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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.1, p. 119–158.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_1_119_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 119-158

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Sunto. – *In questa nota si studia l'esistenza di soluzioni per un sistema di due inclusioni con operatori di Fredholm aventi indice non negativo e multifunzioni «fondamentalmente restringibili» e a valori non necessariamente convessi. Si applica la tecnica della mappa soluzione e, poiché le ipotesi consentono un «difetto di dimensione», l'indice di coincidenza, cioè un invariante omotopico basato sulla teoria della co-omotopia. Si forniscono poi due applicazioni ai problemi ai limiti.*

Summary. – *We consider the existence of solutions to the system of two inclusions involving Fredholm operators of nonnegative index and the so-called fundamentally restrictible maps with not necessarily convex values. We apply the technique of a solution map and, since the assumptions admit a 'dimension defect', we use the coincidence index, i.e. the homotopy invariant based on the cohomotopy theory. Two examples of applications to boundary value problems are included.*

1. – Introduction.

In the paper we study the existence of solutions to the system of inclusions

$$\begin{cases} L_1(y) \in F(x,y) \\ L_2(x) \in G(x,y), \end{cases} \tag{1}$$

where E_1, E_2, E_1', E_2' are Banach spaces, $L_1 : E_1 \to E_1'$, $L_2 : E_2 \to E_2'$ are Fredholm operators of nonnegative index, $\Omega \subset E_2 \times E_1$ and $F : \Omega \multimap E_1'$, $G : \Omega \multimap E_2'$ are multivalued maps. Our approach is as follows: after defining the so-called solution map $\Omega_F : E_2 \multimap E_1$ which assigns to any $x \in E_2$ the (possibly empty) set $\{y \in E_1 \mid L_1(y) \in F(x,y)\}$, we consider the coincidence problem

$$L_2(x) \in \Theta(x) := G(\{x\} \times \Omega_F(x)). \tag{2}$$

A similar attitude has been presented in [26], [8], [31], [12] under the assumption that $E_1 = E_1'$, $E_2 = E_2'$ and $L_1 = id_{E_1}$, $L_2 = id_{E_2}$.

In [14], system (1) was studied in a full generality, however, the maps F and G were assumed to be compact. In the present paper we apply methods from [14] in order to embrace much more general classes of maps, namely the so-called fudamentally restrictible ones. To this end we use a topological invariant, called the generalized coincidence index (see [25], [13] and [14]), which allows to examine the existence of solutions to problems of the form (2). Observe that such problems admit a«dimension defect» between domain and codomain. Therefore the usual approach involving homology fails (this issue has been discussed in detail in [14]): instead some methods of cohomotopy theory initiated in [23, 24, 25] have to be applied.

The studied problem admits a complementary approach. Namely one may consider a coincidence problem of the form $L(x,y) \in \Phi(x,y)$, where $E_2 \times E_1 \ni (x,y) \mapsto L(x,y) := (L_1(x), L_2(y)) \in E_2' \times E_1'$ and $(x,y) \mapsto \Phi(x,y) := (G(x), F(y))$ (see e.g. [27], [21], [35], [15], [2], [25], [10], [32]). It seems however that this approach requires different hypotheses.

The paper is organized as follows. In the next section we shortly recall some preliminaries. The considered class of maps (with examples and some auxiliary facts) is discussed in Section 3. Section 4 is devoted to the main abstract results and in Section 5 we give two examples of its application.

Our paper heavily relies on [14]: in fact it may be considered as its continuation. Therefore we do not repeat algebraic issues leading to the main tool of this paper (see Theorem 4.14) which may be found in [14].

2. – Preliminaries.

All spaces considered in the paper are *metric* and all single-valued maps are *continuous*. If V is a subset of a space, then $\mathrm{cl}\,V$, $\mathrm{int}\,V$, and $\mathrm{bd}\,V$ denote the *closure*, the *interior* and the *boundary* of V, respectively. If V is a subset of a Banach space, then $\mathrm{conv}\,V$ stands for its *convex hull* and $\overline{\mathrm{conv}}\,V = \mathrm{cl}\,\mathrm{conv}\,V$. For $z \in \mathbb{R}^n$, $\varepsilon > 0$, let $B^n(z,\varepsilon) = \{x \in \mathbb{R}^n \mid \|x - z\| < \varepsilon\}$, $D^n(z,\varepsilon) = \mathrm{cl}\,B^n(z,\varepsilon)$. Similarly if z belongs to a Banach space E, then $B^E(z,\varepsilon) = \{x \in E \mid \|x - z\| < \varepsilon\}$, $D^E(z,\varepsilon) = \mathrm{cl}\,B^E(z,\varepsilon)$. By $O_\varepsilon(V)$ we denote the set $\{w \in E \mid \|w - x\| < \varepsilon \text{ for some } x \in V\}$

Let E and E' be Banach spaces. A bounded linear map $L : E \to E'$ is called a *Fredholm operator* if its *kernel* $\mathrm{Ker}\,(L)$ and *cokernel* $\mathrm{Coker}\,(L) := E'/\mathrm{Im}\,(L)$, (where $\mathrm{Im}\,(L)$ is the image of L), are finite dimensional. The Fredholm index of L is then defined as follows:

$$i(L) := \dim \mathrm{Ker}\,(L) - \dim \mathrm{Coker}\,(F)\ (^1).$$

All Fredholm operators considered in this paper will have nonnegative indices.

It is easy to see that $\mathrm{Im}\,(L)$ is a closed subspace of E' (see [16, IV.2.6]). Since both

(1) Observe that if $F : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then F is a Fredholm operator and $i(F) = m - n$.

$\operatorname{Ker}(L)$ and $\operatorname{Im}(L)$ are direct summands in E and E', respectively, we may consider continuous linear projections $P : E \to E$ and $Q : E' \to E'$, such that $\operatorname{Ker} L = \operatorname{Im}(P)$ and $\operatorname{Ker} Q = \operatorname{Im}(L)$. Clearly E, E' split into (topological) direct sums

$$\operatorname{Ker}(P) \oplus \operatorname{Ker}(L) = E, \qquad\qquad \operatorname{Im}(Q) \oplus \operatorname{Im}(L) = E'.$$

Moreover, $L|_{\operatorname{Ker} P}$ is a linear homeomorphism onto $\operatorname{Im}(L)$. By $K_P : \operatorname{Im} L \to \operatorname{Ker} P$ we denote the operator inverse to $L|_{\operatorname{Ker} P}$. Note also that L is proper when restricted to a closed bounded set or, more general, to a closed set X such that $P(X)$ is bounded.

Let X, Y be spaces. By a multivalued map $\varphi : X \multimap Y$ we understand an *upper semicontinuous* transformation which assigns to a point $x \in X$ a *compact nonempty* set $\varphi(x) \subset Y$. We say that φ is *compact* if $\operatorname{cl}\varphi(X)$ is compact.

A compact space A is *cell-like* if there exists an absolute neighborhood retract Y and an embedding $i : A \to Y$ such that the set $i(A)$ is contractible in any of its neighborhoods $U \subset Y$. Let A be a cell-like space, Z be an absolute neighborhood retract and $i : A \to Z$ be an embedding. Then $i(A)$ is contractible in an arbitrary neighborhood (in Z); therefore cell-likeness is an absolute property. Compact convex or contractible, or R_δ-sets (i.e. the intersections of decreasing families of compact contractible sets) are cell-like. Cell-like sets are acyclic; however there are examples of acyclic sets which are not cell-like.

Definition 2.1. – *A proper surjection $p : \Gamma \to X$ is a cell-like map if, for each $x \in X$, the fiber $p^{-1}(x)$ is cell-like. We say that a pair (p, q), where $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ (2), is c-admissible if p is a cell-like map. A set-valued map $\varphi : X \multimap Y$ is c-admissible if it is determined by a c-admissible pair (p, q), i.e. $\varphi(x) = q(p^{-1}(x))$ for all $x \in X$.*

Observe that in the above definition φ is compact if and only if q is compact.

Definition 2.2. – *We say that c-admissible pairs $X \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} Y$, $k = 0, 1$ (or maps determined by them) are homotopic, if there exists a c-admissible pair $X \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} Y$ and homeomorphical embeddings $j_k : \Gamma_k \to \Gamma$, $k = 0, 1$ (3) such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[d, "i_0"'] & \Gamma_0 \arrow[l, "p_0"'] \arrow[d, "j_0"'] \arrow[dr, "q_0"] & \\
X \times [0,1] & \Gamma \arrow[l, "R"'] \arrow[r, "S"] & Y \\
X \arrow[u, "i_1"] & \Gamma_1 \arrow[l, "p_1"] \arrow[u, "j_1"] \arrow[ur, "q_1"'] &
\end{tikzcd}$$

(2) In what follows we write $(p, q) : X \to Y$. The domain of p and q will be stated explicitly if necessary.

(3) i.e. a map $j_k : \Gamma_k \to j_k(\Gamma_k)$ is a homeomorphism.

where $i_k(x) = (x, k)$ *for* $k = 0, 1$ *and* $x \in X$. *The pair* (R, S) *(or the map determined by it) is called a homotopy between pairs* (p_0, q_0) *and* (p_1, q_1) *(or between maps determined by them).*

REMARK 2.3. – Observe that (4)

(*i*) If $\varphi : X \multimap Y$ is a c-admissible map, then its restriction $\varphi|_A$ to $A \subset X$, is c-admissible.

(*ii*) If $f : Y \to Z$ is a single-valued map, then it is c-admissible (determined by (id_Y, f)). Moreover, the composition $f \circ \varphi$, where φ is a c-admissible map, defined by $f \circ \varphi(x) := \{f(y) \mid y \in \varphi(x)\}$ is also c-admissible (since determined by $(p, f \circ q)$).

(*iii*) A set-valued map $\varphi : X \multimap Y$ given, for $x \in X$, by $\varphi(x) = \varphi_1(x) \times \varphi_2(x)$, where $\varphi_1, \varphi_2 : X \multimap Y$ are c-admissible maps, is also a c-admissible one. Similarly the map $\psi : X \multimap Y$, defined by $\psi(x) := \varphi_1(x) + \varphi_2(x) = \{y_1 + y_2 \mid y_1 \in \varphi_1(x), y_2 \in \varphi_2(x)\}$, $x \in X$, is c-admissible. In particular, if a c-admissible pair (p, q) determines a map $\varphi : X \to Y$ and $f : X \to Y$, then the map $f + \varphi : X \multimap Y$, given by $(f + \varphi)(x) = \{f(x) + y \mid y \in \varphi(x)\}$ for $x \in X$, is c-admissible (it is determined by the pair $(p, f \circ p + q)$).

The well known class of set-valued maps with not necessarily convex values consists of maps *admissible in the sense of Górniewicz*, i.e. determined by pairs (p, q) such that p is a Vietoris map (it means a proper surjection with acyclic (5) fibers) (see [18], [17], [5], [6] and [22]). One of the main reasons to study admissible pairs (and multivalued maps determined by them) in classical sense follows from the famous Vietoris-Begle theorem (see e.g. [34]) which states that *if* $p : \Gamma \to X$ *is a Vietoris map, then the induced homomorphism* $p^* : H^*(X) \to H^*(\Gamma)$ *is an isomorphism.* It allows to define a topological degree using homological methods. But, since we admit a «dimensional defect», this result seems to be useless in our framework where no standard (co)homological issues play a role. Therefore we have to consider a less general class of maps satisfying a cohomotopy version of the Vietoris theorem due to W. Kryszewski (see [23], [24], [25], [13]). From the viewpoint of applications, it is sufficient to consider pairs and maps admissible in the sense of Definition 2.1, so results of this paper are stated for them. However all these results stay true for pairs $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ and maps determined by them, such that p is a Vietoris map satisfying the additional condition: $\sup_{x \in X} \dim p^{-1}(x) < \infty$ (see [25], [11], [13]). Observe that if $\dim \Gamma < \infty$, then the above holds automatically; conversely if p is a Vietoris map and $\dim X < \infty$, then $\dim \Gamma < \infty$, too.

