int_\Omega \overline{f}(x) dx < +\infty$$

if and only if $\{f(\cdot, z_{n_k}(\,\cdot\,))\}$ *is equi-integrable. In this case*

$$f(\cdot, z_{n_k}(\,\cdot\,)) \rightharpoonup \overline{f} \text{ in } L^1(\Omega).$$

In the sequel we briefly recall the notion of image of a Young measure and the fiber product of two Young measures.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B})$ be measurable spaces and $p : X \to Y$. Then the image of the measure μ under the mapping p is defined by $p_\sharp\mu(A) := \mu(p^{-1}(A))$ for every $A \in \mathcal{B}$. The fiber product of two Young measures $\upsilon \in L^\infty_w(\Omega; \mathcal{M}(E^t))$ and $\sigma \in L^\infty_w(\Omega : \mathcal{M}(E^s))$ is the Young measure $\mu \in L^\infty_w(\Omega; \mathcal{M}(E^{ts}))$ usually denoted with $\mu = \upsilon \otimes \sigma$ defined by

$$\mu_x = \upsilon_x \otimes \sigma_x,$$

where $\otimes$ denotes the usual tensor product of measures. The following theorem of Balder and Valadier gives a sufficient condition which allows to pass to the limit in the fiber product.

THEOREM 2.2. – *Let $\{\upsilon_n\}$ and $\{\sigma_n\}$ be sequences of Young measures in $L^\infty_w(\Omega; \mathcal{M}(E^t))$ and $L^\infty_w(\Omega; \mathcal{M}(E^s))$ respectively, such that*

$$\upsilon_n \to \upsilon \text{ weakly } * \text{ in } L^\infty_w(\Omega; \mathcal{M}(E^t))$$

while

$$\sigma_n \to \sigma = \delta_{u(\cdot)} \text{ weakly } * \text{ in } L^\infty_w(\Omega; \mathcal{M}(E^s))$$

for a suitable measurable function u. Then

$$\upsilon_n \otimes \sigma_n \to \upsilon \otimes \sigma \text{ weakly } * \text{ in } L^\infty_w(\Omega; \mathcal{M}(E^{ms})).$$

The following result will be exploited in the sequel.

THEOREM 2.3. – *Let Ω be a bounded open set of $\mathbb{R}^k$ with Lipschitz boundary. Assume that $\{z_j\}$ and $\{w_j\}$ are two bounded sequences in $L^p(\Omega)$.*

i) *If $\mathcal{L}^k(\{z_j \neq w_j\}) \to 0$ then the parametrized measure generated from the two sequences is the same.*

ii) *If $|z_j - w_j|_{L^p(\Omega)} \to 0$ then $\{z_j\}$ and $\{w_j\}$ share the parametrized measure.*

To our aims it is worthwhile to recall the notion of $\mathcal{A}$-quasiconvexity (as it has been introduced in [19]) and its relations with Young Measures.

Consider a collection of linear operators $A^{(i)} \in \mathrm{Lin}(E^d, E^l)$, $i = 1, , N$, and define

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, \quad v : E^N \to E^d,$$

$$\mathbb{A}(w) := \sum_{i=1}^N A^{(i)} w_i \in \mathrm{Lin}(E^d, E^l), w \in E^N,$$

where $\mathrm{Lin}(X, Y)$ is the vector space of linear mappings from the vector space X into the vector space Y.

Furthermore assume that $\mathcal{A}$ satisfies the *constant rank* property, i.e. there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r \text{ for all } w \in S^{N-1}.$$

(where S^{N-1} is the unit sphere in $\mathbb{R}^N$).

DEFINITION 2.4. – *A Borel function* $f : E^d \to \mathbb{R}$ *is said to be* $\mathcal{A}$*-quasiconvex if*

$$f(v) \leq \int_Q f(v + w(x))dx \text{ for every } w \in C^\infty_{\text{per}}(E^N; E^d) \cap \operatorname{Ker}\mathcal{A}, \int_Q w(y)dy = 0,$$

Recall the notion of $\mathcal{A}$-quasiconvexification, which extends to the $\mathcal{A}$-free setting (i.e. test functions in ker$\mathcal{A}$) the notion of "quasiconvexification".

DEFINITION 2.5. – *Given a Borel function* $f : E^d \to \mathbb{R}$*, the* $\mathcal{A}$*-quasiconvexification of* f *at* $v \in E^d$ *is given by*

$$\mathcal{Q}_{\mathcal{A}} f(v) := \inf\left\{ \int_Q f(v + w(x))dx : w \in C^\infty_{\text{per}}(\mathbb{R}^N; E^d) \cap \operatorname{Ker}\mathcal{A}, \int_Q w(y)dy = 0 \right\},$$

(where Q is the unit cube in $\mathbb{R}^N$*).*

The next results, due to Fonseca and Müller (see [19]) will be exploited in the sequel.

THEOREM 2.6. – *Let* $1 \leq p < +\infty$ *and let* $\{\upsilon_x\}_{x \in \Omega}$ *be a weakly measurable family of probability measures on* E^d*. There exists a p-equi-integrable sequence* $\{v_n\}$ *in* $L^p(\Omega, E^d)$ *that generates the Young measure* υ *and satisfies* $\mathcal{A}v_n = 0$ *in* Ω *if and only if the following three conditions hold:*

i)
$$\int_\Omega \int_{\mathbb{R}^d} |z|^p d\upsilon_x(z)dx < +\infty;$$

ii) *there exists* $v \in L^p(\Omega, E^d)$ *such that* $\mathcal{A}v = 0$ *and*

$$v(x) = \langle \upsilon_x, \mathrm{id} \rangle \text{ a.e.} x \in \Omega;$$

iii) *for a.e.* $x \in \Omega$ *and all continuous functions* g *that satisfy* $|g(v)| \leq C(1 + |v|^p)$ *for some* C *and all* $v \in E^d$ *one has*

$$\langle \upsilon_x, g \rangle \geq \mathcal{Q}_{\mathcal{A}} g(\langle \upsilon_x, \mathrm{id} \rangle).$$

PROPOSITION 2.7. – *Let* $1 < p < +\infty$*, let* $\{u_n\}$ *be a bounded sequence in* $L^p(\Omega; E^d)$ *such that* $\mathcal{A}u_n \to 0$ *in* $W^{-1,p}(\Omega)$*,* $u_n \rightharpoonup u$ *in* $L^p(\Omega; E^d)$*, and assume* *that* $\{u_n\}$ *generates the Young measure* ν. *Then there exists a p-equi-integrable sequence* $\{v_n\} \subseteq L^p(\Omega; E^d) \cap \ker\mathcal{A}$ *such that*

$$\int_\Omega v_n dx = \int_\Omega u dx, \|v_n - u_n\|_{L^q(\Omega)} \to 0 \;\; \textit{for all } 1 \le q < p$$

and, in particular, $\{v_n\}$ *still generates* ν.

REMARK 2.8. – Note that if $\mathcal{A}$ is "curl" and if $v \in C^\infty_{\text{per}}(E^m; E^d)$ is such that

$$\mathcal{A}v = 0 \; (i.e. \; \text{curl} v = 0) \text{ and } \int_Q v(y)dy = 0,$$

(Q being the unit cube in E^m) then there exists $\varphi \in C^\infty_{\text{per}}(E^m; E^l)$ such that $\nabla\varphi = v$ where $d = ml$, thus Definition 2.4 recovers the well-known notion of quasi-convexity introduced by Morrey in [37]. Moreover, in the curl-free context, Theorem 2.6 corresponds to the characterization of $W^{1,p}$-Gradient Young Measures introduced by Kinderlehrer and Pedregal in [31], (see also [32] and [38]).

The following result, proven in [8], gives an explicit formula for relaxed functionals in the framework of $\mathcal{A}$-quasiconvexity.

Let Ω be a bounded open subset of $\mathbb{R}^k$ and let $1 < p < +\infty$, and consider the functional

$$F : L^p(\Omega, E^d) \times \mathcal{O}(\Omega) \to [0, +\infty[,$$

defined by

$$F(v; D) := \int_D f(v(x))dx,$$

where $\mathcal{O}(\Omega)$ denotes the collection of all open subsets of Ω, and the density f satisfies the following hypothesis

$$\text{(2.4)} \qquad \begin{aligned} &f : E^d \to [0, +\infty[\text{ is a continuous function such that} \\ &\frac{1}{C}|v|^p - C \le f(v) \le C(1 + |v|^p) \end{aligned}$$

for every $v \in E^d$, where $C > 0$.

For $D \in \mathcal{O}(\Omega)$ and $v \in L^q(\Omega; E^d) \cap \ker\mathcal{A}$ define the (L^p – weak) lower semi-continuous envelope

$$\text{(2.5)} \quad \mathcal{F}(v, D) := \inf\left\{ \liminf_{n\to\infty} F(v_n; D) : v_n \in L^p(D; E^d), v_n \rightharpoonup v \text{ in } L^p(D; E^d), v_n \in \ker\mathcal{A} \right\}$$

THEOREM 2.9. – *Under assumption* (2.4) *and the constant rank property, for all* $D \in \mathcal{O}(\Omega), v \in L^p(\Omega; E^d) \cap \ker\mathcal{A}$, *we have*

$$\mathcal{F}(v; D) = \int_D \mathcal{Q}_{\mathcal{A}} f(v(x))dx.$$

The function $\mathcal{Q}_{\mathcal{A}} f(\,\cdot\,)$ in the formula above is $\mathcal{A}$-quasiconvex.

Theorem 2.6 allows us to give the definition below, following the approach of [31] and [32], where Gradient Young Measures have been introduced.

DEFINITION 2.10. – *Let* Ω *be a bounded open subset of* $\mathbb{R}^h$. *A weakly* $*$ *measurable map* $\theta : \Omega \to \mathcal{M}(Sym^k(\mathbb{R}^h) \times M^{h\times k})$ *is a* $W^{2,1,p}$*-Hessian-Gradient Young measure (in the sequel* $W^{2,1,p}$*- HG Young Measure) if there exists a sequence of maps* $(u^n, v^n) : \Omega \to \mathbb{R}^k$ *such that* $(u^n, v^n) \rightharpoonup (u, v)$ *in* $W^{2,p}(\Omega, \mathbb{R}^k) \times W^{1,p}(\Omega, \mathbb{R}^k)$ *and* $\delta_{(D^2u^n(\cdot),\nabla v^n(\cdot))} = \delta_{D^2u^n(\cdot)} \otimes \delta_{\nabla v^n(\cdot)} \to \theta$ *weakly* $*$ *in* $L^\infty_w(\Omega, Sym^k(\mathbb{R}^h) \times M^{h\times k})$. *In this case, the* $W^{2,1,p}$*-HG Young measure* θ *is said to be generated by the sequence* $(D^2u^n, \nabla v^n)$ *and* (u, v) *is called an underlying "deformation" for* θ.

This definition can be read in the light of Theorem 2.6, observing that $W^{2,1,p}$-HG Young measures are in a sort of duality with suitable $\mathcal{A}$-quasiconvex functions.

We will make, first, use of HG Young Measures when $\mathcal{A}$ is the differential operator defined below and $h = k = 3$.

Let Q_3 be the cube $]0,1[^3$, given a function $v = (h, \xi) : Q_3 \to Sym(\mathbb{R}^3) \times M^{3\times 3}$ we define the operator $\mathcal{A}$ as $\mathcal{A} := \mathcal{A}^3$, i.e.

$$\mathcal{A}^3 v := (\mathcal{A}^3_2 h, \mathcal{A}^3_1 \xi) \tag{2.6}$$

where

$$\mathcal{A}^3_2 h := \left(\frac{\partial}{\partial x_i} h^l_{jk} - \frac{\partial}{\partial x_j} h^l_{ik} \right)_{i,j,k,l=1,,3},$$

where $h \in C^\infty_{\text{per}}(Q_3, Sym(\mathbb{R}^3))$.

As in [19],

$$\left\{ h \in C^\infty_{\text{per}}(Q_3; Sym(\mathbb{R}^3)), \mathcal{A}^3_2 h = 0, \int_{Q_3} h dx = 0 \right\} = \left\{ D^2 u : u \in C^\infty_{\text{per}}(Q_3, \mathbb{R}^3) \right\}. \tag{2.7}$$

In fact, for every $l = 1, 2, 3$, if $\mathcal{A}^3_2 h^l = 0$ then $h^l_{jk} = \frac{\partial w^l_j}{\partial x_k}$ for some functions $w^l_j \in C^\infty(Q_3; \mathbb{R}^9)$ with average zero. Note that w^l_j is periodic for every i and j and $\int_{Q_3} h^l dx = 0$. Then by the symmetry of h^l_{jk} with respect to j and k, it results $\operatorname{curl} w^l = 0$ for every $l = 1, 2, 3$ and we conclude that $h^l_{jk} = \frac{\partial^2 u^l}{\partial x_k \partial x_j}$ for some $u^l \in C^\infty_{\text{per}}(Q_3; \mathbb{R})$.

Furthermore it is easily seen that the constant rank condition is satisfied, since for every $w \in S^2$,

$$\ker \mathbb{A}_2^3(w) = \Big\{X \in Sym(\mathbb{R}^3) : w_j X_{jk}^l - w_j X_{ik}^l = 0, i, j, k, l = 1, 2, 3\Big\}$$

$$= \Big\{a \otimes w \otimes w, a \in \mathbb{R}^3\Big\}, \text{ and } \dim\mathrm{Ker}\mathbb{A}_3^2(w) = 3.$$

$$\mathcal{A}_1^3 \xi = \left(\frac{\partial \xi_j^i}{\partial x_k} - \frac{\partial \xi_k^i}{\partial x_j}\right)_{i,j,k=1,2,3}.$$

Furthermore,

$$\text{(2.8)} \quad \Big\{\xi \in C_{\text{per}}^\infty(Q_3, M^{3\times 3}) : \mathcal{A}_1^3 \xi = 0, \int_{Q_3} \xi dx = 0\Big\} = \Big\{D\varphi : \varphi \in C_{\text{per}}^\infty(Q_3, \mathbb{R}^3)\Big\},$$

and it is easy to see that $\mathbb{A}_1^3$ is a constant rank operator and it results

$$\mathrm{Ker}\mathbb{A}_1^3(w) = \Big\{V \in \mathcal{M}^{3\times 3} : \mathbb{A}_1^3(w) V^l = 0,\ l = 1, 2, 3\Big\}$$

$$= \Big\{w_i V_j^l - w_j V_i^l = 0, i, j, l = 1, 2, 3,\Big\} = \Big\{a \otimes w, a \in \mathbb{R}^3\Big\}$$

and dim $\mathrm{Ker}\mathbb{A}_1^3(w) = 3$.

It follows immediately that $\mathcal{A}^3$ is a constant rank operator and for every $w \in S^2$:

$$\text{(2.9)} \quad \mathrm{Ker}\mathbb{A}^3(w) = \Big\{(X, V) \in Sym(\mathbb{R}^3) \times M^{3\times 3} : (X, V) = (b \otimes w^{\otimes 2}, a \otimes w), b \in \mathbb{R}^3, a \in \mathbb{R}^3\Big\},$$

where $w^{\otimes 2}$ stands for $w \otimes w$. For every $v \in Sym(\mathbb{R}^3) \times M^{3\times 3}$, with $v = (h, \xi)$, we have

$$\text{(2.10)} \quad \mathcal{Q}_{\mathcal{A}^3} f(v) = \inf\Big\{\int_{Q_3} f(v + w(x)) dx : w \in C_{per}^\infty(\mathbb{R}^3; Sym(\mathbb{R}^3) \times M^{3\times 3}) \cap Ker\mathcal{A}^3, \int_{Q_3} w dx = 0\Big\},$$

or equivalently

$$\text{(2.11)} \quad \mathcal{Q}_{\mathcal{A}^3} f((h, \xi)) = \inf\Big\{\int_{Q_3} f((h + D^2 u, \xi + D\varphi)) dx : \varphi \in C_0^\infty(Q_3; \mathbb{R}^3), u \in C_0^\infty(Q_3, \mathbb{R}^3)\Big\}.$$

2.1 – *Sequential Γ-convergence.*

We manage our problem in the framework of a variant of Γ-convergence introduced in [18] (see also [16]), namely sequential Γ-convergence, first introduced in [3]. For reader's convenience we recall the main properties of the sequential Γ-convergence that will be used in the sequel.

Let X be a set and let (Y, τ) be a topological space and $q : X \to Y$. Given a sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ and $y \in Y$, let us denote by

$$\Gamma(q, \tau Y) \liminf_{n\to\infty} F_n(y) := \inf \left\{ \liminf_{n\to\infty} F_n(x_n) : q(x_n) \to y \text{ in } \tau \right\},$$
(2.12)
$$\Gamma(q, \tau Y) \limsup_{n\to\infty} F_n(y) := \inf \left\{ \limsup_{n\to\infty} F_n(x_n) : q(x_n) \to y \text{ in } \tau \right\}$$

which are called, respectively, the sequential Γ^--lower limit and the sequential Γ^--upper limit at the point y.

DEFINITION 2.11. – *Given a sequence $\{\varepsilon_n\}$ of positive real numbers we say that a sequence $F_{\varepsilon_n} : X \to [-\infty; +\infty]$ (sequentially) $\Gamma(q, \tau Y)$-converges to a functional $F : Y \to [-\infty, +\infty]$ at a point $y \in Y$, and we write*

$$\Gamma(q, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y)$$

if

$$\Gamma(q, \tau Y) \liminf_{n\to\infty} F_{\varepsilon_n}(y) = \Gamma(q, \tau Y) \limsup_{n\to\infty} F_{\varepsilon_n}(y) = F(y).$$

We say that the family of functionals $F_\varepsilon : X \to [-\infty; +\infty]$ (sequentially) $\Gamma(q, \tau Y)$-converges to a functional $F : Y \to [-\infty; +\infty]$ to a functional $F : Y \to [-\infty; +\infty]$ at a point $y \in Y$, and we write

$$\Gamma(q, \tau Y) \lim_{\varepsilon\to 0} F_\varepsilon(y) = F(y)$$

if for any sequence ε_n of positive reals such that $\lim_{n\to\infty} \varepsilon_n = 0$ we have that

$$\Gamma(q, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y).$$

We say that a family of functionals $\Gamma(q, \tau Y)$-converges on a set if it $\Gamma(q, \tau Y)$-converges at every point of the set.

As it happens for the "classical" Γ-convergence a sequence $F_{\varepsilon_n} : X \to [-\infty; +\infty]$ sequentially Γ-converges to a functional $F : Y \to [-\infty; +\infty]$ if the following two conditions hold:

i) for every sequence $x_n \in X$ such that $q(x_n) \to y$ in τ one has
$$\liminf_{n\to\infty} F_{\varepsilon_n}(x_n) \geq F(y);$$

ii) there exists a sequence $\overline{x_n} \in X$ such that $q(\overline{x_n}) \to y$ in τ and
$$\lim_{n\to\infty} F_{\varepsilon_n}(\overline{x_n}) = F(y).$$

DEFINITION 2.12. – *The family F_ε is said to be $(q, \tau Y)$-equi-coercive if for any real number M there exists a τ-compact and a τ-closed subset K_M of Y such that*

$$\{q(x) : F_\varepsilon(x) \leq M\} \subseteq K_M \ \ \textit{for every } \varepsilon > 0.$$

PROPOSITION 2.13. – *Let us assume that $\Gamma(q, \tau Y) \lim_{\varepsilon \to 0} F_\varepsilon = F$ on Y and that the family F_ε be $(q, \tau Y)$ equi-coercive. Then it results that*

i) *F is τ-lower semicontinuous;*
ii) *F is τ-coercive;*
iii) *if $x_\varepsilon \in X$ satisfy $\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon \to 0} \inf F_\varepsilon$ (e.g. if x_ε minimizes F_ε) then*
   a) *if $\{\varepsilon_n\}$ is a sequence such that $\lim_{n \to \infty} \varepsilon_n = 0$ and if $q(x_{\varepsilon_n}) \to y$ in τ then y is a minimizer of F on Y and $\lim_{n \to \infty} F_{\varepsilon_n}(x_n) = F(y)$;*
   b) *there is a sequence ε_n such that $\lim_{n \to \infty} \varepsilon_n = 0$ and a minimizer y of F on Y such that $q(x_{\varepsilon_n}) \to y$ in τ.*

PROPOSITION 2.14. – *If Y is dual of a separable Banach space, τ is the weak $*$ topology, and $F_n : X \to [-\infty; +\infty]$ is $(q, \tau Y)$-equi-coercive, then*

$$\Gamma(q, \tau Y) \lim_{n \to \infty} F_n(y) = F(y)$$

if and only if

i) $\liminf_{n \to \infty} F_n(x_n) \geq F(y)$ *for every sequence $x_n \in X$ such that $q(x_n) \to y$ in τ;*
ii) *for every sequence $\{n_k\}$ of positive integers there is a subsequence $\{n_{k_p}\}$ and a sequence $x_p \in X$ such that*

$$q(x_p) \to y \ \textit{in } \tau \textit{ and } \lim_{p \to \infty} F_{n_{k_p}}(x_p) = F(y).$$

3. – Formulation of the Problem.

For every $\varepsilon > 0$, let $\Omega_\varepsilon = \{x = (x_a, x_3) \in \mathbb{R}^2 \times \mathbb{R} : x_a = (x_a) \in \omega, |x_3| < \frac{\varepsilon}{2}\}$, $(a = 1, 2)$ where ω is an open, connected, bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. For each $u \in W^{2,p}(\Omega_\varepsilon; \mathbb{R}^3)$, we denote by D_a and D_a^2 the first and second order derivatives, respectively, of u with respect to the planar variables x_a, and by D_3 and D_3^2 the first and the second order derivatives with respect to x_3. Finally for the mixed derivatives we use the symbol $D_{a,3}^2$ or $D_{3,a}^2$ indifferently.

Consider a bulk energy density $W : Sym(\mathbb{R}^3) \to \mathbb{R}$ and assume that W is continuous and satisfies the following growth assumption

$$c(|H|^p - 1) \leq W(H) \leq C(|H|^p + 1), \tag{3.1}$$

for some constants $0 < c \leq C$.

We just focus on the energy I_ε of the body, given by

$$I_\varepsilon : u \in W^{2,p}(\Omega_\varepsilon) \to \int_{\Omega_\varepsilon} W(D^2 u)dx,$$

and suppose that the body is clamped on its lateral boundary $\partial\omega\times]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[$. Equilibria correspond to the minima of the energy I_ε over the set of admissible deformations

$$\mathcal{B}_\varepsilon = \left\{u \in W^{2,p}(\Omega_\varepsilon;\mathbb{R}^3) : u(x) = x \text{ on } \partial\omega\times\left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\right\}.$$

The behavior of the minimizers u_ε is studied through a suitable re-scaling in the transverse direction which maps Ω_ε on a fixed domain Ω, i.e. by considering $v_\varepsilon(x) := u_\varepsilon(x_1, x_2, \varepsilon x_3)$ we are led to the following energy functional

$$I_\varepsilon^\Omega(u) := \int_\Omega W\begin{pmatrix} D_a^2 u & \frac{1}{\varepsilon}D_{a,3}^2 u \\ \frac{1}{\varepsilon}D_{a,3}^2 u & \frac{1}{\varepsilon^2}D_3^2 u \end{pmatrix} dx,$$

obtained re-scaling the energy by $\frac{1}{\varepsilon}$ and where the same notation as in (2.2) has been used.

Accordingly, the set of admissible deformations becomes

$$\mathcal{B}_\varepsilon^\Omega = \left\{u \in W^{2,p}(\Omega;\mathbb{R}^3) : u(x) = (x_1, x_2, \varepsilon x_3) \text{ on } \partial\omega\times\left]-\frac{1}{2},\frac{1}{2}\right[\right\}.$$

Next, consider the extended functional

$$J_\varepsilon(v) := \begin{cases} I_\varepsilon^\Omega(v) & \text{if } v \in W^{2,p}(\Omega,\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

In this setting in [39] it has been considered for any $u \in W^{1,p}(\Omega,\mathbb{R}^3)$, $b \in L^p(\Omega,\mathbb{R}^3)$, the functional

$$\text{(3.2)} \quad J_{\{\varepsilon\}}(u,b) := \inf\left\{ \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) : v_\varepsilon \in W^{2,p}(\Omega,\mathbb{R}^3), v_\varepsilon \to u \text{ in } W^{1,p}(\Omega,\mathbb{R}^3), \right.$$
$$\left. \text{and } \frac{1}{\varepsilon}D_3 v_\varepsilon \to b \text{ in } L^p(\Omega,\mathbb{R}^3)\right\},$$

and the following "Γ-convergence" result has been proven:

$$\text{(3.3)} \qquad J_{\{\varepsilon\}}(u,b) = \begin{cases} \displaystyle\int_\omega \mathcal{Q}_{\mathcal{A}^2} W_0(D_a^2 u, D_a b) dx_a, & \text{if } (u,b) \in W_{\mathcal{B}}^{2,p}(\omega)\times W_{\mathcal{B}}^{1,p}(\omega) \\ +\infty & \text{otherwise} \end{cases}$$

for every $(u,b) \in W^{1,p}(\Omega)\times L^p(\Omega)$, and where $W^{2,p}_{\mathcal{B}}(\omega)$ is the space of functions $u \in W^{2,p}(\omega;\mathbb{R}^3)$ such that $u \equiv (x_1,x_2,0)$ on $\partial\omega$ and $W^{1,p}_{\mathcal{B}}(\omega)$ is the space of functions $b \in W^{1,p}(\omega;\mathbb{R}^3)$ such that $b \equiv (0,0,1)$ on $\partial\omega$.

We first recall that the function u in (3.3) still represents the displacement, while the function b (obtained as limit of $\frac{1}{\varepsilon}D_3u_\varepsilon$) is the Cosserat vector relative to the cross section ω, which keeps memory of the normal to the middle surface of the film prior to the $3D$- $2D$ reduction.