(4) One can find the proof in [14].

(5) A compact space A is *acyclic* with respect to the Čech cohomology H^* with integer coefficients, if $H^*(A) = H^*(pt)$, where pt is a one-point space.

In order to establish hypotheses necessary to state our main results we need some facts concerning the cohomotopy products. Here we briefly recall some preliminaries - the details may be found in [14] (see also [36], [19]).

By a *pair of spaces* we always understand a pair (X,A) where X is a space and $A \subset X$ is *closed*. By $[X,A;K,L]$ we denote the set of homotopy classes $[f]$ of maps $f:(X,A) \to (K,L)$, i.e. $f: X \to K$ and $f(A) \subset L$. Let $\pi^n(X,A) := [X,A;S^n,s_0]$ (6), where S^n stands for the unit n-dimensional sphere and $s_0 = (1,0,\ldots,0) \in \mathrm{R}^{n+1}$ is the base point.

Let $\xi \in \pi^n(X,A)$ and $\eta \in \pi^m(Y,B)$, $n,m \geq 0$, be represented by $u:(X,A) \to (S^n,s_0)$ and $v:(Y,B) \to (S^m,s_0)$, respectively. Then $u \times v : (X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y) \to (S^n \times S^m, S^n \times \{s_0\} \cup \{s_0\} \times S^m)$ is given by $(u \times v) \cdot (x,y) = (u(x),v(y))$. We define

$$\xi \otimes \eta := [g \circ (u \times v)] \in \pi^{n+m}((X,A) \times (Y,B)), \tag{3}$$

where the map $g : (S^n \times S^m, S^n \times \{s_0\} \cup \{s_0\} \times S^m) \to (S^{n+m},s_0)$, is a quotient projection.

The *external product*

$$\otimes : \pi^n(X,A) \times \pi^m(Y,B) \to \pi^{n+m}((X,A)\times(Y,B)) = \pi^{n+m}(X \times Y, X \times N \cup A \times Y)$$

defined above has many properties which are useful in the considerations concerning finite dimensional and compact cases (see [14], [11]). Below we mention only a few facts, strictly connected with this paper.

REMARK 2.4. – If $2n-1 > m \geq n$ and $0 \neq \xi \in \pi^n(S^m,s_0)$, then

$$\xi \otimes \nu^k \neq 0,$$

where ν^k is a homotopy class of the identity map in $\pi^k(S^k,s_0)$.

If $m \leq 2n-1$ we can identify the cohomotopy group $\pi^n(S^m,s_0)$ with the stable homotopy group Π_{m-n} which is finite and abelian for any $m > n$ (see [19]). Therefore if we regard Π_{i_1} and Π_{i_2} as $\pi^n(S^m,s_0)$ and $\pi^l(S^k,s_0)$, where $m-n=i_1$, $k-l=i_2$, $2n-1>m$ and $2l-1>k$, then we may consider an external product $\xi \otimes \eta$ of $\xi \in \Pi_{i_1}$ and $\eta \in \Pi_{i_2}$. Moreover, if $\xi \neq 0$ and ν is an element of Π_{i_2} identified with $\nu^k \in \pi^k(S^k,s_0)$ $(k>1)$, then $\xi \otimes \nu \neq 0$.

3. – Fundamentally restrictible maps.

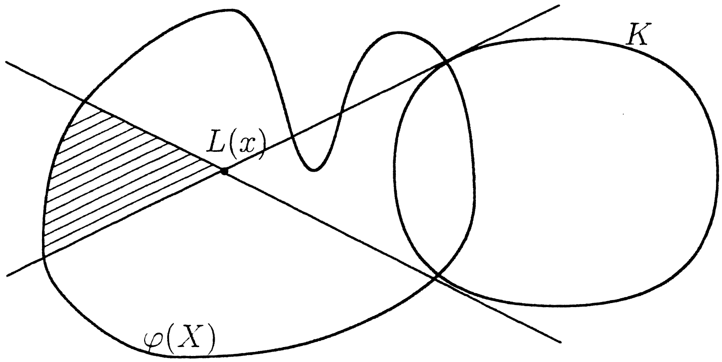
Let E, E' be Banach spaces. Consider a Fredholm operator $L : E \to E'$ of nonnegative index $i(L) = k$ and a set-valued map $\varphi : X \multimap E'$, where $X \subset E$.

(6) We also write $\pi^n(X) := \pi^n(X,\emptyset)$.

DEFINITION 3.1. – *A nonempty closed convex set $K \subset E'$ is L-fundamental for the map φ, if*
(i) $\varphi(L^{-1}(K) \cap X) \subset K$;
(ii) *if for $x \in X$, $L(x) \in \overline{\mathrm{conv}}\,(\varphi(x) \cup K)$, then $L(x) \in K$.*

The whole space E' and $\overline{\mathrm{conv}}\,\varphi(X)$ are the simplest examples of L-fundamental sets for φ.

The geometrical sense of the second condition in Definition 3.1 is as follows: if K is an L-fundamental set for φ, then $\varphi(x)$ must be contained out of the shadowed area.



Below we collect some properties of L-fundamental sets. Let $C := \{x \in X \cdot \mid L(x) \in \varphi(x)\}$.

PROPOSITION 3.2.

(i) *If K is an L-fundamental set for φ, then $C \subset L^{-1}(K)$.*

(ii) *If K_1 and K_2 are L-fundamental sets for φ, then $K_1 \cap K_2$ is empty or L-fundamental for φ. If $K_1 \cap K_2 = \emptyset$, then the set C of coincidence points defined above is empty too.*

(iii) *If K is an L-fundamental set for φ and $P \subset K$, then the set $K' = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup P)$ is also an L-fundamental set for φ.*

(iv) *If S is an intersection of all L-fundamental sets for φ, then*

$$S = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(S) \cap X)).$$

(v) *For any $Z \subset E'$ there is a set K, which is L-fundamental for φ and such that $K = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup Z)$.*

PROOF. – Only the last statement needs a proof, the others are simple consequences of definition.

Consider a nonempty family of sets

$$\mathcal{K} = \{P \subset E' \mid P \text{ is } L\text{-fundamental for } \varphi \text{ and } Z \subset P\}.$$

It is easy to check that the set

$$K = \bigcap_{P \in \mathcal{K}} P$$

is L-fundamental for φ, $Z \subset K$ and, by Definition 3.1 (i),

$$Z \subset K' = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup Z) \subset K.$$

But (*iii*) above implies that $K' \in \mathcal{K}$, therefore, by the definition of K also $K \subset K'$, what ends the proof. □

Observe by the way, that the set K defined above is the smallest L-fundamental set for φ, containing Z.

DEFINITION 3.3. – *A multivalued map $\varphi : X \multimap E'$ is called L-fundamentally restrictible, if there is a compact L-fundamental set for φ, having a nonempty intersection with $L(X)$.*

This definition can be written in another way as follows: «there is an L-fundamental set K such that $\mathrm{cl}\,\varphi(L^{-1}(K) \cap X)$ is compact in E' and $K \cap L(X) \neq \emptyset$». Indeed, then $K' = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup \{y\})$, where $y \in K \cap L(X)$, is a compact L-fundamental set for φ (comp. 3.2 (iii)) and $y \in K' \cap L(X)$. But if one does not assume that E' is a Banach space, but only a locally convex one, then the above conditions are weaker then those in Definition 3.3.

If the set of coincidence points C is nonempty, then each L-fundamental set for φ has nonempty intersection with $L(X)$ (see 3.2 (*i*)), but the inverse fact is not true.

REMARK 3.4. – Let E'' be a Banach space and $L' : E' \to E''$ be a continuous linear isomorphism. Then the composition $L' \circ L$ is a Fredholm operator and $i(L' \circ L) = i(L)$. If $\varphi : X \multimap E'$ is L-fundamentally restrictible, then $L' \circ \varphi$ is $(L' \circ L)$-fundamentally restrictible. Indeed, it is easy to check that if K is a compact L-fundamental set for φ, then $L'(K)$ is a compact $(L' \circ L)$-fundamental set for $L' \circ \varphi$.

In fact, to define the above notions, we do not need the assumption that L is a Fredholm operator (comp. [10]), but further we consider only such situation.

Below we enlist some examples of L-fundamentally restrictible maps.

3.5. COMPACT MAPS. – Recall that φ is compact, if the set $\mathrm{cl}\,\varphi(X)$ is compact in E'. Then the set $K = \overline{\mathrm{conv}}\,(\varphi(X))$ is L-fundamental for φ and compact. If $K \cap L(X) = \emptyset$, then we take

$$K'' = \overline{\mathrm{conv}}\,(\varphi(X) \cup \{v\}) = \overline{\mathrm{conv}}\,(\varphi(L^{-1}(E') \cap X) \cup \{v\}),$$

where $v \in L(X)$. It is clear that K'' satisfies conditions of Definition 3.3.

3.6. L-CONDENSING MAPS. – By a *measure of noncompactness* in a Banach space E' one ussually means any function $\mu : \mathcal{B} \to [0, \infty)$ defined on the family $\mathcal{B}$ of all bounded subsets of E', which has the following properties:

1. $\mu(A) = 0 \Leftrightarrow \mathrm{cl}\, A$ is a compact set;
2. μ is a seminorm, i.e. $\mu(\lambda \cdot a) = |\lambda|\mu(A)$ and $\mu(A_1 + A_2) \leq \mu(A_1) + \mu(A_2)$;
3. If $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$;
4. $\mu(A_1 \cup A_2) \leq \max\{\mu(A_1), \mu(A_2)\}$;
5. $\mu(\mathrm{conv}\, A) = \mu(A)$ and $\mu(\mathrm{cl}\, A) = \mu(A)$;
6. If the sequence of sets $\{A_i\}_{i=1}^{\infty}$ is decreasing and $\lim_{i\to\infty} \mu(A_i) = 0$, then the set $A = \bigcap_{i=1}^{\infty} \mathrm{cl}\, A_i$ is compact and nonempty.

Let $\varphi(X)$ be a bounded set. The map φ is called *L-condensing*, if for any bounded $A \subset X$, the inequality $\mu(\varphi(A)) \geq \mu(L(A))$ implies that $\mathrm{cl}\, A$ is compact.

We shall prove that φ is L-fundamentally restrictible. Indeed, for any $y \in L(X)$ there exists an L-fundamental set K such that $K = \overline{\mathrm{conv}}\, (\varphi(L^{-1}(K) \cap \cap X) \cup \{y\})$ (comp. Prop. 3.2 (v)). Assume for the moment that K is not compact. Then

$$\begin{aligned}\mu(K) &= \mu(\overline{\mathrm{conv}}\, (\varphi(L^{-1}(K) \cap X) \cup \{y\})) = \mu(\varphi(L^{-1}(K) \cap X) \cup \{y\}) \leq \\ &\leq \mu(\varphi(L^{-1}(K) \cap X)) < \mu(L(L^{-1}(K) \cap X)) \leq \mu(K),\end{aligned}$$

a contradiction. Therefore K is compact, and since $y \in K \cap L(X)$, it satisfies conditions of Definition 3.3.