The density $\mathcal{Q}_{\mathcal{A}^2}W_0$ is defined as follows. For every $h \in Sym(\mathbb{R}^2)$ and $\xi \in M^{3\times 2}$:

$$W_0(h,\xi) = \inf_{c\in\mathbb{R}^3} W\begin{pmatrix} h & \xi \\ \xi & c \end{pmatrix}. \tag{3.4}$$

$\mathcal{Q}_{\mathcal{A}^2}W_0$ is the $\mathcal{A}$-quasiconvexification of W_0, defined in (2.5), related to the differential operator $\mathcal{A}^2 := (\mathcal{A}^2_2, \mathcal{A}^2_1)$ given by

$$\mathcal{A}^2 : v \equiv (h,\xi) \in Sym(\mathbb{R}^2)\times M^{3\times 2} \to (\mathcal{A}^2_2 h, \mathcal{A}^2_1\xi) \tag{3.5}$$

where

$$\mathcal{A}^2_2 h = \left(\frac{\partial h^i_{j1}}{\partial x_2} - \frac{\partial h^i_{j2}}{\partial x_1}\right)_{i=1,2,3,\, j=1,2} \quad \text{and } \mathcal{A}^2_1\xi = \left(\frac{\partial \xi^i_1}{\partial x_2} - \frac{\partial \xi^i_2}{\partial x_1}\right)_{i=1,2,3}.$$

REMARK 3.1. – With an argument entirely similar to that of (2.7) and (2.8), it is easily verified that

$$\left\{h \in C^\infty(Q_2; Sym(\mathbb{R}^2)) : \mathcal{A}^2_2 h = 0, \int_{Q_2} h dx = 0\right\} = \left\{D^2_\alpha u : u \in C^\infty_{\text{per}}(Q_2,\mathbb{R}^3)\right\}, \tag{3.6}$$

where Q_2 denotes the cube $]0,1[^2$. Moreover $\ker\mathbb{A}^2_2(w) = \{X \in Sym(\mathbb{R}^2) : w_i X^l_{jk} - w_j X^l_{ik} = 0,\ i,j=1,2, k=1,2, l=1,2,3\} = \{b\otimes w\otimes w, b\in\mathbb{R}^3\}$, so $\dim \mathrm{Ker}\mathbb{A}^2_2(w) = 3$. Also

$$\left\{\xi \in C^\infty(Q_2, M^{3\times 2}) : \mathcal{A}^2_1\xi = 0, \int_{Q_2} \xi dx = 0\right\} = \left\{D_\alpha\varphi : \varphi \in C^\infty_{\text{per}}(Q_2,\mathbb{R}^3)\right\}, \tag{3.7}$$

and for every $w \in S^1$ it results $\mathrm{Ker}\mathbb{A}^2_1(w) = \{V \in M^{3\times 2} : \mathbb{A}_2(w)V^l = 0, l = 1,2,3\} = \{w_i V^l_j - w_j V^l_i = 0, l=1,2,3, i,j=1,2\} = \{a\otimes w, a\in\mathbb{R}^3\}$ and $\dim \mathrm{Ker}\mathbb{A}^2_1(w) = 3$.

It follows immediately that $\mathcal{A}^2$ is a constant rank operator, and for every $w\in S^1$, $\mathrm{Ker}\mathbb{A}^2(w) = \{(X,V)\in Sym(\mathbb{R}^2)\times M^{3\times 2} : (X,V) = (b\otimes w^{\otimes 2}, a\otimes w), b\in\mathbb{R}^3, a\in\mathbb{R}^3\}$, where $w^{\otimes 2}$ stands for $w\otimes w$.

REMARK 3.2. – As in Proposition 1 of [33] it can be shown that W_0 is continuous and satisfies the same growth conditions of order p as W does.

Furthermore we recall that in [39] *it has been proven that* $\mathcal{Q}_{\mathcal{A}^2}W_0$ *is* $\mathcal{A}^2$-*quasiconvex and continuous.*

In the sequel we extend the functional $I_\varepsilon^\Omega(u)$ to the larger space $L_w^\infty(\Omega;\mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$ by setting

$$I_\varepsilon^{\mathcal{M}}(\mu\otimes\upsilon)=\begin{cases} I_\varepsilon^\Omega(u) & \text{if } u\in\mathcal{B}_\varepsilon^\Omega : \mu=\delta_{D^2u(\cdot)} \text{ and } \upsilon=\delta_{D\left(\frac{1}{\varepsilon}D_3u(\cdot)\right)} \\ +\infty & \text{otherwise in } L_w^\infty(\Omega;\mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3})) \end{cases} \tag{3.8}$$

4. – The Asymptotic Behaviour.

Most of the results and the proofs contained in this section follow the structure of the analogous one in [20] with suitable changes in notations and are shown for reader's convenience.

Let $(H|F)$ be an element in $Sym(\mathbb{R}^3)\times M^{3\times 3}$. In the sequel we will exploit the following decomposition of $(H|F)$.

$$(H|F):=\begin{pmatrix} h & \xi & |d \\ \xi & c & |e \end{pmatrix} \tag{4.1}$$

with $h\in Sym(\mathbb{R}^2)$, $\xi\in M^{3\times 2}$, $d\in M^{3\times 2}$, $c\in\mathbb{R}^3$, $e\in\mathbb{R}^3$, according to the notations in (2.1).

With these notations we also consider the following projections

$$\begin{aligned} &\overline{\pi}: Sym(\mathbb{R}^3)\times M^{3\times 3}\to Sym(\mathbb{R}^2)\times M^{3\times 2}, \quad \overline{\pi}(H|F)=\overline{\pi}\begin{pmatrix} h & \xi & |d \\ \xi & c & |e \end{pmatrix}:=(h|d) \\ &\pi^3: Sym(\mathbb{R}^3)\times M^{3\times 3}\to (M^{3\times 2}\times\mathbb{R}^3)\times\mathbb{R}^3, \quad \pi^3(H|F)=\pi^3\begin{pmatrix} h & \xi & |d \\ \xi & c & |e \end{pmatrix}:=(\xi,c|e) \end{aligned} \tag{4.2}$$

and by $\overline{\pi}_\sharp\upsilon_x$ and $\pi^3_\sharp\upsilon_x$ the usual image measures by the corresponding projection maps.

We denote by $\partial_L\Omega$ the lateral boundary of Ω, i.e. $\partial\omega\times]-\frac{1}{2},\frac{1}{2}[$ and by $\mathcal{Y}^{2,1,p}_{\partial_L\Omega}(\Omega,\mathbb{R}^3)$ the set of $W^{2,1,p}(\Omega;\mathbb{R}^3)$ HG-Young measures where the underlying deformation (u,b) is such that $(u(x),b(x))=(x_1,x_2,0),(0,0,1))$ on $\partial_L\Omega$, and by $\mathring{\mathcal{Y}}^{2,1,p}_{\partial_L\omega}(\Omega,\mathbb{R}^3)$ the subset of $\mathcal{Y}^{2,1,p}_{\partial_L\Omega}(\Omega,\mathbb{R}^3)$ of the Young measures υ with $\pi^3_\sharp\upsilon=\delta_0$ (here δ_0 is a vector valued Dirac mass centered at 0), i.e.

$$\mathring{\mathcal{Y}}^{2,1,p}_{\partial_L\Omega}(\Omega,\mathbb{R}^3):=\{\upsilon\in\mathcal{Y}^{2,1,p}_{\partial_L\Omega}(\Omega,\mathbb{R}^3):\pi^3_\sharp\upsilon=\delta_0\}.$$

The following lemma establishes the equi-coerciveness of the sequence of functionals $I_\varepsilon^{\mathcal{M}}$.

LEMMA 4.1. – *Let* $\varepsilon_n \to 0$ *and* $\{(\mu^n, \upsilon^n)\} \subseteq L^\infty_w(\Omega, \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$ *be such that*

$$\sup_n I^{\mathcal{M}}_{\varepsilon_n}(\mu^n \otimes \upsilon^n) < +\infty.$$

Then there exist $\theta \in \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega, \mathbb{R}^3)$ *and a subsequence* $\{(\mu^{n_k}, \upsilon^{n_k})\}$ *such that*

$$\mu^{n_k} \otimes \upsilon^{n_k} \to \theta \text{ weakly } * \text{ in } L^\infty_w(\Omega, \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3})).$$

Moreover, denoted by $(u_{n_k}, \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k})$ *and* (u, b) *the pairs of underlying deformations of* $(\mu^{n_k}, \upsilon^{n_k})$ *and* θ *which satisfy the boundary conditions* $u_{n_k}(x) = (x_1, x_2, \varepsilon_{n_k} x_3)$ *and* $(u(x), b(x)) = ((x_1, x_2, 0), (0, 0, 1))$, *respectively, we have*

$$u_{n_k} \rightharpoonup u \qquad \text{weakly in } W^{2,p}(\Omega, \mathbb{R}^3) \text{ and } D_3 u = 0,$$

$$\frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \quad \text{weakly in } W^{1,p}(\Omega, \mathbb{R}^3) \text{ and } D_3 b = 0.$$

PROOF. – Since $\sup_n I^{\mathcal{M}}_{\varepsilon_n}(\mu^n \otimes \upsilon^n) < +\infty$, then $(\mu^n_x, \upsilon^n_x) = (\delta_{D^2 u_n(x)}, \delta_{(D\frac{1}{\varepsilon_n} D_3 u_n(x))})$ where $u_n \in W^{2,p}(\Omega, \mathbb{R}^3)$ and $u_n(x) = (x_1, x_2, \varepsilon_n x_3)$ on $\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[$. By the growth assumptions on W (3.1), we get, for every n large enough so that $\varepsilon_n < 1$

$$\text{(4.3)} \quad I^{\mathcal{M}}_{\varepsilon_n}(\mu^n \otimes \upsilon^n) = \int_\Omega W \begin{pmatrix} D^2_a u_n & \frac{1}{\varepsilon_n} D^2_{3,a} u_n \\ \frac{1}{\varepsilon_n} D^2_{3,a} u_n & \frac{1}{\varepsilon_n^2} D^2_3 u_n \end{pmatrix} dx$$

$$\geq C \int_\Omega \left(|D^2_a u_n|^p + \frac{1}{\varepsilon_n^p} |D^2_{3,a} u_n|^p + \frac{1}{\varepsilon_n^{2p}} |D^2_3 u_n|^p \right) dx - c.$$

Thus, Poincarè inequality and Rellich theorem guarantee that there exist a subsequence $\{n_k\}$ and two functions $u \in W^{2,p}(\Omega, \mathbb{R}^3)$ and $b \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that

$$\text{(4.4)} \qquad u_{n_k} \rightharpoonup u \quad \text{in } W^{2,p}(\Omega, \mathbb{R}^3) \text{ and } \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \text{ in } W^{1,p}(\Omega, \mathbb{R}^3).$$

Moreover, since $\varepsilon_n \to 0$, (4.3) implies that $D^2_3 u = 0$. Hence $D_3 u = A(x_a)$ and so $u = A(x_a) x_3 + B(x_a)$, and $D_a u = D_a A(x_a) x_3 + D_a B(x_a)$. In addition $D^2_{3,a} u = 0$, hence $A(x_a) = C$. Thus $u = C x_3 + B(x_a)$ and the convergence in the trace space gives $u = (x_1, x_2, 0)$ on $\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[$, that is $C = 0$, and we deduce that $u - (x_1, x_2, 0) \in W^{2,p}_0(\omega, \mathbb{R}^3)$.

Thus, up to subsequences,

$$u_n \rightharpoonup u \text{ weakly in } W^{2,p}(\Omega, \mathbb{R}^3) \text{ and } D_3 u_n \to 0 \text{ strongly in } W^{1,p}(\Omega, \mathbb{R}^3).$$

It follows that

$$u = u(x_1, x_2) \text{ and } u(x) = (x_1, x_2, 0) \text{ on } \partial\omega \times \left] -\frac{1}{2}, \frac{1}{2} \right[. \tag{4.5}$$

For what concerns the second convergence in (4.4), one can observe that (4.3) also implies $D_3 b = 0$. Furthermore the convergence in the trace space guarantees that $b(x) = (0, 0, 1)$ on $\partial\omega \times]-\frac{1}{2}, \frac{1}{2}[$. Hence, up to subsequences, it results that

$$\frac{1}{\varepsilon_n} D_3 u^n \rightharpoonup b \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^3) \text{ and } \frac{1}{\varepsilon_n} D_3^2 u^n \to 0 \text{ strongly in } L^p(\Omega, \mathbb{R}^3).$$

Finally,

$$b = b(x_1, x_2) \text{ and } b = (0, 0, 1) \text{ on } \partial\omega \times]-1, 1[. \tag{4.6}$$

From (4.5) and (4.6), Theorem 2.6, applied to functional $\mathcal{A}^3$ in (3.5), it results $\pi^3_\sharp \theta = \delta_0$. □

It can be observed that if $\theta \in \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)$ then it can be proved that the center of mass of θ_x does not depend on the x_3 variable, but we are able to show in Example 4.6, that in general θ_x does depend also on x_3. Thus we will make use of Γ convergence with respect to the weak * convergence of the averages with respect to the third variable.