3.7. L-CONTRACTIONS.. – A bounded map φ is an *L-contraction* with $k > 0$ with respect to the measure of noncompactness μ, if for any bounded set $A \subset X$, $\mu(\varphi(A)) \leq k\mu(L(A))$. For $k \in (0, 1)$ it is of course an L-condensing map, hence an L-fundamentally restrictible one.

3.8. L-LIMIT COMPACT MAPS. – Consider a family of sets:

$$K_1 := \overline{\mathrm{conv}}\, (\varphi(X)),$$

$$K_a := \overline{\mathrm{conv}}\, (\varphi(L^{-1}(K_{a-1}) \cap X)), \text{ if there is an ordinal number } a - 1,$$

$$K_a := \bigcap_{\beta < a} K_\beta, \text{ when } > a \ > \text{is a limit number.}$$

Since the sequence $\{K_a\}$ is decreasing, there exists an ordinal number γ, such that $K_\gamma = K_{\gamma+1}$, hence $K_\gamma = K_\beta$ for all $\beta \geq \gamma$. Let $K := K_\gamma$.

The map φ is *L-limit compact* if the set K defined above is compact and nonempty.

It is easy to check that the set K is L-fundamental for φ since each K_a is so, and that $K = \overline{\text{conv}}\, \varphi(L^{-1}(K) \cap X)$. Moreover, if $K \cap L(X) = \emptyset$, then also $L^{-1}(K) \cap X = \emptyset$ and $K = \emptyset$, but we know that $K \neq \emptyset$, so $K \cap L(X) \neq \emptyset$ and φ is L-fundamentally restrictible.

Observe that if $E = E'$ and $L = id_E$, then the maps described above became condensing, k-set contractions or limit compact ones in old well known sense (comp. e.g. [7], [33]).

In the coincidence theory for Fredholm operators with index 0, started by Mawhin (e.g. [28]) and continued by many authors (see [9], [2], [29], [21], [20] and others) one considers also L-compact or L-condensing maps, but in different sense (here we call them L_M-compact and L_M-condensing). Nevertheless, they are also L-fundamentally restrictible.

3.9. L_M-CONDENSING MAPS. – Let P, Q, K_P be the respective linear operators for L (see Preliminaries) and $I' := id_{E'}$, $I := id_E$. Assume that $(I' - Q) \circ \varphi(X)$ is bounded and denote by ν a measure of noncompactness in E. The map φ is L_M-*condensing*, if for any bounded set $A \subset X$, the condition $\nu((K_P \circ (I' - Q) \circ \varphi)(A)) \geq \nu(A)$ implies that $\operatorname{cl} A$ is compact. We will prove that φ is L-fundamentally restrictible provided X is closed and $P(X)$ is bounded.

In order to simplify the notation let $\Phi := K_P \circ (I' - Q) \circ \varphi$. Take an arbitrary $y_0 \in L(X)$ and put $x_0 := K_P \circ (I' - Q)(y_0)$. Consider the family of sets

$$\begin{aligned} \Lambda := \quad & \{Z_s \mid Z_s = \overline{\text{conv}}\,(Z_s) \subset \operatorname{Ker} P,\ \overline{\text{conv}}\,(\Phi((Z_s + \operatorname{Ker} L) \cap X) \cup \{x_0\}) \subset Z_s \text{ and} \\ & (I - P)(x) \in \overline{\text{conv}}\,(\Phi(x) \cup Z_s) \Rightarrow (I - P)(x) \in Z_s \text{ for any } x \in X\}. \end{aligned}$$

Λ is nonempty since $Z_1 = \overline{\text{conv}}\,(\Phi(X) \cup \{x_0\}) \in \Lambda$. Moreover, observe that $Z := \bigcap_{Z_s \in \Lambda} Z_s \neq \emptyset$ since $x_0 \in Z$, and that $Z \in \Lambda$. We shall prove that

$$Z = W := \overline{\text{conv}}\,[\Phi((Z + \operatorname{Ker} L) \cap X) \cup \{x_0\}]. \tag{4}$$

It is easy to see that $W \subset Z$. Indeed, for every set $Z_s \in \Lambda$, $W \subset \overline{\text{conv}}\,(\Phi((Z_s + \operatorname{Ker} L) \cap X) \cup \{x_0\}) \subset Z_s$, hence $W \subset \bigcap_{Z_s \in \Lambda} Z_s$.

To prove that $Z \subset W$, we check that $W \in \Lambda$. At first observe that, since $W \subset Z$,

$$\overline{\text{conv}}\,(\Phi((W + \operatorname{Ker} L) \cap X) \cup \{x_0\}) \subset \overline{\text{conv}}\,(\Phi((Z + \operatorname{Ker} L) \cap X) \cup \{x_0\}) = W.$$

Moreover, if $(I - P)(x) \in \overline{\text{conv}}\,[\Phi(x) \cup W]$ for some $x \in X$, then $(I - P)(x) \in$ $\in \overline{\text{conv}}\,[\Phi(x) \cup Z]$, so $(I - P)(x) \in Z$ and hence $x \in (Z + \operatorname{Ker} L) \cap X$. It follows that

$$\Phi(x) \subset \Phi((Z + \operatorname{Ker} L) \cap X) \subset W$$

and $\overline{\text{conv}}\,(\Phi(x) \cup W) = W$. Since $W = \overline{\text{conv}}\, W \subset \operatorname{Ker} P$, we have just proved that $W \in \Lambda$ and finally that $W = Z$.

Observe that, since $(I' - Q) \circ \phi(X)$ is a bounded set, (4) implies that Z is bounded and so is $(Z + \operatorname{Ker} L) \cap X$ (because also $P(X)$ is bounded). Therefore $(Z + \operatorname{Ker} L) \cap X$ is compact. Now it is easy to see that

$$K = \overline{\operatorname{conv}}\,[\varphi((Z + \operatorname{Ker} L) \cap X) \cup (I' - Q)(\varphi((Z + \operatorname{Ker} L) \cap X)) \cup \{y_0\}]$$

is compact, L-fundamental for φ (7) and $K \cap L(X) \neq \emptyset$.

3.10. L-COMPACT MAPS. – A map φ is *L-compact*, if $K_P \circ (I' - Q) \circ \varphi(X)$ is a relatively compact subset of E. Then it is obviously an L_M-condensing map, so L-fundamentally restrictible.

3.11. $(L, \mathcal{K}_n)$-OPERATORS. – This notion is a generalization of so-called $\mathcal{K}_n$-operators considered in [1]. Let $\mathcal{A} \subset 2^{E'}$. The map φ is an *$(L, \mathcal{A})$-operator*, if for any $Z \in \mathcal{A}$ and any $M \subset E'$ the following condition holds:

$$\text{if} \quad \overline{\operatorname{conv}}\,(\varphi(L^{-1}(M) \cap X) \cup Z) = M, \quad \text{then} \quad M \text{ is compact.}$$

By $\mathcal{K}_n$, $\mathcal{K}_\infty$ and $\mathcal{K}_c$ we denote the families of all n-element, all finite, and all compact subsets of E', respectively. It is easy to see that any $(L, \mathcal{K}_c)$-operator is an $(L, \mathcal{K}_\infty)$-operator, any $(L, \mathcal{K}_\infty)$-operator is an $(L, \mathcal{K}_n)$-operator for arbitrary $n \in \mathbb{N}$ and any $(L, \mathcal{K}_n)$-operator is also an $(L, \mathcal{K}_m)$-operator provided $m \leq n$.

If $\mathcal{A}$ denotes any family from among $\mathcal{K}_1, \mathcal{K}_n, \ldots \mathcal{K}_\infty, \mathcal{K}_c$ and φ is an $(L, \mathcal{A})$-operator, then, using Proposition 3.2 *(v)*, one can easily check that φ is L-fundamentally restrictible.

REMARK 3.12. – Observe that any L-condensing or L_M-condensing map φ (then also L-contraction, compact, L-compact) is an (L, K_c) operator. Indeed, by Proposition 3.2 *(v)*, for an arbitrary compact set Z there is K being a fundamental set for φ and such that $\overline{\operatorname{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup Z) = K$. Assume for the moment that it is not compact. If φ is L_M-condensing, then it means that also a closed set $L^{-1}(K) \cap X$ is not compact. Since $L^{-1}(K) \cap X \subset (K_P \circ (I' - Q)(K) \oplus \operatorname{Ker} L) \cap X$,

$$\begin{aligned}
&\nu(L^{-1}(K) \cap X) \leq \nu((K_P \circ (I' - Q)(K) \oplus \operatorname{Ker} L) \cap X) = \\
&= \nu(K_P \circ (I' - Q)((\overline{\operatorname{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup Z)) \oplus \operatorname{Ker} L) \cap X) \leq \\
&\leq \nu(K_P \circ (I' - Q)(\overline{\operatorname{conv}}\,(\varphi(L^{-1}(K) \cap X) \cup Z)) \cap X) + \nu(\operatorname{Ker} L \cap X) \leq \\
&\leq \nu(\overline{\operatorname{conv}}\,(K_P \circ (I' - Q)(\varphi(L^{-1}(K) \cap X) \cup Z)) \cap X) + \nu(P(X)) \leq \\
&\leq \nu(K_P \circ (I' - Q)(\varphi(L^{-1}(K) \cap X))) < \nu(L^{-1}(K) \cap X),
\end{aligned}$$

(7) K is nonempty, because it contains y_0.

a contradiction. Then $L^{-1}(K) \cap X$ is compact and so is K. If φ is L-condensing, one can repeat arguments from example 3.6 putting Z instead of $\{y\}$.

Finally we give an example of an L-fundamentally restrictible map, which is not contained in any class defined earlier.

3.13. Let $E = E' = l^2$ and $X = \mathrm{cl}\, B^{l^2}(0,1)$. Define $L : l^2 \to l^2$ and $\varphi : \mathrm{cl}\, B^{l^2}(0,1) \to l^2$ by

$$L((x_1, x_2, x_3, \ldots, x_n, \ldots)) = \left(\frac{1}{2}(x_1 + x_2), 0, x_3, \ldots, x_n, \ldots\right),$$

$$\varphi((x_1, x_2, x_3, \ldots, x_n, \ldots)) = (x_2, x_3, \ldots, x_{n-1}, \ldots).$$

It is easy to see that e.g. $K = \{(a, 0, 0, \ldots) \mid a \in [-1, 1]\}$ is a compact L-fundamental set for φ and $K \cap L(X) \neq \emptyset$. Observe that $L(B^{l^2}(0,1)) \subset B^{l^2}(0,1) \subset \varphi(B^{l^2}(0,1))$ and $B^{l^2}(0,1) \subset L^{-1}(B^{l^2}(0,1))$.

φ is not an L-limit compact map, because any set K_a contains $B^{l^2}(0,1)$; it is not an $(L, \mathcal{K}_1)$-operator (hence also it is not an $(L, \mathcal{K}_v)$-operator for $v > 1$, $v = \infty$ or $v = c$), because the smallest L-fundamental set for φ containing $B^{l^2}(0,1)$ satisfies respective condition (see Prop. 3.2 *(v)* and definition of $(L, \mathcal{K})$-operator) for any $T = \{m\} \subset B^{l^2}(0,1)$ and it is not compact; it is not L-condensing and L_M-condensing, since it is not $(L, \mathcal{K}_c)$ operator (see Remark 3.12).

Compact associated maps.

Denote by $\mathcal{FD}_L(X, E')$ the family of c-admissible L-fundamentally restrictible maps. Below we introduce two definitions of homotopy in $\mathcal{FD}_L(X, E')$. The first one seems to be more natural, but it does not guarantee useful properties of homotopy (e.g. as an equivalence relation), so it has only a technical character.

DEFINITION 3.14. – *Let K be a compact convex and nonempty subset of E'. We say that maps $\varphi_0, \varphi_1 \in \mathcal{FD}_L(X, E')$ are (L, K)-homotopic (and denote $\varphi_0 \simeq_K \varphi_1$), if there exists a c-admissible pair (R, S) being a homotopy between φ_0 and φ_1 in the sense of Definition 2.2 and such that K is an L-fundamental set for all maps of the form $S(R^{-1}(\cdot, t))$, where $t \in [0, 1]$.*

DEFINITION 3.15. – *We say that maps $\varphi_0, \varphi_1 \in \mathcal{FD}_L(X, E')$ are L-homotopic (and denote $\varphi_0 \simeq \varphi_1$), if there exists a finite sequence of compact convex and nonempty sets $K_1, \ldots, K_n$ and maps $\psi_1, \ldots, \psi_{n-1} \in \mathcal{FD}_L(X, E')$ such that*

$$\varphi_0 \simeq_{K_1} \psi_1 \simeq_{K_2} \cdots \simeq_{K_{n-1}} \psi_{n-1} \simeq_{K_n} \varphi_1.$$

Obviously, if two maps are (L, K)-homotopic, then they are L-homotopic.

Moreover, a c-admissible pair of maps $X \times [0,1] \xleftarrow{R} \varGamma \xrightarrow{S} E'$ determining a compact homotopy is also an (L, K)-homotopy, with $K = \overline{\mathrm{conv}}\,(S(R^{-1}(X \times [0,1])))$, hence simply an L-homotopy.

In order to introduce a homotopy invariant for L-fundamentally restrictible maps, let us first describe some connections with compact maps. Suppose that $\varphi \in \mathcal{FD}_L(X, E')$ is determined by a c-admissible pair $X \xleftarrow{p} \varGamma \xrightarrow{q} E'$ and has a compact L-fundamental set K such that $K \cap L(X) \neq \emptyset$.

DEFINITION 3.16.
(i) *A c-admissible pair* $(p, \overline{q})$ *is* K*-associated with* (p, q)*, if* $\overline{q} : \varGamma \to K$ *and* $\overline{q}|_{p^{-1}(L^{-1}(K)\cap X)} = q|_{p^{-1}(L^{-1}(K)\cap X)}$.
(ii) *A c-admissible map* $\overline{\varphi}$ *determined by* $(p, \overline{q})$ *is* K*-associated with* φ.

Observe that the map $\overline{\varphi}$ defined above is compact. Moreover, for any $\varphi \in \mathcal{FD}_L(X, E')$ and its arbitrary compact L-fundamental set K such that $K \cap L(X) \neq \emptyset$, one can find a K-associated map. Indeed, if the pair (p, q) determines φ, then the nonempty set $p^{-1}(L^{-1}(K) \cap X)$ is closed in $\varGamma$ and by the Dugundji Theorem (see e.g. Th. (7.1), III [3]) the map $q|_{p^{-1}(L^{-1}(K)\cap X)} :$ $: p^{-1}(L^{-1}(K) \cap X) \to K$ has a compact extension $\overline{q} : \varGamma \to K$. It is clear that φ may admit other associated maps (for the same or different L-fundamental set). Some of their properties are described in the following two theorems.

THEOREM 3.17. – *Let* K_0, K_1 *be compact sets* L*-fundamental for* $\varphi \in \mathcal{FD}_L(X, E')$ *and such that* $L(X) \cap K_0 \cap K_1 \neq \emptyset$*. Assume that* φ_i *is a compact c-admissible map* K_i*-associated with* φ *for* $i = 0, 1$*. Then there is a compact homotopy between* φ_0 *and* φ_1.

If, additionally, L *and* φ *do not have coincidence points in some set* $X' \subset X$*, then also this homotopy does not have coincidence points with* L *in the set* $X' \times [0,1]$.

PROOF. – Observe that $K_2 = K_0 \cap K_1$ is a nonempty L-fundamental set for φ. Denote by $r_i : E' \to K_i$, $i = 0, 1, 2$ the respective retractions (8) and by $(p, \overline{q_0})$, $(p, \overline{q_1})$ pairs determining φ_0 and φ_1, respectively. Let $R : \varGamma \times [0,1] \to X \times [0,1]$, $R(w,t) = (p(w), t)$ and $S : \varGamma \times [0,1] \to E'$,

$$S(w,t) = \begin{cases} (1-4t)\overline{q_0}(w) + 4t(r_0 \circ q)(w), & \text{for } t \in [0, \frac{1}{4}] \\ (2-4t)(r_0 \circ q)(w) + (4t-1)(r_2 \circ q)(w), & \text{for } t \in (\frac{1}{4}, \frac{1}{2}] \\ (3-4t)(r_2 \circ q)(w) + (4t-2)(r_1 \circ q)(w), & \text{for } t \in (\frac{1}{2}, \frac{3}{4}] \\ (4-4t)(r_1 \circ q)(w) + (4t-3)\overline{q_1}(w), & \text{for } t \in (\frac{3}{4}, 1]. \end{cases}$$

(8) By the Dugundji Theorem any closed convex subset of a Banach space is a retract of this space.

Observe that S is compact since $S(\Gamma \times [0,1]) \subset K_0 \cup K_1$, and R is a cell-like map. Moreover the pair (R,S) is a compact homotopy between c-admissible pairs $(p,\overline{q_0})$ and $(p,\overline{q_1})$ (then it also determines a compact homotopy between φ_0 and φ_1).

Suppose now, that φ and L do not have coincidence points in X' and for some $t \in [0,1]$, $x \in X$,

$$L(x) \in S(R^{-1}(x,t)) = S(\{(w,t) \mid w \in \Gamma, p(w) = x\}).$$

If $t \in [0,\frac{1}{4}]$, then it means that there is $w \in \Gamma$ such that

$$p(w) = x \qquad \text{and} \qquad L(x) = (1-4t)\overline{q_0}(w) + 4t(r_0 \circ q)(w).$$

Therefore $L(x) \in K_0$, so $x \in L^{-1}(K_0) \cap X$. This implies that $w \in p^{-1}(L^{-1}(K_0) \cap X)$. But then $\overline{q_0}(w) = q(w)$ and $r_0 \circ q(w) = q(w)$, because $q(w) \in K_0$. Finally $S(w,t) = q(w)$, and hence $x \notin X'$.

Similar arguments we can use for $t \in (\frac{3}{4},1]$.

If $t \in (\frac{1}{4},\frac{1}{2}]$, then again there is $w \in \Gamma$, such that

$$p(w) = x \qquad \text{and} \qquad L(x) = (2-4t)(r_0 \circ q)(w) + (4t-1)(r_2 \circ q)(w).$$

Then, like earlier $L(x) \in K_0$, so $w \in p^{-1}(L^{-1}(K_0) \cap X)$ and $q(w) \in K_0$. Therefore $r_0 \circ q(w) = q(w)$, what means that $r_0 \circ q(w) \in q(p^{-1}(x))$. But since $r_2 \circ q(w) \in K_2$, we state that $L(x) \in \overline{\operatorname{conv}}\,(q(p^{-1}(x)) \cup K_2)$. It follows that $L(x) \in K_2$ (because K_2 is L-fundamental for (p,q)), and hence $r_2 \circ q(w) = q(w)$. Once more we get $S(w,t) = q(w)$, so $x \notin X'$.

The same proof works when $t \in (\frac{1}{2},\frac{3}{4}]$. □

THEOREM 3.18. – *Assume that (p,q) determines $\varphi \in \mathcal{FD}_L(X,E')$ and K is a compact L-fundamental set for φ such that $K \cap L(X) \neq \emptyset$. If $\overline{\varphi}$ is determined by $(p,\overline{q})$ - a compact pair K-associated with (p,q), then $\varphi \simeq_K \overline{\varphi}$.*

PROOF. – Since $\overline{\varphi} \in \mathcal{FD}_L(X,E')$, one can easily check that the pair $X \times [0,1] \xleftarrow{R} \Gamma \times [0,1] \xrightarrow{S} E'$, where $R(w,t) = (p(w),t)$ and $S(w,t) = (1-t)q(w) + t\overline{q}(w)$ is an (L,K)-homotopy between φ and $\overline{\varphi}$. □

REMARK 3.19. – Observe that, if the pair (p,q) does not have coincidence points with L in some set X', then also any map of the form $x \mapsto S(R^{-1}(x,t))$, where (R,S) is defined in the proof above, does not have coincidence points with L in X' for any t.

In particular, if K_0 and K_1 are compact L-fundamental sets for $\varphi \in \mathcal{FD}_L(X,E')$ such that $K_0 \cap K_1 = \emptyset$, then none of maps K_0- or K_1-associated with φ has coincidence points with L, because then also φ does not have coincidence points with L (comp. Th. 3.17).

The coincidence index $\operatorname{Ind}_L((p,q),X)$ of a compact c-admissible pair

$X \xleftarrow{p} \Gamma \xrightarrow{q} E'$ is defined in [25] or, in a bit different way, in [13]. We do not repeat this construction here, but assuming, that it is given, shortly recall the generalization of this notion for maps belonging to $\mathcal{FD}_L(X, E')$. More details one can find in [13] or in [10]. But we would like to stress that, because of possible «dimensional defect», this coincidence index is an element of respective stable homotopy group. However, if there is no «dimensional defect» between domain and codomain, i.e. $i(L) = 0$, then the coincidence index is in fact equivalent to the fixed point index for c-admissible maps, when $L = id$ (comp. [25]), or to index of Mawhin in a general case (see [29], [28] and comp. Th. 3.21 (v)).

Let $X \subset E$, $\operatorname{int} X \neq \emptyset$, and $\varphi \in \mathcal{FD}_L(X, E')$. Assume that the set of coincidence points $C = \{x \in X \mid L(x) \in \varphi(x)\}$ is bounded closed and contained in $\operatorname{int} X$ (9). Take any c-admissible pair (p, q) determining φ.

DEFINITION 3.20. – *By a generalized index of the pair* (p, q) *we understand an element*

$$\operatorname{Ind}_L((p, q), X) := \operatorname{Ind}_L((p, \overline{q}), X) \in \Pi_k,$$

where $(p, \overline{q})$ *is a compact pair* K*-associated with* (p, q)*, and* K *is an arbitrary compact* L*-fundamental set for* (p, q)*.*

The correctness of this definition follows from Theorem 3.17 and the homotopy property of the generalized index for compact pairs (see [25], [13]).

If (p, q) is a compact pair, then it is obviously $(\overline{\operatorname{conv}}\, q(p^{-1}(X))$-associated to itself. Then both indices: defined above and that for compact pairs are the same.

The index defined above heavily depend on the orientations of $\operatorname{Ker} L$ and $\operatorname{Coker} L$, (which we have to fix at the beginning, as in the compact case). Namely, any change of these orientations may effect a change of a «sign» of the index but does not change its nontriviality.