LEMMA 4.2. – *If θ is a $W^{2,1,p}(\Omega, \mathbb{R}^3)$-HG Young measure generated by the sequence of $(D^2 u_\varepsilon, D v_\varepsilon)$ and η and ζ are the Young measures generated by $(D_a^2 u_\varepsilon, D_a v_\varepsilon)_{(a=1,2)}$ and $(D_3(D_a u_\varepsilon), D_3 v_\varepsilon)_{(a=1,2)}$, respectively, then*

$$\overline{\pi}_\sharp \theta = \eta \text{ and } \pi^3_\sharp \theta = \zeta.$$

PROOF. – We sketch the proof for reader's convenience.

It will be enough to apply the definitions and the classical integration formula with respect to an image measure.

Indeed, since $\overline{\pi}(D^2 u_\varepsilon, D v_\varepsilon) = (D_a u_\varepsilon, D_a v_\varepsilon)_{(a=1,2)}$, for every $\varphi \in C_0(Sym(\mathbb{R}^3) \times M^{3\times 2})$ and for every $g \in L^1(\Omega)$, one gets

$$\int_\Omega \langle \overline{\pi}_\sharp \theta_x, \varphi \rangle g(x) dx = \int_\Omega \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} \varphi \begin{pmatrix} h & \xi & | d \\ \xi & c & | e \end{pmatrix} d\overline{\pi}_\sharp \theta_x g(x) dx$$

$$= \int_\Omega \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} \varphi(\overline{\pi} \begin{pmatrix} h & \xi & | d \\ \xi & c & | e \end{pmatrix}) d\theta_x \begin{pmatrix} h & \xi & | d \\ \xi & c & | e \end{pmatrix} g(x) dx = \lim_{\varepsilon \to 0} \int_\Omega \varphi(\overline{\pi}(D^2 u_\varepsilon, D v_\varepsilon)) g(x) dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega \varphi(D_a^2 u_\varepsilon, D_a v_\varepsilon) g(x) dx = \int_\Omega \langle \eta, \varphi \rangle g(x) dx.$$

This proves the first part of the statement. The other relation can be proved analogously. □

REMARK 4.3. – Arguing as in Lemma 4.2 *one can easily check that if* $\{u^\varepsilon\}$ *is a bounded sequence in* $W^{2,p}(\Omega)$ *then the Young measure generated by* $D^2_{ij}u_\varepsilon$ *coincides with the Young measure generated by* $D^2_{ji}u_\varepsilon$ *for every* $i, j = 1, 2, 3$.

REMARK 4.4. – Lemma 4.2 *will be applied in the sequel to the sequence* $\left\{\left(D^2 u_\varepsilon, D\left(\frac{1}{\varepsilon}D_3 u_\varepsilon\right)\right)\right\}$.

LEMMA 4.5. – *If* $\theta \in L^\infty_w(\Omega, \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$, *with* $\pi^3_\sharp\theta = \delta_0$, *is generated by the sequence* $(D^2u_\varepsilon, Dv_\varepsilon)$ *with underlying deformation* (u, b), *and* η *is the Young measure generated by* $(D^2_a u_\varepsilon, D_a v_\varepsilon)$, *then*

$$\theta = \eta \otimes \delta_0.$$

Moreover, the functions (u, b) *and the center of mass of the measures* θ_x, *and* η_x *for almost every* $x \in \Omega$ *do not depend on the variable* x_3. *In particular, with an abuse of notation, we shall write*

$$\langle \theta_x, \mathrm{id}\rangle = (D^2u(x_a), Db(x_a)), \text{ and } \langle \eta_x, \mathrm{id}\rangle = (D^2_a u(x_a), D_a b(x_a)).$$

PROOF. – From $\pi^3_\sharp\theta = \delta_0$, and Lemmas 4.1 and 4.2 it can be easily deduced that $\{(D^2_{3,a}u^\varepsilon, D^2_{3,3}u^\varepsilon, D_3v^\varepsilon)\}_\varepsilon$ generate the Young measures δ_0. Thus, the first part of the statement follows from Theorem 2.2.

Consequently, from Lemma 4.2 it follows that

$$\begin{aligned}(D^2u(x), Db(x)) &= \langle \theta_x, \mathrm{id}\rangle = \langle (\eta_x \otimes \delta_0), \mathrm{id}\rangle \\ &= \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} \begin{pmatrix} h & \xi & |d \\ \xi & c & |e \end{pmatrix} d(\eta_x\otimes\delta_0)\begin{pmatrix} h & \xi & |d \\ \xi & c & |e \end{pmatrix} = \left(\langle\eta_x, \mathrm{id}\rangle \middle| \begin{pmatrix} \cdot & 0 & |\cdot \\ 0 & 0 & |0 \end{pmatrix}\right),\end{aligned}$$

for a.e. $x \in \Omega$. Hence it results $D^2_{3,i}u = 0$, for $(i = 1, 2, 3)$, and so u is linear with respect to x_3, actually of the kind $u = c \cdot x_3 + B(x_a)$, analogously it results $D_3 b = 0$ and $b(x) = b(x_a)$.

Consequently we can write that

$$\langle \eta_{x_a,x_3}, \mathrm{id}\rangle = (D^2_a u(x_a), Db(x_a)).$$

□

EXAMPLE 4.6. – We build a measure $\theta \in \mathring{\mathcal{Y}}^{2,1,p}_{\partial_L\Omega}(\Omega;\mathbb{R}^3)$ such that θ depends also on x_3. Let $\rho(s)$ be the 2-periodic function equaling $\frac{s^2}{2} - \frac{1}{2}s$ on $[0, 1]$ and $-\frac{s^2}{2} + \frac{3}{2}s - 1$ on $[1, 2]$. Define the functions $z_\varepsilon := \left(\frac{\varepsilon^2 x_3^2}{2}\rho\left(\frac{x_1}{\varepsilon}\right), 0, 0\right)$. It is easily seen that $\frac{1}{\varepsilon}D_3 z_\varepsilon = \left(\varepsilon x_3\rho\left(\frac{x_1}{\varepsilon}\right), 0, 0\right) \to 0$ and $D^2_3 z_\varepsilon = \left(\varepsilon^2\rho\left(\frac{x_1}{\varepsilon}\right), 0, 0\right) \to 0$ in $L^\infty(\Omega;\mathbb{R}^3)$.

Moreover $D^2_{2,j} z_\varepsilon = 0$ ($j = 1, 2, 3$). Furthermore

$$D^2_1 z_\varepsilon = \left(\frac{x_3^2}{2}\rho''\left(\frac{x_1}{\varepsilon}\right), 0, 0\right) \text{ generates the Young measure } \frac{1}{2}\delta_{-\frac{x_3^2}{2}} + \frac{1}{2}\delta_{\frac{x_3^2}{2}}.$$

On the other hand it results that $D_3\left(\frac{1}{\varepsilon}D_3 z_\varepsilon\right) = \left(\varepsilon\rho\left(\frac{x_1}{\varepsilon}\right), 0, 0\right) \to 0$ and $D_3\left(\frac{1}{\varepsilon}D_2 z_\varepsilon\right) \equiv 0$. Finally $D_3\left(\frac{1}{\varepsilon}D_1 z_\varepsilon\right) = \left(x_3\rho'\left(\frac{x_1}{\varepsilon}\right), 0, 0\right)$ and it can be verified that it weakly converges to 0 and generates a Young measure which depends on x_3. Indeed formula (2.3), applied to Ω with test functions g only depending on x_3 gives

$$\int_0^2\int_0^2 \varphi\left(x_3\rho\left(\frac{x_1}{\varepsilon}\right)\right)g(x_3)dx_1dx_3 \to \int_0^2 g(x_3)\int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(x_1x_3)dx_1dx_3.$$

Thus considering the sequence of functions $u_\varepsilon := (x_1, x_2, \varepsilon x_3) + z_\varepsilon(x)\min\{1, \operatorname{dist}\left(x, \frac{\partial\Omega}{\varepsilon}\right)\}$, it turns out that this sequence is in $\mathcal{B}^\Omega_\varepsilon$ and generates a measure $\theta \in \mathring{\mathcal{Y}}^{2,1,p}_{\partial_L\Omega}$ which effectively depends on x_3.

The functions z_ε can be also used in order to show that both the sequences $D^2_{i,j}z_\varepsilon$ and $D_j\frac{1}{\varepsilon}D_3 z_\varepsilon$ ($i, j = 1, 2, 3$) generate Young measures whose baricenter is 0 but which both depend on x_3, say μ_ε and υ_ε, converging to μ and υ respectively, thus their fiber product $\delta_{D^2_{i,j}z_\varepsilon} \otimes \delta_{D_j\frac{1}{\varepsilon}D_3 z_\varepsilon}$ converges to $\theta \neq \mu \otimes \upsilon$.

DEFINITION 4.7. – *Let X be either $Sym(\mathbb{R}^3)\times M^{3\times 3}$ or $Sym(\mathbb{R}^2)\times M^{3\times 2}$. If $\upsilon \in L^\infty_w(\Omega, \mathcal{M}(X))$ we define*

$$\mathrm{Av}^3\upsilon : \omega \to \mathcal{M}(X),$$

the average of υ with respect to the variable x_3, by

$$\langle \mathrm{Av}^3_{x_a}\upsilon, \varphi\rangle := \langle \mathrm{Av}^3\upsilon(x_a), \varphi\rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \upsilon_{(x_a,x_3)}, \varphi\rangle dx_3,$$

for every $\varphi \in C_0(X)$.

Fubini's theorem guarantees that the map $x_a \to \langle \mathrm{Av}^3_{x_a}\upsilon, \varphi\rangle$ is measurable, and since it is also essentially bounded, we can think of Av^3 as a mapping

$$\mathrm{Av}^3 : L^\infty_w(\Omega, X) \to L^\infty_w(\omega, X).$$

We also recall that this map is continuous, i.e.

$$\upsilon^\varepsilon \to \upsilon \text{ weakly } * \text{ in } L^\infty_w(\Omega, \mathcal{M}(X)) \Rightarrow \mathrm{Av}^3\upsilon^\varepsilon \to \mathrm{Av}^3\upsilon \text{ weakly } * \text{ in } L^\infty_w(\omega, \mathcal{M}(X)),$$

furthermore Av^3 maps Young measures to Young measures.

We will make also use of the average-projection mapping

$$q : L^\infty_w(\Omega, \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3})) \to L^\infty_w(\Omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2})), \tag{4.7}$$

defined by $q = \overline{\pi}_\sharp \circ \mathrm{Av}^3 = \mathrm{Av}^3 \circ \overline{\pi}_\sharp$, where the commutativity of composition is a consequence of the definitions. Furthermore the continuity of the average and the projection mapping ensure that q is continuous.

We denote by $\mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3)$ the subset of $W^{2,1,p}$-HG Young measures with underlying deformation (u, b) such that $(u(x), b(x)) = ((x_1, x_2, 0), (0, 0, 1))$ on $\partial\omega$.

In the sequel we state the main theorems which hold under the continuity and coercivity assumptions made on W in section 3. The proofs will be given in the sequel.

THEOREM 4.8. – *The functionals I^M_ε introduced in* (3.8) *are* $(q, w^* L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 2}))$*-equicoercive, according to Definition* 2.12.

THEOREM 4.9. – *Let $I^{\mathcal{M}}_\varepsilon$ be the family of functionals defined in* (3.8). *Then*

$$\Gamma(q, w^* L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2})) \lim_{\varepsilon\to 0} I^{\mathcal{M}}_\varepsilon(\sigma) = I(\sigma)$$

with

$$I(\sigma) = \begin{cases} \displaystyle\int_\omega \langle \sigma_{x_a}, W_0\rangle dx_a & \text{if } \sigma \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3) \\ +\infty & \text{otherwise in } L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2})), \end{cases} \tag{4.8}$$

where the function W_0 has been defined in (3.4).

To make precise the variational character of the limit energy we state the following result:

THEOREM 4.10. – *Let I be the functional defined in Theorem* 4.9. *We have:*

i) *I is weakly $*$ lower semicontinuous and weakly $*$ coercive; hence it admits minimum on the space $\mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3)$;*

ii) *if $u_\varepsilon \in W^{2,p}(\Omega, \mathbb{R}^3)$ satisfies the boundary condition $u_\varepsilon(x) = (x_1, x_2, \varepsilon x_3)$ on the lateral boundary $\partial_L\Omega$ and $\liminf_{\varepsilon\to 0} I^\Omega_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \inf I^\Omega_\varepsilon$ (i.e. if u_ε minimizes I^Ω_ε) then*

a) *if $\varepsilon_n \to 0$ and if $q(\delta_{(D^2 u_{\varepsilon_n}, \frac{1}{\varepsilon_n} D(D_3 u_{\varepsilon_n}))}) \to \sigma$ weakly $*$ in $L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$ then σ is a minimizer of I on $\mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3)$ and $\lim_{n\to\infty} I^\Omega_{\varepsilon_n}(u_{\varepsilon_n}) = I(\sigma)$;*

b) *there is a sequence $\varepsilon_n \to 0$ and a minimizer σ of I on $\mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3)$ such that $q(\delta_{(D^2 u_{\varepsilon_n}, \frac{1}{\varepsilon_n} D(D_3 u_{\varepsilon_n}))}) \to \sigma$ weakly $*$ in $L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$.*

5. – Relaxation Results.