THEOREM 3.21. – *The generalized index defined above has the following properties:*

(i) *(existence) If* $\operatorname{Ind}_L((p, q), X) \neq 0$*, then* $C \neq \emptyset$*, i.e. there is* $x \in X$*, such that* $L(x) \in q(p^{-1}(x))$*.*

(ii) *(localization) If* $X' \subset X$ *is open and* $C \subset X'$*, then* $\operatorname{Ind}_L((p, q), X')$ *is well defined and equal to* $\operatorname{Ind}_L((p, q), X)$*.*

(iii) *(homotopy invariance) If* $X \times [0, 1] \xleftarrow{R} \Gamma \xrightarrow{S} E'$ *is an* L*-homotopy between pairs of maps* (p_0, q_0) *and* (p_1, q_1) *(in the sense of Definition* 3.15*) and the*

(9) We can omit the assumption that C is bounded when X is closed and $P(X)$ is bounded, where P is a respective projection (comp. Preliminaries).

set $\{x \in X \mid L(x) \in S(R^{-1}(x,t))$ *for some* $t \in [0,1]\}$ *is closed bounded and contained in* $\operatorname{int} X$*, then*

$$\operatorname{Ind}_L((p_0,q_0),X) = \operatorname{Ind}_L((p_1,q_1),X).$$

(iv) *(additivity) If* X_1, $X_2 \subset X$ *are disjoint, have nonempty interiors and* $C \subset X_1 \cup X_2$, *then*

$$\operatorname{Ind}_L((p,q),X) = \operatorname{Ind}_L((p,q),X_1) + \operatorname{Ind}_L((p,q),X_2).$$

(v) *Let* E'' *be a Banach space and* $\Lambda : E' \to E''$ *be a linear homeomorphism. In the space* $\operatorname{Ker} \Lambda \circ L = \operatorname{Ker} L$ *the orientation is given. Fix the orientation in the space* $\operatorname{Coker} \Lambda \circ L$. *Then*

$$\operatorname{Ind}_{\Lambda\circ L}((p,\Lambda\circ q),X) \neq 0 \quad \Leftrightarrow \quad \operatorname{Ind}_L((p,q),X) \neq 0.$$

Moreover, if the isomorphism $\overline{\Lambda} : \operatorname{Coker} L \to \operatorname{Coker}(\Lambda \circ L)$, *given by* $\overline{\Lambda}([z]) = [\Lambda(z)]$ *saves the orientation, then respective indices are equal; if* $\overline{\Lambda}$ *does not save the orientation, then respective indices differ in parameter* (-1).

(vi) *(restriction) If* $q(p^{-1}(X)) \subset Y$ *and* Y *is a closed subspace of* E' *such that* $\dim(\operatorname{Im} L + Y) > i(L) + 2$, *then* $\operatorname{Ind}_L((p,q),X) = \operatorname{Ind}_{L'}((p',q'),X\cap T)$, *where* $T := L^{-1}(\operatorname{Im} Q + Y)$, $p' = p|_{p^{-1}(\operatorname{cl} X\cap T)}$, $q' = q|_{p^{-1}:(\operatorname{cl} X \cap T)}$ *and* $L' = L|_T : T \to \operatorname{Im} Q + Y$.

(vii) *(finite dimensional restriction) If we know that* $q(p^{-1}(X)) \subset Y = \operatorname{Im} Q \oplus W'$, *and* W' *is a finite dimensional subspace of* $\operatorname{Im} L$, *then*

$$\operatorname{Ind}_L((p,q),X) = \operatorname{Deg}((p_1, L'\circ p_1 - q_1), X\cap T, 0),$$

where $T := L^{-1}(Y)$, $L' = L|_T$ *and* $p_1 = p|_{p^{-1}(\operatorname{cl} X\cap T)} : p^{-1}(\operatorname{cl} X \cap T) \to T$, $q_1 = q|_{p^{-1}(\operatorname{cl} X\cap T)} p^{-1}(\operatorname{cl} X\cap T) \to Y$. *By* Deg *we understand the stable generalized degree defined in* [25] *(see also* [14], [11]*).*

One can find the proof in [13] or in [10].

Remark 3.22.

(i) By the generalized index of φ determined by (p,q) we understand

$$\operatorname{Ind}_L(\varphi,X) := \operatorname{Ind}_L((p,q),X).$$

This index depends on the choice of (p,q), or, more precisely, on a choice of an L-homotopy class of pairs determining φ.

(ii) An alternative definition of the generalized index for $\varphi \in \mathcal{FD}_L(X,E')$ is the following:

$$\mathbf{Ind}_L(\varphi,X) := \{\operatorname{Ind}_L((p,q),X) \mid \varphi(x) = q(p^{-1}(x)) \text{ and } (p,q) \text{ is c-admissible}\}.$$

Or, if we widen the notion of a c-admissible map, i.e. admit ones having multivalued selections determined by a c-admissible pair, we can put:

$$\mathbf{Ind}_L(\varphi, X) := \{\operatorname{Ind}_L((p,q),X) \mid \varphi(x) \supset q(p^{-1}(x)) \text{ and } (p,q) \text{ is c-admissible}\}.$$

If any of indices defined above is nontrivial ([10]), then $C \neq \emptyset$.

4. – Main result.

Let E_1, E_2 E_1', E_2' be Banach spaces, $\mathcal{D}$ be an open subset of E_2 and $X \subset \mathcal{D} \times E_1$. We consider a system of inclusions:

$$\begin{cases} L_1(y) \in F(x,y), \\ L_2(x) \in G(x,y), \end{cases} \tag{5}$$

where $L_1 : E_1 \to E_1'$, $L_2 : E_2 \to E_2'$ are Fredholm operators and $F : X \multimap E_1'$, $G : X \multimap E_2'$ are multivalued maps.

Observe that the operator $L : E_2 \times E_1 \to E_2' \times E_1'$ given by $L(x,y) = (L_2(x), L_1(y))$ is a Fredholm one with $i(L) = i(L_1) + i(L_2)$. If the map $X \ni (x,y) \mapsto \Phi(x,y) := (G(x,y), F(x,y))$ is c-admissible compact or L-fundamentally restrictible, and $\operatorname{Ind}_L(\Phi, X)$ is defined and nontrivial, then some additional technical assumptions guarantee the existence of solutions to the considered problem (comp. e.g. [25], [10]). But, as we have mentioned in Introduction, we are interested in the another method, using a so-called solution map, which let us to relax the main assumptions concerning the coincidence index and the L-fundamental restrictibility (see Th. 4.15). In further considerations we use some results concerning a similar problem with compact maps on the right hand side proved in [14].

Let $X(a) := \{y \in E_1 | (a,y) \in X\}$ for $a \in E_2$ and $\operatorname{pr} : E_2 \times E_1 \to E_2$ be the projection.

DEFINITION 4.1.

(i) *A multivalued map $F : X \multimap E_1'$ is L_1-fundamentally restrictible, if there is a compact convex set $K \subset E_1'$, being L_1-fundamental for all maps of the form $F(a,\cdot) : X(a) \multimap E_1'$ where $a \in \operatorname{pr}(X)$.*

(ii) *A multivalued map $F : X \multimap E_1'$ is L_1-fundamentally restrictible with respect to the second variable, if for any compact set $A \subset \operatorname{pr}(X)$ there is a compact convex set $K \subset E_1'$, being L_1-fundamental for all maps of the form $F(a,\cdot) : X(a) \multimap E_1'$ where $a \in A$.*

Observe that any L_1-fundamentally restrictible map is L_1-fundamentally

([10]) It means that $\mathbf{Ind}_L(\varphi, X)$ contains a nontrivial element.

restrictible with respect to the second variable, but the inverse fact is not true. One can easily get some examples of maps satisfying the above definition by obvious modification of those given in the previous section.

Let

$$C(F) := \{(x,y) \in X \mid L_1(y) \in F(x,y)\} \quad \text{and} \quad \mathcal{D}_F := \mathrm{pr}\,(C(F)).$$

It is clear that $C(F)$ is closed in X and coincides with the graph of the *solution map* Ω_F defined by

$$\mathcal{D}_F \ni x \mapsto \Omega_F(x) = \{y \in E_1 | L_1(y) \in F(x,y)\} \subset E_1.$$

LEMMA 4.2. – *Ω_F is a multivalued map (i.e. is u.s.c. with compact values) if and only if* $\mathrm{pr}\,|_{C(F)} : C(F) \to \mathcal{D}_F$ *is a proper map.* □

Let P_1, Q_1, K_{P_1} be the respective linear operators related to L_1 (see Preliminaries), $I_1' = id_{E_1'}$, and $\mathcal{P} : E_2 \times E_1 \to E_1$ be given by the formula: $\mathcal{P}(x,y) = P_1(y)$.

LEMMA 4.3. – *If the map F is L_1-fundamentally restrictible with respect to the second variable, X is closed in $\mathcal{D} \times E_1$ and such that the set $\mathcal{P}(X)$ is bounded, then* $\mathrm{pr}\,|_{C(F)} : C(F) \to \mathcal{D}_F$ *is a proper map. In particular $\Omega_F : \mathcal{D}_F \to E_1$ is a multivalued map.*

PROOF. – We shall prove that for any compact $A \subset \mathcal{D}_F$, the set $(\mathrm{pr}\,|_{C(F)})^{-1}(A) = \{(x,y) \in X | L_1(y) \in F(x,y)\}$ is compact, too.

Let K_0 be compact and L-fundamental for maps of the form $F(x,\cdot)$ for $x \in A$. The properties of fundamental sets imply that, if $L_1(y) \in F(x,y)$, then $L_1(y) \in K_0$, so also $y \in L_1^{-1}(K_0)$. Therefore $(\mathrm{pr}\,|_{C(F)})^{-1}(A) \subset A \times L_1^{-1}(K_0)$.

Consider a sequence $\{(x_n,y_n)\}$ contained in $(\mathrm{pr}\,|_{C(F)})^{-1}(A)$. Without loss of generality, assume that the whole sequence $\{x_n\}$ converges to $x_0 \in \mathcal{D}_F \subset \mathcal{D}$. Let $y_n = v_n + z_n$, where $v_n = K_{P_1}(L_1(y_n)) \subset K_{P_1} \circ (I_1' - Q_1)(K_0)$, and $z_n \in \mathcal{P}(X)$. Then, since $(I_1' - Q_1)(K_0)$ is compact, there is $v_0 \in K_{P_1} \circ (I_1' - Q_1)(K_0)$ and $z_0 \in \mathrm{cl}\,(\mathcal{P}(X))$, such that $y_0 := v_0 + z_0$ is the limit of some subsequence of $\{y_n\}$. Since X is closed in $\mathcal{D} \times E_1$, $(x_0,y_0) \in X$.

Assume that the whole $\{y_n\}$ converges to y_0, i.e. $\{(x_n,y_n)\} \to (x_0,y_0)$. If $y_n' := L_1(y_n) \in F(x_n,y_n)$ then, since L_1 is continuous, $\{y_n'\} \to L_1(y_0)$, and, since F is u.s.c., $L_1(y_0) \in F(x_0,y_0)$. Finally $(x_0,y_0) \in (\mathrm{pr}\,|_{C(F)})^{-1}(x_0)$ and $(\mathrm{pr}\,|_{C(F)})^{-1}(x_0)$ is compact. □

REMARK 4.4. – In particular, if F is a compact map, the above lemma stays true. Moreover, if $E_1 = \mathbb{R}^m$, $E_1' = \mathbb{R}^{m'}$ where $m' \leq m$, X is closed and locally bounded over E_2 ([11]) and F is an arbitrary multivalued map, then $\mathrm{pr}\,|_{C(F)} : C(F) \to \mathcal{D}$ is a proper map.

Further on we assume that

4.5. *F is a c-admissible multivalued map, L_1-fundamentally restrictible with respect to the second variable.*

4.6. *X is closed in $\mathcal{D} \times E_1$ with nonempty interior and the set $\mathcal{P}(X)$ is bounded (or, more generally, $\mathrm{pr}\,|_{C(F)} : C(F) \to \mathcal{D}_F$ is proper);*

4.7. *$C(F) \subset \operatorname{int} X$;*

4.8. *$\operatorname{int} X(a) \neq \emptyset$ for any $a \in \mathcal{D}$.*

Suppose that a c-admissible pair $X \xleftarrow{p} \Gamma \xrightarrow{q} E_1'$ determines F and let $a \in \mathcal{D}$. Let $\Gamma_a = \{(y,w) \in X(a) \times \Gamma \mid p(w) = (a,y)\}$. Define maps $p_a : \Gamma_a \to X(a)$, $q_a : \Gamma_a \to E_1'$ by

$$p_a(y,w) = y, \qquad q_a(y,w) = q(w).$$

Observe that (p_a, q_a) is c-admissible, $q(p^{-1}(a,y)) = q_a(p_a^{-1}(y))$ for any $y \in X(a)$ and (p_a, q_a) determines an L_1-fundamentally restrictible map. Note that in view of the above assumptions, $\{y \in X(a) \mid L_1(y) \in F(a,y)\}$ is compact and contained in $\operatorname{int} X(a)$. If $\operatorname{Ker} L_1$ and $\operatorname{Coker} L_1$ are oriented in an arbitrary, but fixed manner, then $\operatorname{Ind}_{L_1}((p_a,q_a), X(a))$ is well defined. Let

$$\operatorname{Ind}_{L_1}((p,q)(a,\cdot), X(a)) := \operatorname{Ind}_{L_1}((p_a,q_a), X(a)).$$

Assume that

4.9. *$\eta_0 := Ind_{L_1}((p,q)(a,\cdot), X(a)) \neq 0$,*

LEMMA 4.10. – *Under the above assumptions, if $\mathcal{D}_0$ is the pathwise component of $\mathcal{D}$ containing a, then $\mathcal{D}_0 \subset \mathcal{D}_F$.*

PROOF. – Since $\Omega_F(a)$ is compact, there are open bounded sets V and W, such that

$$\{a\} \times \Omega_F(a) \subset V \times W \subset \operatorname{cl} V \times \operatorname{cl} W \subset \operatorname{int} X.$$

The upper semicontinuity of Ω_F implies that there is $r > 0$ such that $B^{E_2}(a,r) \subset V$ and $\Omega_F(\mathcal{D}_F \cap B^{E_2}(a,r)) \subset W$. We shall prove that $B^{E_2}(a,r) \subset \mathcal{D}_F$.

([11]) We say that X is locally bounded over E_2 if each $x \in E_2$ has a neighborhood N in E_2 such that $N \times E_1 \cap X$ is bounded.

The existence property of the generalized index Ind_{L_1} (Th. 3.21 (i)) implies that $a \in \mathcal{D}_F$. Take an arbitrary $b \in B^{E_2}(a, r)$, then $\operatorname{cl} W \subset \operatorname{int} X(b)$. Let

$$\begin{array}{ll} \Gamma_0 = \Gamma_a \cap (\operatorname{cl} W \times \Gamma), & \Gamma_1 = \Gamma_b \cap (\operatorname{cl} W \times \Gamma) \\ p_0 = p_a|_{\Gamma_0}, \quad q_0 = q_a|_{\Gamma_0}, & p_1 = p_b|_{\Gamma_1}, \quad q_1 = q_b|_{\Gamma_1}. \end{array}$$

The following diagram

$$\begin{tikzcd}
\operatorname{cl} W \arrow[d, "i_0"'] & \Gamma_0 \arrow[l, "p_0"'] \arrow[d, "j_0"'] \arrow[dr, "q_0"] & \\
\operatorname{cl} W \times [0,1] & \Gamma \arrow[l, "R"'] \arrow[r, "S"] & E_1' \\
\operatorname{cl} W \arrow[u, "i_1"] & \Gamma_1 \arrow[l, "p_1"'] \arrow[u, "j_1"] \arrow[ur, "q_1"'] &
\end{tikzcd}$$

where

$$\Gamma = \{(y, \gamma, t) \subset W \times \Gamma \times [0,1] \mid p(\gamma) = ((1-t)a + tb, y)\},$$

$$R(y, \gamma, t) = (y, t), \quad S(y, \gamma, t) = q(\gamma) \text{ for } (y, \gamma, t) \in \Gamma,$$

$$j_k(y, \gamma) = (y, \gamma, k) \text{ for } (y, \gamma) \in \Gamma_k, \quad k = 0, 1$$

is commutative. Moreover, the pair (R, S) determines the L_1-fundamentally restrictible map (with L_1-fundamental set K equal to a compact L_1-fundamental set for the family of maps $y \mapsto q(p^{-1}(z, y))$, where z belongs to the (compact) segment connecting points a and b). It means that the maps determined by (p_0, q_0) and (p_1, q_1) are homotopic in $\mathcal{FD}_{L_1}(W, E_1')$ (more precisely (K, L_1)-homotopic). Observe additionally that if for some $t \in [0,1]$, $L_1(y) \in S(R^{-1}(y, t))$, then $(1-t)a + tb \in \mathcal{D}_F \cap B^E(a, r)$ and $y \in \Omega_F((1-t)a + tb)$. Hence $y \in W$ and the set $\{y \in E' \mid L_1(y) \in S(R^{-1}(y, t))\}$ does not intersect $\operatorname{bd} W$. By Theorem 3.21 (*ii*), (*iii*),

$$0 \neq \operatorname{Ind}_{L_1}((p_a, q_a), X(a)) = \operatorname{Ind}_{L_1}((p_a, q_a), W) =$$

$$= \operatorname{Ind}_{L_1}((p_b, q_b), W) = \operatorname{Ind}_{L_1}((p_b, q_b), X(b)).$$

Therefore $b \in \mathcal{D}_F$.

We have proved that the map $a \mapsto \operatorname{Ind}_{L_1}((p_a, q_a), X(a))$ is locally constant, what implies that it is constant on path components of $\mathcal{D}$. Hence, for any $c \in \mathcal{D}_0$, $\operatorname{Ind}_{L_1}(F(c, \cdot), X(c)) = \operatorname{Ind}_{L_1}(F(a, \cdot), X(a)) \neq 0$, and by (*i*) Th. 3.21 $c \in \mathcal{D}_F$. □

Additionally we assume that

4.11. *There is* $r > 0$ *and a compact c-admissible map* $\Psi : D^{E_2}(a,r) \multimap E_2'$ *determined by a c-admissible pair* (u,v) *such that* $D^{E_2}(a,r) \subset \mathcal{D}$ *and, for* $(x,y) \in C(F) \cap (S^{E_2}(a,r) \times E_1)$, *if* $\mu(L_2(x) - \Psi(x)) \cap (L_2(x) - G(x,y)) \neq \emptyset$, *then* $\mu \geq 0$.

4.12. *If* $x \in S^{E_2}(a,r)$, *then* $L_2(x) \notin \Psi(x)$, $\xi := \mathrm{Ind}_{L_2}((u,v), D^{E_2}(a,r)) \neq 0 \in \in \Pi_{i(L_2)}$ *and* $\xi \otimes \eta_0 \neq 0 \in \Pi_{i(L_1)+i(L_2)}$.

REMARK 4.13. – Note that the last two assumptions are a sort of *a priori* bounds condition.

If L_2 is an isomorphism and $a = 0$, then for $\Psi \equiv 0$ assumption 4.12 is obviously satisfied. Indeed, observe, that

$$\mathrm{Ind}_{L_2^{-1}\circ L_2}(L_2^{-1} \circ \Psi, D^{E_2}(0,r)) = \mathrm{Ind}_{id_{E_2}}(0, D^{E_2}(0,r)) \neq 0\,(^{12}),$$

hence, by Th. 3.21 (v), also $\xi = \mathrm{Ind}_{L_2}(\Psi, D^{E_2}(0,r)) \neq 0$. Moreover, in fact $\xi = \pm\,\nu \in \Pi_0$, so, by 2.4, $\eta \otimes \xi \neq 0$.

Assumption 4.11 takes then the form: there is $r > 0$ such that for any $x \in S^{E_2}(0,r)$, if $(1-\mu)L_2(x) \in T(x) := G(\{x\} \times \Omega_F(x))$, then $\mu > 0$. Sometimes it can be checked directly, but there are also some sufficient conditions. The simplest one is the following: if $\mathcal{D} = E_2$, G is bounded, i.e. $|G(x,y)| \leq R$ and $\mathrm{dist}(\{0\}, L_2(S^{E_2}(0,1)) > 0$, then 4.11 is satisfied, provided r is large enough. Another possibility is to consider the family of systems:

$$\begin{cases} L_1(y) \in F(x,y) \\ L_2(x) \in \lambda G(x,y). \end{cases} \tag{6}$$

When the set of all solutions for $\lambda \in (0,1)$ is bounded, then 4.11 is satisfied. The same holds, when there is $r > 0$ such that for any solution (x,y) of the system with some $\lambda > 0$, if $\|x\| = r$, then $\lambda \geq 1$.

If L_2 is not an isomorphism, but $i(L_2) = 0$, then $\Psi := -J \circ P_2|_{D^{E_2}(a,r)}$ satisfies assumption 4.12, where $J : \mathrm{Ker}\, L_2 \to \mathrm{Im}\, Q_2$ is a linear isomorphism which saves the orientation (induced on $\mathrm{Im}\, Q_2$ by the given one on $\mathrm{Coker}\, L_2$). Indeed, observe that $(L_2|_{\mathrm{Ker}\, P_2} + J \circ P_2)^{-1} : E_2' \to E_2$ is also a linear isomorphism and $(L_2|_{\mathrm{Ker}\, P_2} + J \circ P_2)^{-1} \circ L_2 = id_{E_2} - P_2 : E_2 \to E_2$, hence

$$\mathrm{Ind}_{L_2}(\Psi, D^{E_2}(0,r)) = \mathrm{Ind}_{id_{E_2}-P_2}((L_2|_{\mathrm{Ker}\, P_2} + J \circ P_2)^{-1} \circ (-J \circ P_2), D^{E_2}(0,r)) =$$

$$= \mathrm{Ind}_{id_{E_2}-P_2}(-P_2, D^{E_2}(0,r)) = \pm\,\nu \neq 0 \in \Pi_0(^{13}).$$

([12]) Since it is simply a fixed point index for compact admissible maps (see the construction of the generalized index in e.g. [25] or [13]).

([13]) See the construction of the generalized index for compact maps in e.g. [25] or [13].

If additionally E_2' is a Hilbert space, then 4.11 is satisfied if e.g. for $(x, y) \in C(F) \cap S^{E_2}(0, r)$,

$$\sup_{z \in G(x,y)} \langle L_2(x) + \Psi(x), z \rangle \leq |L_2(x)|^2$$

Let us now recall the main result of [14], which is basic for our further consideration ([14]).

THEOREM 4.14. – *Under assumptions* 4.6-4.8, 4.9, 4.11 *and* 4.12, *if F and G are compact, then* (5) *has a solution* (x, y), *such that* $x \in D^{E_2}(a, r)$. □

As we have mentioned, by using the solution map to solve problem of the form (5), we can reduce the assumption concerning the generalized index of maps, i.e. we assume 4.9 instead of $\mathrm{Ind}_L(\Phi, X) \neq 0$, where $\Phi : (x, y) \mapsto (G(x, y), F(x, y))$. Another aim of this paper is to relax also assumptions concerning the compactness. In the previous section we have shown that the class of fundamentally restrictible maps is a quite large and natural extension of compact maps. Moreover, observe that we assume less then the map Φ is L-fundamentally restrictible. It allows us to apply the below theorem in a bit different situations then earlier (comp. Ex 1 in the next section and [10]).