In the sequel we prove a series of results that will be exploited in the next section and which give our Γ-limits as relaxed functionals of suitable ones.

With the notations of Section 3, consider the functional

$$E(u,b) = \int_\omega W_0(D_a^2 u, D_a b)dx_a, (u,b) \in W_{\mathcal{B}}^{2,p}(\omega;\mathbb{R}^3)\times W_{\mathcal{B}}^{1,p}(\omega;\mathbb{R}^3). \tag{5.1}$$

and its relaxed one

$$\overline{E}(u,b) := \inf\left\{ \liminf_{n\to\infty} \int_\omega W_0(D_a^2 u_n, D_a b_n)dx_a : (u_n,b_n) \in W_{\mathcal{B}}^{2,p}(\omega;\mathbb{R}^3)\times W_{\mathcal{B}}^{1,p}(\omega;\mathbb{R}^3), (u_n,b_n) \rightharpoonup (u,b) \text{ in } W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)\right\}. \tag{5.2}$$

THEOREM 5.1. – *Let E be the functional defined in* (5.1) *and let* $\bar{E}$ *be its relaxed functional in* (5.2). *The following representation formula holds*

$$\overline{E}(u,b) = \int_\omega Q_{\mathcal{A}^2} W_0(D_a^2 u, D_a b)dx_a$$

for every $(u,b) \in W_{\mathcal{B}}^{2,p}(\omega;\mathbb{R}^3)\times W_{\mathcal{B}}^{1,p}(\omega;\mathbb{R}^3)$, *where* $\mathcal{Q}_{\mathcal{A}^2} W_0$ *is the* $\mathcal{A}$*-quasiconvexification of* W_0 *relative to the operator* $\mathcal{A}^2$, *namely*

$$\mathcal{Q}_{\mathcal{A}^2} W_0((h,\xi)) = \inf\left\{ \int_{Q_2} W_0((h + D_a^2\psi, \xi + D_a\varphi))dx : \varphi \in C_0^\infty(Q_2;\mathbb{R}^3), \psi \in C_0^\infty(Q_2,\mathbb{R}^3)\right\}. \tag{5.3}$$

PROOF. – Standard relaxation arguments show that $\overline{E}$ coincides with the functional

$$\overline{\overline{E}}(u,b) := \inf\left\{ \liminf_{n\to\infty} \int_\omega W_0(D_a^2 u_n, D_a b_n)dx_a : (u_n,b_n) \in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3), (u_n,b_n) \rightharpoonup (u,b) \text{ in } W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)\right\} \tag{5.4}$$

on the set $W_{\mathcal{B}}^{2,p}(\omega;\mathbb{R}^3)\times W_{\mathcal{B}}^{1,p}(\omega;\mathbb{R}^3)$.

An argument entirely similar to that of Theorem 1.3 in [8] guarantees that the functional $\overline{\overline{E}}$ in (5.4) can be obtained, in the same class, as

$$\overline{\overline{E}}(u,b) := \inf\left\{ \liminf_{n\to\infty} \int_\omega W_0(w_n, d_n)dx_a, (w_n,d_n) \in L^p(\omega; Sym(\mathbb{R}^2)\times M^{3\times 2}) \cap \mathrm{Ker}\mathcal{A}^2, (w_n,d_n) \rightharpoonup (D_a^2 u, D_a b) \text{ in } L^p(\omega; Sym(\mathbb{R}^2)\times M^{3\times 2}), w_n \text{ and } d_n \text{ are } p-\text{equi-integrable}\right\}.$$

Again as in the proof of Corollary 3.2 in [8] one can avoid to use p-equi-integrable sequences, thus obtaining that

$$(5.5)\ \overline{\overline{E}}(u,b):=\inf\left\{\liminf_{n\to\infty}\int_\omega W_0(w_n,d_n):(w_n,d_n)\in L^p(\omega;Sym(\mathbb{R}^2)\times M^{3\times 2})\cap\ker\mathcal{A}^2,\right.$$
$$\left. v_n\rightharpoonup(D_a^2u,D_ab)\text{ in }L^p(\omega;Sym(\mathbb{R}^2)\times M^{3\times 2})\right\}$$

for every $(u,b)\in W^{2,p}_{\mathcal{B}}(\omega;\mathbb{R}^3)\times W^{1,p}_{\mathcal{B}}(\omega;\mathbb{R}^3)$. Remark 3.2, (5.4), (5.5) and Theorem 2.9 applied to $f:=W_0$ with the differential operator $\mathcal{A}^2$ and to $v:=(w,d)$ give the desired representation

$$\bar{E}(u,b)=\int_\omega Q_{\mathcal{A}^2}W_0(D_a^2u,D_ab)dx_a$$

for every $(u,b)\in W^{2,p}_{\mathcal{B}}(\omega;\mathbb{R}^3)\times W^{1,p}_{\mathcal{B}}(\omega;\mathbb{R}^3)$. □

LEMMA 5.2. – *Let* $\{(v_j,d_j)\}$ *be a bounded sequence in* $W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$. *There exists a sequence* $\{(u_j,b_j)\}\in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ *such that* $\{(|D_a^2u_j|^p,|D_ab_j|^p)\}$ *is equi-integrable and the two sequences* $\{(D_a^2v_j,D_ad_j)\}$ *and* $\{(D_a^2u_j,D_ab_j)\}$ *have the same underlying parametrized* $W^{2,1,p}$*-measure.*

PROOF. – Because of Theorem 3.8 consider first a subsequence $\{(D_a^2v_k,D_ad_k)\}$ of $\{(D_a^2v_j,D_ad_j)\}$ that generates the Young measure υ_x.

By Proposition 2.7 there exists a p-equi-integrable sequence $\{(w_k,r_k)\}\subset L^p(\omega;\mathbb{R}^3)\times L^p(\omega;\mathbb{R}^3)\cap\mathrm{Ker}\mathcal{A}^2$ (where $\mathcal{A}^2$ is the differential operator introduced by (3.5)) such that $(w_k,r_k)\rightharpoonup(D_a^2u,D_ab)$ in $L^p(\omega;\mathbb{R}^3)$, and

$$\int_\omega w_kdx=\int_\omega D_a^2udx,\ \|w_k-D_a^2v_k\|_{L^s(\omega)}\to 0\text{ for all }1\le s<q$$
$$\int_\omega r_kdx=\int_\omega D_abdx,\ \|r_k-D_ad_k\|_{L^s(\omega)}\to 0\text{ for all }1\le s<q.$$

We observe that a finite induction argument similar to that of (2.7) and (2.8) shows that $\mathcal{A}^2(v_k,r_k)=0$ if and only if there exists $(\hat{u}_k,\hat{b}_k)\in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ such that $(D_a^2\hat{u}_k,D_a\hat{b}_k)=(v_k,r_k)$. By Lemmas 1.1-1.3 in [29], for any $\gamma\in W^{s,p}(\omega;\mathbb{R}^3)$, $s=1,2$ we may find a unique function $P\in C^\infty(\mathbb{R}^2;\mathbb{R}^3)$ whose components are polynomials of degree $s-1$ such that

$$\int_\Omega D^l(\gamma-P)dx=0,\ \ 0\le l\le s-1, \tag{5.6}$$

and a constant $C=C(2,s,p,\text{ meas }(\omega))$ such that the following Poincarè type inequality holds

$$\|\gamma-P\|_{W^{s,p}(\omega;\mathbb{R}^3)}\le C\|D^s\gamma\|_{L^p(\omega;\mathbb{R}^3)}. \tag{5.7}$$

Let P_k, P, Q_k, Q be the functions associated with $\hat{u}_k u, \hat{b}_k$ and b respectively, and satisfying (5.6) and (5.7). Since $D^2_a \hat{u}_k \rightharpoonup D^2_a u$ and $D_a \hat{b}_k \rightharpoonup D_a b$ in $L^p(\omega; Sym(\mathbb{R}^2))$ and $L^p(\omega; M^{3\times 2})$, we have that

$$\hat{u}_k - P_k \rightharpoonup u - P \text{ in } W^{2,p}(\omega; \mathbb{R}^3),$$
$$\hat{b}_k - Q_k \rightharpoonup b - Q \text{ in } W^{1,p}(\omega; \mathbb{R}^3),$$

hence

$$u_k := \hat{u}_k - P_k + P \rightharpoonup u \text{ in } W^{2,p}(\omega; \mathbb{R}^3),$$
$$b_k := \hat{b}_k - Q_k + Q \rightharpoonup b \text{ in } W^{1,p}(\omega; \mathbb{R}^3),$$

which concludes the proof.

The following lemma generalizes to our context well-known arguments for parametrized measures generated by gradients.

LEMMA 5.3. – *Let ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary, and let $1 < p < \infty$ and let $\{(v_j, d_j)\}$ be a bounded sequence in $W^{2,p}(\omega; \mathbb{R}^3)\times W^{1,p}(\omega; \mathbb{R}^3)$ such that the sequence $\{(D^2_a v_j, D_a d_j)\}$ generates the parametrized measure $\sigma = \{\sigma_x\}_{x\in\omega}$. Furthermore let*

$$w = (\tilde{h}, \xi) = (D^2_a u, D_a b)(x) = \int_{Sym(\mathbb{R}^2)\times M^{3\times 3}} (h|d) d\sigma_x(h|d) \in Sym(\mathbb{R}^2)\times M^{3\times 2},$$

with $(\tilde{h}, \xi) \in \mathrm{Ker}\mathcal{A}^2$ (where $\mathcal{A}^2$ is the differential operator in (3.5)), that is $(u, b) \in W^{2,p}(\omega; \mathbb{R}^3) \times W^{1,p}(\omega; \mathbb{R}^3)$, so that $(v_j, d_j) \rightharpoonup (u, b)$ in $W^{2,p}(\omega; \mathbb{R}^3)\times W^{1,p}(\omega; \mathbb{R}^3)$. There exists a new sequence $\{(u_k, b_k)\}$ bounded in $W^{2,p}(\Omega; \mathbb{R}^3)\times W^{1,p}(\omega; \mathbb{R}^3)$ such that $\{(D^2_a u_k, D_a b_k)\}$ generates the same Young measure σ and $(u_k - u, b_k - b) \in W^{2,p}_0(\omega)\times W^{1,p}_0(\omega)$ for all k. If $\{(|D^2_a v_j|^p, |D_a d_j|^p)\}$ is equi-integrable, so is $\{|D^2_a u_k|^p, |D_a b_k|^p\}$.

PROOF. – Observe first that Lemma 5.2 guarantees the existence of a sequence $\{(u_{j_n}, b_{j_n})\} \subseteq W^{2,p}(\omega; \mathbb{R}^3) \times W^{1,p}(\omega; \mathbb{R}^3)$ such that $u_{j_n} \rightharpoonup u$ in $W^{2,p}(\omega; \mathbb{R}^3)$ and $b_{j_n} \rightharpoonup b$ in $W^{1,p}(\omega; \mathbb{R}^3)$, the sequence $\{(|D^2_a u_{j_n}|^p, |D_a b_{j_n}|^p)\}$ is equi-integrable, and (u, b) is the underlying deformation for θ. Then recall that Theorem 2.3 entails that $\{u_{j_n}, b_{j_n}\}$ still generates θ (but it does not preserve the boundary conditions). Finally a standard argument, considering two sequences of smooth cut-off functions and a diagonalization process, leads to the statement. □

The analogous of Theorem 4.4 in [38] for Gradient Young Measures follows from Lemma 5.2 and 5.3.

THEOREM 5.4. –

$$\inf E(u, b) = \inf I(\sigma) = \inf \overline{E}(u, b)$$

where E, $\overline{E}$ are the functionals defined in (5.1) *and* (5.2)*, respectively and I is the functional in* (4.8).

LEMMA 5.5. – *The functional E in* (5.1) *admits a minimizing sequence $\{(u_k, b_k)\}$ such that $\{(|D_a^2 u_k|^p,\ |D_a b_k|^p)\}$ is weakly convergent in $L^1(\omega)$.*

PROOF. – Let $\{(v_k, d_k)\}$ be any minimizing sequence for E. From (4.3), Remark 3.2, this sequence is a bounded sequence in $W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$. Let $\sigma=\{\sigma_x\}_{x\in\omega}$ denote the $W^{2,1,p}$-HG Young measure associated to $\{(D^2 v_k, Dd_k)\}$. By Lemma 5.2, σ can also be generated by some other sequence $\{(D_a^2 w_k, D_a r_k)\}$ such that $\{|D_a^2 w_k|^p\}$ and $\{|D_a r_k|^p\}$ are weakly convergent in $L^1(\omega)$. In particular both the sequences have the same weak limit in $W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$, u, and $w_k\to u$ strongly in $W^{1,p}(\omega;\mathbb{R}^3)$ and $r_k\to b$ strongly in $L^p(\omega;\mathbb{R}^3)$. By Lemma 5.3 we can find $\{(u_k, b_k)\}$ admissible for E (in the sense of boundary data) and still being p-equi-integrable.