THEOREM 4.15. – *Theorem* 4.14 *stays true, if instead of compactness of F and G we assume* 4.5 *and* 4.16. *G is L_2-fundamentally restrictible such that there is a compact set M, L_2-fundamental for any map of the form $G(\cdot, y)$, which contains $\Psi(D^{E_2}(a, r)) \cup \{L_2(a)\}$.*

REMARK 4.16. – Observe that if maps $G(\cdot, y)$ are e.g. L_2-condensing, $(L_2)_M$-condensing, (L_2, K_c)-operators then G satisfies assumption 4.16, since for such maps one can find a compact L_2-fundamental set containing an arbitrary compact subset.

PROOF OF THEOREM 4.15. – Observe that $a \in B_M := \mathrm{cl}\, B^{E_2}(a, r) \cap L_2^{-1}(M)$. Since B_M is compact, the assumptions concerning F imply that there is a compact convex set K, L_1-fundamental for maps of the form $F(x, \cdot)$, where $x \in B_M$.

Let $\overline{q} : \Gamma \to K$ be an extension of $q|_{p^{-1}(B_M \times L_1^{-1}(K)) \cap X}$ and $\overline{F}$ be a compact c-admissible map determined by $(p, \overline{q})$. Assumption 4.7 implies that for any $v \in B_M$ there is $\varepsilon_v > 0$ such that $B^{E_2}(v, \varepsilon_v) \times O_{\varepsilon_v}(\Omega_F(v)) \subset \mathrm{int}\, X$. Since both Ω_F and $\Omega_{\overline{F}}$ are u.s.c. with compact values (see Lemma 4.3), for $v \in B_M$ one can find

([14]) Remember that compact maps are fundamentally restrictible, so all previous lemmas stay true.

$\delta_v > 0$ and $\overline{\delta_v} > 0$ such that

$$\Omega_F(B^{E_2}(v,\delta_v)) \subset O_{\varepsilon_v}(\Omega_F(v)) \quad \text{and} \quad \Omega_{\overline{F}}(B^{E_2}(v,\overline{\delta_v})) \subset O_{\varepsilon_v}(\Omega_{\overline{F}}(v)).$$

Moreover, for any $v \in B_M$, $\Omega_F(v) = \Omega_{\overline{F}}(v)$, hence for $\vartheta_v = \min(\varepsilon_v, \delta_v, \overline{\delta_v})$,

$$\Omega_F(B^{E_2}(v,\vartheta_v)) \cup \Omega_{\overline{F}}(B^{E_2}(v,\vartheta_v)) \subset O_{\varepsilon_v}(\Omega_F(v)) = O_{\varepsilon_v}(\Omega_{\overline{F}}(v)).$$

Take a finite covering $B^{E_2}(v_1,\vartheta_{v_1}),\dots,B^{E_2}(v_n,\vartheta_{v_n})$ of the set B_M and observe that

$$\operatorname{dist}(\operatorname{bd}(\bigcup_{i=1}^{n} B^{E_2}(v_i,\vartheta_{v_i})), B_M) = 2\zeta > 0.$$

Then $\operatorname{cl} O_\zeta(B_M) \subset \bigcup_{i=1}^{n} B^{E_2}(v_i,\vartheta_{v_i})$, $O_\zeta(B_M)$ is convex and $\{z\} \times \Omega_{\overline{F}}(z) \subset \operatorname{int} X$ for any $z \in O_\zeta(B_M)$.

Let $\widehat{X} := ((O_\zeta(B_M) \times E_1) \cap X) \cup \{(x,y) \in \mathcal{D} \times E_1 \mid (d(x),y) \in (\operatorname{bd}(O_\zeta(B_M)) \times E_1) \cap X\}$, where $d : E_2 \to \operatorname{cl}(O_\zeta(B_M))$ is a retraction, and $\widehat{\Gamma} := \{(x,y,w) \in \widehat{X} \times \Gamma \mid p(w) = (d(x),y)\}$. The map $\widehat{F} : \widehat{X} \multimap K$ determined by $(\widehat{p},\widehat{q})$, where $\widehat{p} : \widehat{\Gamma} \to \widehat{X}$, $\widehat{p}(x,y,w) = (x,y)$ and $\widehat{q} : \widehat{\Gamma} \multimap K$, $\widehat{q}(x,y,w) = \overline{q}(w)$ satisfies all assumptions of Theorem 4.14 concerning the first map. In particular $C(\widehat{F}) \subset \operatorname{int} \widehat{X}$ and, since $\widehat{p}_a = p_a$ $\widehat{q}_a = q_a$, obviously

$$\operatorname{Ind}_{L_1}((\widehat{p}_a,\widehat{q}),\widehat{X}(a)) \neq 0.$$

Assume that G is determined by $X \xleftarrow{r} \Delta \xrightarrow{s} E_2'$ and let $\overline{s} : \Delta \to M$ be a compact extension of $s|_{r^{-1}((L_2^{-1}(M) \cap \operatorname{cl} O_\zeta(B_M) \times E_1) \cap X)}$. Like earlier put $\widehat{\Delta} := \{(x,y,\delta) \in \widehat{X} \times \Delta \mid r(\delta) = (d(x),y)$ and $\widehat{r} : \widehat{\Delta} \to \widehat{X}$, $\widehat{r}(x,y,\delta) = (x,y)$, $\widehat{s} : \widehat{\Delta} \to E_2'$, $\widehat{s}(x,y,\delta) = \overline{s}(\delta) \in M$. The map $\widehat{G}$ determined by $(\widehat{r},\widehat{s})$ is compact and c-admissible.

We have to prove that $\widehat{G}$ satisfies assumption 4.11. To this end let $x \in S^{E_1}(a,r)$, $y \in \Omega_{\widehat{F}}(x)$ and $\mu(L_2(x) - \Psi(x)) \cap (L_2(x) - \widehat{G}(x,y)) \neq \emptyset$. We check that $\mu \geq 0$.

If $x \in L_2^{-1}(M) \cap S^{E_1}(a,r)$, then $x \in B_M$ and $d(x) = x$. Therefore

$$\widehat{G}(x,y) = \widehat{s}(\widehat{r}^{-1}(x,y)) = \widehat{s}(\{(x,y)\} \times r^{-1}(x,y)) = \overline{s}(r^{-1}(x,y)) = s(r^{-1}(x,y)) = G(x,y),$$

and $\mu \geq 0$ since G satisfies assumption 4.11.

If $x \in S^{E_1}(a,r) \setminus L_2^{-1}(M)$, then there are $z_1 \in \Psi(x)$ and $z_2 \in \widehat{G}(x,y)$ such that $\mu(L_2(x) - z_1) = L_2(x) - z_2$, i.e. $(1-\mu)L_2(x) = z_2 - \mu z_1$. Assume for the moment that $\mu < 0$. Then, since z_1, $z_2 \in M$ and M is convex, $(1-\mu)L_2(x) \in (1-\mu)M$. Hence $L_2(x) \in M$, a contradiction. Therefore again $\mu \geq 0$.

By Theorem 4.14, the system of inclusions

$$\begin{cases} L_1(y) \in \widehat{F}(x,y) \\ L_2(x) \in \widehat{G}(x,y), \end{cases} \tag{7}$$

has a solution (x_0,y_0), such that $x_0 \in D^{E_2}(a,r)$. But since then $L_2(x_0) \in M$, one can easily see that $x_0 \in B_M$ and hence $\widehat{G}(x_0, y_0) = \overline{G}(x_0, y_0) = G(x_0, y_0)$. Moreover, $L_1(y_0) \in \widehat{F}(x_0, y_0) \subset K$, what means that $(x_0, y_0) \in B_M \times L_1^{-1}(K)$. Therefore $\widehat{F}(x_0, y_0) = \overline{F}(x_0, y_0) = F(x_0, y_0)$ and (x_0, y_0) is also a solution of (5), what ends the proof. $\square$

REMARK 4.17. – As we have mentioned in Remark 4.13, if L_2 is an isomorphism and $a = 0$, instead of 4.11 and 4.12 we assume that for $x \in S^{E_2}(0, r)$

$$\text{if } (1-\mu)L_2(x) \in T(x) := G(\{x\} \times \Omega_F(x)), \text{ then } \mu > 0. \tag{8}$$

Moreover, assumption 4.16 is a little bit too strong. In fact we need only the following one:

4.18. T is L_2-fundamentally restrictible such that there is a compact set M, L_2-fundamental for T which contains $\Psi(D^{E_2}(a, r)) \cup \{L_2(a)\}$.

Then in the above proof we have to define $\bar{s}$ as a compact extension of $s|_{r^{-1}((L_2^{-1}(M) \cap \mathrm{cl}\, O_\zeta(B_M) \times E_1) \cap C(F))}$ and $\widehat{G}$ like earlier. Next we consider the map $\widehat{T}(x) := \widehat{G}(\{x\} \times \Omega_{\widehat{F}}(x))$. Observe that

$$\{x \in B_M \mid L_2(x) \in \widehat{T}(x)\} = \{x \in B_M \mid L_2(x) \in T(x)\}.$$

Since for $x \in S^{E_1}(a, r)$ and $y \in \Omega_{\widehat{F}}(x)$, $\widehat{G}(x, y) \subset \widehat{T}(x)$, arguments similar to those used in the proof lead to the conclusion.

5. – Applications.

EXAMPLE 1. – Let E and E' be Banach spaces with given measures of noncompactness χ and χ' respectively. Denote by μ and μ' the Hausdorff measures of noncompactness in spaces of integrable (in Bochner sense) maps $\mathcal{L} := L^1([0, T], E)$ and $\mathcal{L}' := L^1([0, T], E')$ respectively with usual norms, i.e. $\|u\|_{\mathcal{L}} = \int_0^T \|u(s)\|_E ds$ and $\|w\|_{\mathcal{L}'} = \int_0^T \|w(s)\|_{E'} ds$.

Let $f : [0, T] \times E \times E \to E'$ be a map satisfying the following natural assumptions:

(f_1) f is a Carathéodory map, i.e. $f(\cdot, u, v)$ is measurable for all $(u, v) \in E \times E$ and $f(t, \cdot, \cdot)$ is continuous for almost all $t \in [0, T]$,

(f_2) there is an integrable function $\lambda_1 : [0, T] \to [0, \infty)$ and a continuous function $\lambda_2 : [0, T] \to [0, \infty)$, such that for almost all $t \in [0, T]$, and any $u_1, u_2, v_1, v_2 \in E$,

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_E \leq \lambda_1(t)\|u_1 - u_2\|_E + \lambda_2(t)\|v_1 - v_2\|_E,$$

(f_3) (sublinear growth) there are integrable functions $m, n : [0,T] \to [0,\infty)$, such that for almost all $t \in [0,T]$ and any $u, v \in E$, $\|f(t,u,v)\| \leq m(t) + n(t)\|u\|$.

Consider the following boundary value problem

$$\begin{cases} L_1(u')(t) = f(t, u(t), u'(t)), & \text{for a.a. } t \in [0,T], \\ au(0) + bu(T) \in a(u(0)), \end{cases} \tag{9}$$

where $L_1 : \mathcal{L} \to \mathcal{L}'$ is a Fredholm operator with nonnegative index, $a, b \in \mathbb{R}$ and $a : E \multimap E$ is a compact c-admissible map. By a solution we understand an absolutely continuous map satisfying (9) for almost all $t \in [0,T]$.