Since $\{(v_k, d_k)\}$ is minimizing

$$\lim_{k\to\infty}\int_\omega W_0(D_a^2 v_k, D_a d_k)dx \le \lim_{k\to\infty}\int_\omega W_0(D_a^2 u_k, D_a b_k)dx$$
$$=\int_\omega\ \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0((h|d))d\sigma_x(h|d)dx.$$

Since Theorem 2.1 prevents strict inequality in the first two terms, $\{(u_k, b_k)\}$ is also minimizing. □

THEOREM 5.6. – *There exists a υ admissible for I in* (4.8) *such that*

$$I(\upsilon)=\inf I(\sigma).$$

PROOF. – Take a minimizing sequence for E in (5.1), $\{(u_k, b_k)\}$, and let υ be the parametrized measure associated to $\{(D_a^2 u_k, D_a b_k)\}$. By Lemma 5.5 we may assume without loss of generality that $\{|D_a^2 u_k|^p\}$ and $\{|D_a b_k|^p\}$ are weakly convergent in $L^1(\omega)$, so that

$$\tilde{m}=m\lim_{k\to\infty}E(u_k, b_k)=I(\upsilon).$$

□

COROLLARY 5.7. – *Define*

$$(5.8)\quad I_\infty(\sigma)=\begin{cases} E(u,b) & \text{if } \sigma=\delta_{(D_a^2 u, D_a b)} : (u,b)\in W_{\mathcal{B}}^{2,p}(\omega;\mathbb{R}^3)\times W_{\mathcal{B}}^{1,p}(\omega;\mathbb{R}^3)\\ +\infty & \text{otherwise,}\end{cases}$$

where E is the functional in (5.1)*. Then the relaxed functional of I_∞ with respect to the weak $*$ convergence of $L_w^\infty(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$ is given by I in* (4.8).

PROOF. – The thesis follows from Lemma 5.5 and Theorem 5.6. □

THEOREM 5.8. – *Let υ be a minimizer for I. If*

$$(D^2_a u(x_a), D_a b(x_a)) = \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} (h|d) d\upsilon_x(h|d), \text{ for a.e. } x \in \omega, \tag{5.9}$$

for $u \in W^{2,p}(\omega;\mathbb{R}^3)$ and $b \in W^{1,p}(\omega;\mathbb{R}^3)$, then (u,b) is a minimizer for $\bar{E}$ in (5.2) *and*

$$\mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b) = \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0(h|d) d\upsilon_x(h|d) \text{ for a.e. } x \in \omega. \tag{5.10}$$

Conversely, if (u,b) is a minimizer for $\bar{E}$ and $\upsilon = \{\upsilon_x\}_{x\in\omega}$ is an admissible $W^{2,1,p}$-HG Young measure such that (5.9) *and* (5.10) *hold, then υ is a minimizer for I.*

PROOF. – It is enough to observe that (u,b) is admissible, and by Jensen's inequality the following chain of inequalities can be written

$$\bar{m} \le \bar{E}(u,b) = \int_\omega \mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b) dx \le \int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} \mathcal{Q}_{\mathcal{A}^2} W_0(h|d) d\upsilon_x(h|d) dx$$

$$\le \int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0(h|d) d\upsilon_x(h|d) dx = I(\upsilon) = \tilde{m} = \bar{m}.$$

Therefore (u,b) is a minimizer for $\bar{E}$ and (5.10) holds true. The same reasoning holds for the converse. □

6. – Comparison with the classical approaches.

Let $J_{\{\varepsilon\}}$ be the functional introduced in (3.3).

THEOREM 6.1. – *Let $(u,b) \in W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\Omega,\mathbb{R}^3)$. Then*

$$J_{\{\varepsilon\}}(u,b) = \inf\Big\{ I(\sigma) : \sigma \in L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}), \langle \sigma, \mathrm{id}\rangle = (D^2_a u(x_a), D_a(b(x_a)) u(x) = (x_1,x_2,0) \text{ on } \partial\omega, b(x) = (0,0,1) \text{ on } \partial\omega \Big\}, \tag{6.1}$$

where $\inf\emptyset = +\infty$ *and the infimum is attained and $J_{\{\varepsilon\}}$ is the functional in* (3.3).

PROOF. – Let $(u, b) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathbb{R}^3)$. The infimum is attained since from item (i) of Theorem 4.10 the functional I turns out to be weakly $*$ coercive and weakly $*$ lower semicontinuous and the center of mass is continuous with respect to the weak $*$ convergence in $L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2) \times M^{3\times 2}))$.

For convenience define the right hand side of (6.1) by $\tilde{I}(u, b)$.

We claim that $J_{\{\varepsilon\}}(u, b) \leq \tilde{I}(u, b)$. Assume that $\tilde{I}(u, b) < +\infty$, whence there exists $\sigma \in L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2) \times M^{3\times 2}))$ such that $I(\sigma) = \tilde{I}(u, b)$ and (u, b) satisfies the boundary value problem $(D^2_a u, D_a b) = \langle \sigma, \mathrm{id} \rangle$, $(u(x), b(x)) = ((x_1, x_2, 0), (0, 0, 1))$ on $\partial\omega$. Consequently $(u, b) = (u(x_a), b(x_a))$.

Since $I(\sigma) < +\infty$, we have that $\theta \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\omega, \mathbb{R}^3)$, hence $(u, b) \in W^{2,p}_{\mathcal{B}}(\omega, \mathbb{R}^3) \times W^{1,p}_{\mathcal{B}}(\omega, \mathbb{R}^3)$. The latter entails $J_{\{\varepsilon\}}(u, b) = \int_\omega \mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b) dx_a$ and by (iii) of Theorem 2.6 it results $\mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b) \leq \langle \sigma, W_0 \rangle$, thus $J_{\{\varepsilon\}}(u, b) \leq I(\sigma) \leq \tilde{I}(u, b)$.

In order to prove the converse inequality, it is convenient to observe that the involved functionals are essentially Γ-limits (cf. [39]).

Since $J_{\{\varepsilon\}}(u, b) = \Gamma(W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3)) - \lim_{\varepsilon \to 0} J_\varepsilon(u)$ there exist $\varepsilon_n \to 0$, $u_n \rightharpoonup u$ in $W^{2,p}(\Omega, \mathbb{R}^3)$ and $\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b$ in $W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\lim_{n\to\infty} J_{\varepsilon_n}(u_n) = J_{\{\varepsilon\}}(u, b)$. Without loss of generality we may assume $J_{\{\varepsilon\}}(u, b) < +\infty$, and from the coerciveness of W (3.1), up to pass to subsequences, one can assume that

i) $\left(D^2 u_n, D\left(\frac{1}{\varepsilon_n} D_3 u_n\right)\right)$ generates a $W^{2,1,p}(\Omega, \mathbb{R}^3)$-HG Young measure γ with underlying deformation (u, b);

ii) $\left(D^2_a u_n, D_a\left(\frac{1}{\varepsilon_n} D_3 u_n\right)\right)$ generates a Young measure σ_Ω;

iii) $\left(D^2_3 u_n, D_3\left(\frac{1}{\varepsilon_n} D_3 u_n\right)\right) \to (0, 0)$ strongly in L^p, hence generates the measure δ_0.

Lemma 4.2 guarantees that $\overline{\pi}_\sharp \gamma = \sigma_\Omega$ and $\pi^3_\sharp \gamma = \delta_0$, thus, by definition $\gamma \in \mathring{\mathcal{Y}}^{2,1,p}_{\partial_L \Omega}(\Omega, \mathbb{R}^3)$.

Furthermore, applying Lemma 4.5 we obtain

$$\gamma = \sigma_\Omega \otimes \delta_0, \text{ and } \langle \sigma_\Omega, \mathrm{id} \rangle = (D^2_a u, D_a b).$$

Let $\gamma_n := \delta_{\left(D^2 u_n, D\left(\frac{1}{\varepsilon_n} D_3 u_n\right)\right)}$ and $\sigma := q(\gamma) = \mathrm{Av}^3(\sigma_\Omega)$. Consequently $\sigma \in \mathcal{Y}^{2,1,p}(\omega, \mathbb{R}^3)$ and one has $\langle \sigma, \mathrm{id} \rangle = (D^2_a u, D_a b)$. Since q is continuous $q(\gamma_n) \rightharpoonup * \sigma$ in $L^\infty_w(\omega, \mathcal{M}(Sym(\mathbb{R}^2) \times M^{3\times 2}))$. From (3.8), Theorem 4.9 and the fact that $J_{\varepsilon_n}(u_n) = I^{\mathcal{M}}_{\varepsilon_n}(\gamma_n)$ one gets that $J_{\{\varepsilon\}}(u, b) = \lim_n J_{\varepsilon_n}(u_n) = \liminf_{n\to\infty} I^{\mathcal{M}}_{\varepsilon_n}(\gamma_n) \geq I(\gamma) \geq \tilde{I}(u, b)$, and this concludes the proof. □

For what concerns the relations between the minimizers of the two functionals $J_{\{\varepsilon\}}$ and I one can prove the following theorem.

THEOREM 6.2. – *The following propositions hold for the functionals I and* $J_{\{\varepsilon\}}$.

i) $\min\left\{J_{\{\varepsilon\}}(u,b) : (u,b) \in W^{2,p}_{\mathcal{B}}(\omega,\mathbb{R}^3)\times W^{1,p}_{\mathcal{B}}(\omega,\mathbb{R}^3)\right\} =$

$$\min\left\{I(\sigma) : \sigma \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\omega,\mathbb{R}^3)\right\}.$$

ii) *Let* σ *be a minimizer of I. If* $(u,b) \in W^{2,p}_{\mathcal{B}}(\omega,\mathbb{R}^3)\times W^{1,p}_{\mathcal{B}}(\omega,\mathbb{R}^3)$ *is such that* $(D^2_a u, D_a b) = \langle \sigma_{x_a}, \mathrm{id}\rangle$ *for a.e.* $x_a \in \omega$ *then* (u,b) *is a minimum point for* $J_{\{\varepsilon\}}$ *and* $\mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b) = \langle \sigma_{x_a}, W_0\rangle$ *for a.e.* $x_a \in \omega$.

iii) *Let* (u,b) *a minimum point for* $J_{\{\varepsilon\}}$. *If* $\sigma \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\omega,\mathbb{R}^3)$ *which satisfies* $(D^2_a u, D_a b)(x_a) = \langle \sigma_{x_a}, \mathrm{id}\rangle$ *and* $\mathcal{Q}_{\mathcal{A}^2} W_0(D^2_a u, D_a b)(x_a) = \langle \sigma_{x_a}, W_0\rangle$ *for a.e.* x_a *in* ω, *then* σ *is a minimum point for I.*

PROOF. – The proof follows from the fact that $J_{\{\varepsilon\}}$ and I are the relaxed functionals of the extensions to $+\infty$ of (5.1) with respect to the strong topology of $W^{1,p}(\omega,\mathbb{R}^3)\times L^p(\omega,\mathbb{R}^3)$ and the weak $*$ of $L^\infty_w(\omega;\, \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$, respectively. The statement follows from Lemma 5.5, Theorems 5.4, 5.6 and 5.8. □

REMARK 6.3. – With an argument similar to that proposed in section 7 of [20] we may propose an energy density for the thin film obtained through our asymptotic analysis. Indeed it is possible to see that W_0 and

$$\tilde{E}(u,b) = \int_\omega W_0(D^2_a u, D_a b)\, dx_a$$

are the energy density and the total energy of the thin film, respectively.

To this end, one may replace the boundary data (x_1, x_2, x_3) by considering more general data, (say tensors $(\mathcal{B})_1 \equiv (B'|B'') \in Sym(\mathbb{R}^3)$, where $B' \in Sym(\mathbb{R}^2)$ and $B'' \in M^{3\times 2}$ according to decomposition (2.4)) i.e. quadratic ones, more explicitly with no term of the kind x_3^2 and prove the Γ-convergence results above (i.e. the analogous of (3.3) and Theorem 4.8). On the other hand one can observe that also the analogues of Theorem 5.1 and Corollary 5.7 hold in this framework obtaining two relaxed functionals $\overline{E}_{\mathcal{B}_1}$ and $I_{\mathcal{B}_1}$, respectively .

Finally one can prove that W_0 is the unique continuous integrand satisfying the growth condition in (4.3) such that $I_{\mathcal{B}_1}$ is the relaxed functional of I_∞ for every such boundary data. To show this it is easily seen that if there were another energy density W_1, such that $I_{\mathcal{B}_1}$ were the relaxed functional then $\int_\omega \langle \sigma_{x_a}, W_0 - W_1\rangle dx_a = 0$ for every σ and for every data. By simply taking $\sigma = \delta_{B',B''}$ the statement follows.

Hence the energy functional I determines a unique energy density W_0. This selection is not provided by the classical relaxation formulas in the Sobolev framework, i.e. with $\mathcal{Q}_{\mathcal{A}^2} W_0$: indeed, in general, there exists an infinite number of functionals W_1 than can represent $\overline{E}_{\mathcal{B}_1}$ for every "quadratic" boundary data $\mathcal{B}_1$.

7. – Proofs of the main results.

LEMMA 7.1. – *Let $\sigma \otimes \delta_0$ be a $W^{2,1,p}(\Omega, \mathbb{R}^3)$-HG Young measure. Then* $\mathrm{Av}^3\sigma$ *is a $W^{2,1,p}(\omega, \mathbb{R}^3)$-HG Young measure, and* $\langle \mathrm{Av}^3\sigma, \mathrm{id}\rangle = \langle \sigma_{(x_a,x_3)}, \mathrm{id}\rangle$ *for a.e. $x \in \Omega$. Conversely, for every $W^{2,1,p}(\omega, \mathbb{R}^3)$-HG Young measure $\overline{\sigma}$, there exists a $W^{2,1,p}(\Omega, \mathbb{R}^3)$-HG Young measure $\sigma \otimes \delta_0$ such that $\overline{\sigma} = \mathrm{Av}^3\sigma$.*

PROOF. – In order to show that $\mathrm{Av}^3\sigma$ is a $W^{2,1,p}(\omega, \mathbb{R}^3)$-hessian-gradient Young measure it is possible to adopt the characterization given from Theorem 2.6. Indeed it is sufficient to show that

(i) $\int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} |h|d|^p d\mathrm{Av}^3_{x_a}\sigma(h|d)dx_a < +\infty$, where by $(h|d)$ we denote an element of $Sym(\mathbb{R}^2)\times M^{3\times 2}$ according to (2.1), (4.1) and (4.2);

(ii) $\langle \mathrm{Av}^3_{x_a}\sigma, \mathrm{id}\rangle = (D^2_a u, D_a b)(x_a)$ and $(u, b) \in W^{2,p}(\omega, \mathbb{R}^3)\times W^{1,p}(\omega, \mathbb{R}^3)$;

(iii) for a.e. $x \in \Omega$ and every continuous function f, such that $|f(v)| \le C(1 + |v|^p)$ for some $C > 0$ and every $v \in \mathbb{R}^d$ one has

$$\langle \mathrm{Av}^3_{x_a}\sigma, f\rangle \ge \mathcal{Q}_{\mathcal{A}^2} f(\langle \mathrm{Av}^3_{x_a}\sigma, \mathrm{id}\rangle),$$

where $\mathcal{A}^2$ is the operator introduced in (3.5).

In order to see that property (i) holds one can observe that by Definition 4.7

$$\int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} |h|d|^p d\mathrm{Av}^3_{x_a}\sigma(h|d)dx_a = \int_\omega \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} |h|d|^p d\sigma_{x_a,x_3}(h|d)dx_a dx_3$$

$$= \int_\Omega \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} \left| \begin{matrix} h & \xi & | & d \\ \xi & c & | & e \end{matrix} \right|^p d\sigma_x \otimes \delta_0 \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e \end{pmatrix} dx < +\infty.$$

To prove (ii) it is worthwhile to observe that if $(u, b) \in W^{2,p}(\Omega, \mathbb{R}^3)\times W^{1,p}(\Omega; \mathbb{R}^3)$ is an underlying deformation for $\sigma \otimes \delta_0$, then Lemma 4.5 guarantees that $u \equiv cx_3 + \overline{u}(x_a)$, $b \equiv b(x_a)$ for some constant c, and $\langle \sigma_{(x_a,x_3),\mathrm{id}}\rangle = (D^2_a u(x_a), D_a b(x_a))$. Hence it follows that

$$\langle \mathrm{Av}^3_{x_a}\sigma, \mathrm{id}\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \sigma_{x_a,x_3}, \mathrm{id}\rangle dx_3 = (D^2_a u(x_a), D_a b(x_a)), \tag{7.1}$$

which is (ii).

For each function $g : Sym(\mathbb{R}^2)\times M^{3\times 2} \to \mathbb{R}$, define $g^\sharp : Sym(\mathbb{R}^3)\times M^{3\times 3} \to \mathbb{R}$, in such a way that

$$g^\sharp(H|F) := g(h|d), \tag{7.2}$$

for each $H := \begin{pmatrix} h & \xi \\ \xi & c \end{pmatrix}$ and $F := (d|e)$.

Assume that $f : Sym(\mathbb{R}^2)\times M^{3\times 2} \to \mathbb{R}$ is a continuous function such that $|f(h|d)| \leq C(|(h|d)|^p + 1)$.

Let $U \subseteq \mathbb{R}^2$ be a bounded open set such that $|\partial U| = 0$, then we recall that

$$\int_U f((h|d) + (D_a^2\phi|D_a\psi))dx_a \geq Q_{\mathcal{A}^2}f(h|d)|U|,$$

for every $\phi \in C_0^2(U;\mathbb{R}^3)$ and $\psi \in C_0^1(U;\mathbb{R}^3)$ and for every $h \in Sym(\mathbb{R}^2)$ and $d \in M^{3\times 2}$, where $\mathcal{A}^2$ is the operator defined by (3.5) and $Q_{\mathcal{A}^2}f$ is the $\mathcal{A}$-quasiconvexification introduced in (5.3) but relative to the function f.

Considering the function f above and its extension $f^\sharp$ to $Sym(\mathbb{R}^3)\times M^{3\times 3}$ in the sense of (7.2), it can be observed that

$$Q_{\mathcal{A}^3}(f^\sharp)(H|F) \geq (Q_{\mathcal{A}^2}f)^\sharp(H|F) \tag{7.3}$$

for every $(H|F) \in Sym(\mathbb{R}^3)\times M^{3\times 3}$, and where $Q_{\mathcal{A}^3}(f^\sharp)$ is the $\mathcal{A}^3$-quasiconvexification of $f^\sharp$ given by (2.6). Indeed it results

$$Q_{\mathcal{A}^3}(f^\sharp)(H|F) = \inf\left\{\int_{Q_3} f^\sharp(H + D^2\Phi, F + D\Psi)dx : \Phi \in C_0^2(Q_3;\mathbb{R}^3), \Psi \in C_0^1(Q;\mathbb{R}^3)\right\}$$

$$= \inf\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_3 \int_{Q_a}^{f} (h + D_a^2\Phi, d + D_a\Psi)dx_a : \Phi \in C_0^2(Q_3;\mathbb{R}^3), \Psi \in C_0^1(Q_3;\mathbb{R}^3)\right\}$$

$$\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} Q_{\mathcal{A}^2}f(h,d)dx_3 = Q_{\mathcal{A}^2}f(h,d) = (Q_{\mathcal{A}^2}f)^\sharp(H|F).$$

where it has been used the fact that $\Psi(x_a, x_3)$ and $\Phi(x_a, x_3) \in C_0^2(Q_a;\mathbb{R}^3)$ and $C_0^1(Q_a;\mathbb{R}^3)$ for every $x_3 \in]-\frac{1}{2},\frac{1}{2}[$.

By (iii) of Theorem 2.6 applied to the $W^{2,1,p}$-HG Young measure $\sigma \otimes \delta_0$, and from (7.2),(7.3), (7.1) it results that

$$\langle \mathrm{Av}^3_{x_a}\sigma, f\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} f(h|d)d\sigma_{x_a,x_3}(h|d)dx_3$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} f^\sharp\begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e\end{pmatrix} d\sigma_{(x_a,x_3)} \otimes \delta_0 \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e\end{pmatrix} dx_3$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \sigma_{(x_a,x_3)} \otimes \delta_0, f^\sharp\rangle dx_3 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} Q_{\mathcal{A}^3}(f^\sharp)(\langle \sigma_{(x_a,x_3)} \otimes \delta_0, \mathrm{id}\rangle)dx_3$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} Q_{\mathcal{A}^3}(f^\sharp)\begin{pmatrix} D_a^2u(x_a) & 0 & | & D_ab(x_a) \\ 0 & 0 & | & 0\end{pmatrix} dx_3$$

$$\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} (Q_{\mathcal{A}^2} f)^\sharp \begin{pmatrix} D_a^2 u(x_a) & 0 & | & D_a b(x_a) \\ 0 & 0 & | & 0 \end{pmatrix} dx_3$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} Q_{\mathcal{A}^2} f(D_a^2 u(x_a), D_a b(x_a)) dx_3 = Q_{\mathcal{A}^2} f(\langle \mathrm{Av}^3_{x_a} \sigma, \mathrm{id} \rangle),$$

which completes the proof of (iii).

Finally, concerning the last part of the Theorem, consider $\overline{\mu}$ a $W^{2,1,p}(\omega, \mathbb{R}^3)$-HG Young measure. Obviously $\mu = \overline{\mu} \otimes \delta_0$ turns out to be a $W^{2,1,p}(\Omega, \mathbb{R}^3)$-HG Young measure. Moreover, clearly $\mathrm{Av}^3(\mu) = \overline{\mu}$, and this completely proves the theorem. □

The next result relates the spaces $\overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)$ and $\mathcal{Y}^{2,1,p}_{\partial \omega}(\omega; \mathbb{R}^3)$.

Lemma 7.2. – *It results*

$$\mathrm{Av}^3 \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3) = \mathcal{Y}^{2,1,p}_{\partial \omega}(\omega; \mathbb{R}^3) \otimes \delta_0.$$

Proof. – Let $\theta \in \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)$. Then Lemma 4.5 entails $\theta = \sigma \otimes \delta_0$, for a suitable σ. Let $\varphi \in C_0(Sym(\mathbb{R}^3) \times M^{3\times 3})$, then by Definition 4.7

$$\langle \mathrm{Av}^3_{x_a} \theta, \varphi \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Sym(\mathbb{R}^3)\times M^{3\times 3}} \varphi \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e \end{pmatrix} d\sigma_{(x_a,x_3)} \otimes \delta_0 \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e \end{pmatrix} dx_3$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}}^{\varphi} \begin{pmatrix} h & 0 & | & d \\ 0 & 0 & | & 0 \end{pmatrix} d\sigma_{x_a,x_3} \begin{pmatrix} h & 0 & | & d \\ 0 & 0 & | & 0 \end{pmatrix} dx_3$$

$$= \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}}^{\varphi} \begin{pmatrix} h & 0 & | & d \\ 0 & 0 & | & 0 \end{pmatrix} d\mathrm{Av}^3_{x_a} \sigma(h|d)$$

$$= \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}}^{\varphi} \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e \end{pmatrix} d\mathrm{Av}^3_{x_a} \sigma \otimes \delta_0 \begin{pmatrix} h & \xi & | & d \\ \xi & c & | & e \end{pmatrix} = \langle (\mathrm{Av}^3{}_{x_a} \sigma) \otimes \delta_0, \varphi \rangle$$

Hence $\mathrm{Av}^3(\sigma \otimes \delta_0) = (\mathrm{Av}^3 \sigma) \otimes \delta_0$, and the statement follows applying Lemma 7.1. □

Remark 7.3. – From Lemma 7.2 it follows the very useful equality

$$q(\overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)) = \mathcal{Y}^{2,1,p}_{\partial \omega}(\omega; \mathbb{R}^3),$$

where q is the mapping defined in (4.7).

7.1 – *Proof of Theorem* 4.8.

Since $L^\infty_w(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$ is the dual of the separable Banach space $L^1(\omega;\, C_0(Sym(\mathbb{R}^2)\,\times M^{3\times 2}))$ the proof follows from the following result.

PROPOSITION 7.4. – *If* $\{\varepsilon_n\}$ *is a sequence such that* $\varepsilon_n \to 0$ *and if* $\{\theta_n\} \subseteq L^\infty_w(\Omega;\, \mathcal{M}(Sym(\mathbb{R}^3)\,\times M^{3\times 3}))$ *satisfies*

$$\sup_n I^{\mathcal{M}}_{\varepsilon_n}(\theta^n) < +\infty$$

then there exists a subsequence $\{n_k\}$ *and a Young measure* $\sigma \in L^\infty_w(\omega;\, \mathcal{M}(Sym(\mathbb{R}^2))\times M^{3\times 2})$ *such that* $q(\theta^{n_k}) \to \sigma$ *weakly* $*$ *in* $L^\infty_w(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$ *and* $\sigma \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\Omega;\mathbb{R}^3)$.

PROOF. – Clearly Lemma 4.1 guarantees that there exists a Young measure $\theta \in \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L\Omega}(\Omega;\mathbb{R}^3)$ and a subsequence $\{\theta^{n_k}\}$ such that $\theta^{n_k} \overset{*}{\rightharpoonup} \theta$ in $L^\infty_w(\Omega;\mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$. Let $\sigma := q(\theta)$. From the continuity of q it easily follows that $q(\theta^{n_k}) \overset{*}{\rightharpoonup} \sigma$ in $L^\infty_w(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$, which proves the first part of the statement. Finally, Remark 7.3 ensures that $\sigma \in \mathcal{Y}^{2,1,p}_{\partial\omega}(\omega;\mathbb{R}^3)$. □

7.2 – *Proof of Theorem* 4.9.

Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Since, by Theorem 4.8 the sequence $\{I^{\mathcal{M}}_{\varepsilon_n}\}$ is $L^\infty_w(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$-weakly coercive, Proposition 2.14 applies.

First we claim the existence a recovery sequence for a suitable subsequence $\{I^{\mathcal{M}}_{\varepsilon_{n_k}}\}$, i.e. for every $\sigma \in \mathcal{Y}^{2,1,p}(\omega;\mathbb{R}^3)$ one can determine a sequence $u_k \in W^{2,p}(\Omega;\mathbb{R}^3)$, with $u_k(x) = (x_1, x_2, \varepsilon_{n_k} x_3)$ on Γ, such that the projection of the averages with respect to x_3 of the associated Young measures $\theta^k := \delta_{\left(D^2 u_k, \frac{1}{\varepsilon_{n_k}} D(D_3 u_k)\right)}$ converge to σ weakly $*$ in $L^\infty_w(\omega;\mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$, and

$$\limsup_{k\to\infty} I^{\mathcal{M}}_{\varepsilon_{n_k}}(\theta^k) \geq I(\sigma).$$

Let $(u,b) \in W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ such that $u \equiv (x_1, x_2, 0)$ and $b \equiv (0,0,1)$ on $\partial\omega$.

By virtue of Lemma 5.2, there exists a sequence $\{(u_{\varepsilon_j}, b_{\varepsilon_j})\} \subseteq W^{2,p}(\omega;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ such that

$$u_{\varepsilon_j} \rightharpoonup u \text{ in } W^{2,p}(\omega;\mathbb{R}^3) \text{ and } b_{\varepsilon_j} \rightharpoonup b \text{ in } W^{1,p}(\omega;\mathbb{R}^3)$$

and the sequence $\{(|D^2_\alpha u_{\varepsilon_j}|^p, |D_\alpha b_{\varepsilon_j}|^p)\}$ is equi-integrable. On the other hand (u,b) is the underlying deformation for σ and $\{u_{\varepsilon_j}, b_{\varepsilon_j}\}$ still generates σ, but they do not preserve the boundary condition.

Applying lemma 5.3 one can construct $w_{\varepsilon_{j_k}}$ and $p_{\varepsilon_{j_k}}$ such that $|D_a^2 w_{\varepsilon_{j_k}}|^p$ and $|D_a p_{\varepsilon_{j_k}}|^p$ are also equi-integrable and $\{(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}})\}$ still generates σ.

By (3.4) and a measurable selection argument (see [11], proposition 2.2.7), it follows that there exists a sequence of measurable functions $\{z_k\}$, $z_k : \omega \to \mathbb{R}^3$ such that

$$W_0(D_a^2 w_{\varepsilon_{j_k}}(x_a), D_a p_{\varepsilon_{j_k}}(x_a)) = \min\{W(D_a^2 w_{\varepsilon_{j_k}}(x_a), D_a p_{\varepsilon_{j_k}}(x_a), z(x_a)), z \in \mathbb{R}^3\}$$
$$= W(D_a^2 w_{\varepsilon_{j_k}}(x_a), D_a p_{\varepsilon_{j_k}}(x_a), z_k(x_a)) \text{ for a.e. } x_a \in \omega.$$

From the growth condition on W it results that

$$W_0(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}}) = W(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}}, z_k) \leq W(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}}, 0)$$
$$\leq j_k|^p + |D_a p_{\varepsilon_{j_k}}|^p + 1)$$

and

$$W_0(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}}) = W(D_a^2 w_{\varepsilon_{j_k}}, D_a p_{\varepsilon_{j_k}}, z_k)$$
$$\geq c(|D_a^2 w_{\varepsilon_{j_k}}|^p + |D_a p_{\varepsilon_{j_k}}|^p + |z_k|^p - 1) \geq c(|D_a^2 w_{\varepsilon_{j_k}}|^p + |D_a p_{\varepsilon_{j_k}}|^p - 1)$$

almost everywhere, and therefore the sequence $W_0(D_a^2 w^{\varepsilon_{j_k}}, D_a p^{\varepsilon_{j_k}})$ is equi-integrable. Moreover the continuity of W_0 and Theorem 2.1 entail

$$\text{(7.4)} \quad \lim_k \int_\Omega W(D_a^2 w_{\varepsilon_{j_k}}(x_a), D_a p_{\varepsilon_{j_k}}(x_a), z_k(x_a))dx = \lim_k \int_\omega W_0(D_a^2 w_{\varepsilon_{j_k}}(x_a), D_a p_{\varepsilon_{j_k}}(x_a))dx_a$$
$$= \int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0(h|d)d\sigma_{x_a}(h|d)dx_a.$$

Furthermore, again from (4.3) it follows that also $\{z_k\}$ are p-equi-integrable.

Set $v_k := w_{\varepsilon_{j_k}}$ and let $\bar{w_k}$ and $\bar{b_k}$ be functions in $C_0^\infty(\omega; \mathbb{R}^3)$ such that

$$\|z_k - \bar{w_k}\|_p \leq \frac{1}{k}, \quad \|p_{\varepsilon_{j_k}} - (0,0,1) - \bar{b_k}\|_p \leq \frac{1}{k}$$

and let ε_{n_k} a subsequence of $\{\varepsilon_n\}$ such that

$$\text{(7.5)} \qquad \|\varepsilon_{n_k} D_a \bar{w_k}\|_p \to 0, \ \|\varepsilon_{n_k}^2 D_a^2 \bar{w_k}\|_p \to 0, \ \text{and } \|\varepsilon_{n_k} D_a^2 \bar{b_k}\|_p \to 0.$$

Next, define $b_k := (0,0,1) + \bar{b_k}$. Then

(7.6) $b_k = (0,0,1)$ and $\bar{w}_k = (0,0,0)$ on $\partial\omega, b_k$ and w_k are $p-$ equi-integrable,

and we claim that

$$\text{(7.7)} \quad \lim_{k\to\infty} \int_\Omega \Big[W(D_a^2 v_k, D_a b_k, z_k)$$
$$- W\Big(D_a^2 v_k + \varepsilon_{n_k} x_3 D_a^2 b_k + \frac{\varepsilon_{n_k}}{2} x_3^2 D_a^2 \bar{w}_k, D_a b_k + \varepsilon_{n_k} x_3 D_a \bar{w}_k, \bar{w}_k\Big)\Big] dx = 0.$$

Indeed the sequences $\{(D_a^2 v_k, D_a b_k, \bar{w}_k)\}$ and $\{(D_a^2 v_k + \varepsilon_{n_k} x_3 D_a^2 b_k + \varepsilon_{n_k}^2 \frac{x_3^2}{2} D_a^2 \bar{w}_k, D_a b_k + \varepsilon_{n_k} x_3 D_a \bar{w}_k, \bar{w}_k)\}$ generate the same Young measure and, being p-equi-integrable, (4.3) ensures that the sequences

$$W(D_a^2 v_k, D_a b_k, \bar{w}_k) \text{ and } W\left(D_a^2 v_k + \varepsilon_{n_k} x_3 D_a^2 b_k + \varepsilon_{n_k}^2 \frac{x_3^2}{2} D_a^2 \bar{w}_k, D_a b_k + \varepsilon_{n_k} x_3 D_a \bar{w}_k, \bar{w}_k\right)$$

are p-equi-integrable as well.

Again by Theorem 2.1 follows (7.7).

Let us define

$$u_k(x_a, x_3) := v_k(x_a) + \varepsilon_{n_k} x_3 b_k(x_a) + \frac{\varepsilon_{n_k}^2}{2} x_3^2 \bar{w}_k(x_a).$$

Thus (7.7) and (7.4) guarantee that

$$\lim_{k\to\infty} \int_\Omega W\left(D_a^2 u_k, \frac{1}{\varepsilon_{n_k}} D_{a,3}^2 u_k, \frac{1}{\varepsilon_{n_k}^2} D_3^2 u_k\right) dx$$

$$= \lim_{k\to\infty} \int_\Omega^W \left(D_a^2 v_k + \varepsilon_{n_k} x_3 D_a^2 b_k + \varepsilon_{n_k}^2 \frac{x_3^2}{2} D_a^2 w_k, D_a b_k + \varepsilon_{n_k} x_3 D_a w_k, w_k\right) dx$$

$$= \lim_{k\to\infty} \int_\Omega W(D_a^2 v_k, D_a b_k, w_k) dx = \int_\omega \int_{Sym(\mathbb{R}^2)\times M^{3\times 2}} W_0(h|d) d\sigma_{x_a}(h|d) dx_a.$$

Moreover it is easily seen that, from (7.5)

$$u_k \rightharpoonup u \text{ in } W^{2,p}(\Omega;\mathbb{R}^3), \frac{1}{\varepsilon_{n_k}} D_3 u_k \rightharpoonup b \text{ in } W^{1,p}(\Omega;\mathbb{R}^3) \text{ and}$$

$$u(x) \equiv u(x_a), b(x) \equiv b(x_a) \text{ and } u_k(x_a, x_3) = (x_1, x_2, \varepsilon_{n_k} x_3) \text{ on } \partial\omega \times \left(\frac{\varepsilon_{n_k}}{2}, \frac{\varepsilon_{n_k}}{2}\right)$$

which, in turn, entails

$$\theta^k := \left(\delta_{D^2 u_k, \frac{1}{\varepsilon_{n_k}} D(D_a u_k)}\right) \to \sigma \otimes \delta_0 \text{ weakly } * \text{ in } L_w^\infty(\Omega; \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$$

and

$$q(\theta^k) \to q(\sigma\otimes\delta_0) = \sigma \text{ weakly} * \text{ in } L_w^\infty(\omega; \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2})),$$

where the last convergence follows from the continuity of q and from Remark 7.3. This proves the claim.

To conclude the proof it remains to show the liminf inequality for the sequence $I_{\varepsilon_n}^{\mathcal{M}}$. Let $\theta^n \in L_w^\infty(\omega; \mathcal{M}(Sym(\mathbb{R}^3)\times M^{3\times 3}))$ be a sequence such that $q(\theta^n)$ weakly $*$ converges to σ in $L_w^\infty(\omega; \mathcal{M}(Sym(\mathbb{R}^2)\times M^{3\times 2}))$, we must prove that

$$\liminf_{n\to\infty} I_{\varepsilon_n}^{\mathcal{M}}(\theta^n) \geq I(\sigma).$$

Without loss of generality one may also assume that the left hand side of the inequality above is finite and that the lower limit is indeed a limit. Then $\sup_n I^{\mathcal{M}}_{\varepsilon_n}(\theta^n) < +\infty$.

Hence, by Lemma 4.1, there exists $\theta \in \overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)$ and a subsequence θ^{n_k} such that $\theta^{n_k} \rightharpoonup \theta$ weakly $*$ in $L^\infty_w(\Omega; \mathcal{M}(Sym(\mathbb{R}^3) \times M^{3\times 3}))$ and the corresponding underlying deformations u_{n_k} and (u, b) which satisfy the boundary conditions $u_{n_k}(x) = (x_a, \varepsilon_{n_k} x_3)$ and $(u(x), b(x)) = ((x_1, x_2, 0), (0, 0, 1))$ are such that

$$u_{n_k} \rightharpoonup u \text{ weakly in } W^{2,p}(\Omega; \mathbb{R}^3) \text{ and } \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^3).$$

Hence by Remark 7.3 we see that $\sigma \in q(\overset{\circ}{\mathcal{Y}}{}^{2,1,p}_{\partial_L \Omega}(\Omega; \mathbb{R}^3)) = \mathcal{Y}^{2,1,p}(\omega; \mathbb{R}^3)$.

Finally, by (3.8), (3.4), the definition of $\overline{\pi}_\sharp$ in section 4, Definition 4.7, (4.7)

$$\begin{aligned}
\liminf_{n\to\infty} I^{\mathcal{M}}_{\varepsilon_n}(\theta^n) &= \lim_{k\to\infty} I^{\mathcal{M}}_{\varepsilon_{n_k}}(\theta^{n_k}) = \lim_{k\to\infty} \int_\Omega W\left(D^2_a u_{n_k}, \frac{1}{\varepsilon_{n_k}} D_a D_3 u_{n_k}, \frac{1}{\varepsilon^2_{n_k}} D^2_3 u_{n_k}\right) dx \\
&\geq \limsup_{k\to\infty} \int_\Omega W_0\left(D^2_a u_{n_k}, \frac{1}{\varepsilon_{n_k}} D_a D_3 u_{n_k}\right) dx = \liminf_{k\to\infty} \int_\Omega \langle \overline{\pi}_\sharp \theta^{n_k}_x, W_0\rangle dx \\
&= \liminf_{k\to\infty} \int_\omega \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \overline{\pi}_\sharp \theta^{n_k}, W_0\rangle dx_3 \right) dx_a = \liminf_{k\to\infty} \int_\omega \langle \mathrm{Av}^3 \overline{\pi}_\sharp \theta^{n_k}, W_0\rangle dx_a \\
&= \liminf_{k\to\infty} \int_\omega \langle q(\theta^{n_k})_{x_a}, W_0\rangle dx_a \geq \int_\omega \langle \sigma_{x_a}, W_0\rangle dx_a
\end{aligned}$$

where in the last inequality it has been used Theorem 2.1.

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