Similar Floquet-type problem was considered in [10]. But now we base on the different abstract results then earlier, so the technical assumptions concerning the map f are not the same. (comp. Th. 5.1) and Rem. 5.2.

One can write (9) as a coincidence problem in the following form

$$\begin{cases} L_1(y) = F(x,y) \\ L_2(x) \in G(x,y), \end{cases} \tag{10}$$

where $F : E \times \mathcal{L} \to \mathcal{L}'$, $L_2 : E \to E$, $G : E \times \mathcal{L} \to E$ and

$$F(x,y) = f\left(\cdot, x + \int_0^{(\cdot)} y(s)ds,\ y(\,\cdot\,) \right)$$

$$L_2(x) = a \cdot x,$$

$$G(x,y) = -b \cdot \left(x + \int_0^T y(s)ds \right) + a(x).$$

Indeed, if (x,y) is a solution to problem (10), then $u(t) = x + \int_0^t y(s)ds$ is a solution to problem (9).

Let P, Q and K_P be the respective linear maps for L_1 (comp. Preliminaries) and let $N = \int_0^T n(s)ds$, $M = \int_0^T m(t)dt$, $\Lambda_1 = \int_0^T \lambda_1(s)ds$, $\Lambda_2 = \sup_{t\in[0,T]} \lambda_2(t)$.

THEOREM 5.1. – *Assume that f satisfies conditions* (f_1) – (f_3) *and*
(f_4) $(\Lambda_1 + \Lambda_2)\|K_P\| < 1$,
(f_5) L_1 *is a linear isomorphism,*
(f_6) $\|\frac{a}{b}\| > max\{e, e^{\|K_P\|N}\}$.
Then the problem (10) *has a solution.*

REMARK 5.2. – Since we want to make our consideration more clear, we resign from explaining the most general situation. Below we present another (weaker) possible assumptions (comp. [11]).

Assumption (f_5) implies that L_1 and F do not have coincidence points on the boundary of some ball and that the respective generalized index is nontrivial. It can be replaced as follows:

there is $R' > 0$ such that, if $y \in \operatorname{Ker} L_1$ and $\|y\|_{\mathcal{L}} > R'$, then $Q \circ F(z, y) \neq 0$ for $z \in E$ and

(f_5') $i(L_1) = 0$ and the Leray-Schauder degree of the map $J \circ Q \circ F(0, \cdot)\|_{\operatorname{Ker} L_1}$ is not equal to 0, where $J : \operatorname{Im} Q \to \operatorname{Im} P$ is a linear isomorphism;

or

(f_5'') $\operatorname{Deg}(Q \circ F(0, \cdot)\|_{\operatorname{Ker} L_1}, B^{\mathcal{L}}(0, R) \cap \operatorname{Ker} L_1, 0) \neq 0$ (15)

Assumptions (f_4) and (f_6) are technical. They also can be relaxed when one accepts more technical complications in the proof. For instance, instead of (f_4) we can assume

(f_4') $(\Lambda_1 + \Lambda_2)\|K_P\|^{(\mu',\mu)} < 1$,

where $\|K_P\|^{(\mu',\mu)} := \inf Z$ and Z is the set of all positive numbers k such that K_P is a (k, μ', μ)-contraction (16) (comp. [1]). It is easy to check, that $\|K_P\|^{(\mu',\mu)} \leq \|K_P\|$.

Then instead of (f_6) we can assume that

(f_6') $\|\frac{a}{b}\| > \max\{\exp(\int_0^T \frac{\lambda_1(s)\|K_P\|}{1-\lambda_2(s)\|K_P\|} ds), \exp(\|K_P\|N)\}$.

Observe that $\int_0^T \frac{\lambda_1(s)\|K_P\|}{1-\lambda_2(s)\|K_P\|} ds$ may be less then 1 if we assume (f_4'), but assumption (f_4) implies that $\int_0^T \frac{\lambda_1(s)\|K_P\|}{1-\lambda_2(s)\|K_P\|} ds \geq 1$.

PROOF OF THEOREM 5.1.

Step 1. Observe that F is a continuous map. Indeed, if for $(x_0, y_0) \in E \times \mathcal{L}$ and any $\varepsilon > 0$, we take $\delta < \min(\frac{\varepsilon}{3\Lambda_1}, \frac{\varepsilon}{3\Lambda_2})$ and assume that for some $(x, y) \in E \times \mathcal{L}$,

$$\delta > \|(x_0, y_0) - (x, y)\|_{E\times\mathcal{L}} = \max(\|x_0 - x\|_E, \|y_0 - y\|_{\mathcal{L}})$$

then

$$\|F(x_0, y_0) - F(x, y)\|_{\mathcal{L}} = \left\| f\left(\cdot, x_0 + \int_0^{\cdot} y_0(s)ds, y_0(\cdot)\right) - f\left(\cdot, x + \int_0^{\cdot} y(s)ds, y(\cdot)\right) \right\|_{\mathcal{L}} \leq$$

$$\leq \int_0^T \lambda_1(t) \left\| x_0 - x + \int_0^t (y_0(s) - y(s))ds \right\|_E dt + \int_0^T \lambda_2(t)\|y_0(t) - y(t)\|_E dt < \Lambda_1 \cdot 2\delta + \Lambda_2\delta < \varepsilon.$$

(15) By Deg we understand the stable generalized degree defined in [25].
(16) i.e. For any bounded set $B \subset E'$, $\mu(K_P(B)) \leq k\mu'(B)$.

Since G is the sum of a continuous single-valued map and a c-admissible one, both F and G are c-admissible (see Remark 2.3 (ii), (iii)).

Step 2. We prove that F is L_1-condensing with respect to the second variable (hence L_1-fundamentally restrictible with respect to the second variable).

Let Z be a compact subset of E, and V be a bounded subset of $\mathcal{L}$ such that $\mu(V)=\delta$. Observe that

$$\mu(V)=\mu(K_P\circ L_1(V))\le \|K_P\|\mu'(L_1(V)),$$

what implies that

$$\mu'(L_1(V))\ge \frac{\mu(V)}{\|K_P\|}. \tag{11}$$

Take arbitrary $\varepsilon>0$, $\delta_1>0$. Of course Z has a finite ε-net while V has a finite $(\delta+\delta_1)$-net. Let x_l and y_k be elements of the respective nets and assume that, for some $x\in Z$ and $y\in V$, $\|x_l-x\|_E<\varepsilon$ and $\|y_k-y\|_{\mathcal{L}}<\delta+\delta_1$. Then

$$\begin{aligned}
&\|F(x_l,y_k)-F(x,y)\|_{\mathcal{L}'}=\int_0^T \|F(x_l,y_k)(t)-F(x,y)(t)\|_{E'}dt=\\
&=\int_0^T\left\|f\left(t,x_l+\int_0^t y_k(s)ds,y_k(t)\right)-f\left(t,x+\int_0^t y(s)ds,y(t)\right)\right\|_{E'}dt\le\\
&\le\int_0^T\lambda_1(t)\|x_l-x\|_E dt+\int_0^T\lambda_1(t)\left\|\int_0^t(y_k(s)-y(s))ds\right\|_E dt+\int_0^T\lambda_2\|y_k(t)-y(t)\|_E dt<\\
&<\Lambda_1\varepsilon+\Lambda_1(\delta+\delta_1)+\Lambda_2(\delta+\delta_1).
\end{aligned}$$

It means that elements $F(x_l,y_k)$ form a finite $(\Lambda_1\varepsilon+\Lambda_1(\delta+\delta_1)+\Lambda_2(\delta+\delta_1))$-net in the set $F(Z\times V)$. But since ε and δ_1 can be arbitrary small, (11) and assumption (f_4) imply

$$\mu'(F(Z\times V))\le(\Lambda_1+\Lambda_2)\delta<\frac{\mu(V)}{\|K_P\|^{(\mu',\mu)}}\le\mu'(L_1(V)).$$

Step 3. We prove that for any $r>0$, the set $\{\|y\|_{\mathcal{L}}\mid L_1(y)=F(x,y),\ \|x\|_E<r\}$ is bounded.

If $\|x\|_E<r$ and $L_1(y)\in F(x,y)$, then for almost all $t\in[0,T]$

$$L_1(y)(t)=f\left(t,x+\int_0^t y(s)ds,y(t)\right).$$

Hence

$$K_P\circ L_1(y)=K_P\left(f(\cdot,x+\int_0^{\cdot} y(s)ds,y(\,\cdot\,))\right),$$

and, by assumption (f_5),

$$y = K_P\left(f(\cdot, x + \int_0^{\cdot} y(s)ds, y(\,\cdot\,))\right).$$

Therefore, for any $t \in [0, T]$,

$$\begin{aligned}\int_0^t \|y(s)\|_E ds &\le \int_0^t \|K_P\| \left\| f\left(s, x + \int_0^s y(\tau)d\tau, y(s)\right)\right\|_{E'} ds \le \\ &\le \|K_P\| \int_0^t \left(m(s) + n(s)\left\| x + \int_0^s y(\tau)d\tau \right\|_E \right) ds \le \\ &\le \|K_P\| \left(M + N\|x\|_E + \int_0^t n(s)\left(\int_0^s \|y(\tau)\|_E d\tau\right) ds\right) \le \\ &\le \|K_P\|(M + N\|x\|_E) + \|K_P\| \int_0^t n(s) \left(\int_0^s \|y(\tau)\|_E d\tau\right) ds,\end{aligned}$$

and by the Gronwall inequality

$$\int_0^t \|y(s)\|_E ds \le \|K_P\|(M + N\|x\|_E) \exp\left(\|K_P\| \int_0^t n(s)ds\right).$$

It means that

$$\|y\|_{\mathcal{L}} \le \|K_P\|(M + N\|x\|_E) \exp(\|K_P\|N) \le \|K_P\|(M + Nr) \exp(\|K_P\|N).$$

In particular, for $x = 0$ and $L_1(y) = F(0, y)$,

$$\|y\|_{\mathcal{L}} \le \|K_P\|M \exp(\|K_P\|N) \le \|K_P\|M \exp(\|K_P\|N) + 1 =: R,$$

and consequently

$$\Omega_F(0) \subset B^{\mathcal{L}}(0, R).$$

Therefore $\operatorname{Ind}_{L_1}(F(0, \cdot), B^{\mathcal{L}}(0, R))$ is well defined and, since by assumption (f_5) the map $K_P : \mathcal{L}' \to \mathcal{L}$ is a linear isomorphism,

$$\operatorname{Ind}_{L_1}(F(0, \cdot), B^{\mathcal{L}}(0, R)) = \operatorname{Ind}_{id_{\mathcal{L}}}\big(K_P \circ F(0, \cdot), B^{\mathcal{L}}(0, R)\big).$$

It is easy to see that the map

$$H : \operatorname{cl} B^{\mathcal{L}}(0, R) \times [0, 1] \to \mathcal{L}$$

given by

$$H(y,\lambda) = \lambda \cdot K_P \circ F(0,y)$$

is a homotopy in the class of single-valued fundamentally restrictible maps, such that for any $\lambda \in [0,1]$,

$$\{y \mid y \in H(y,\lambda)\} \subset B^{\mathcal{L}}(0,R).$$

Hence following equalities hold:

$$\begin{aligned}&\operatorname{Ind}_{id_{\mathcal{L}}}\big(K_P \circ F(0,\cdot), B^{\mathcal{L}}(0,R)\big) = \operatorname{Ind}_{id_{\mathcal{L}}}(H(\cdot,1), B^{\mathcal{L}}(0,R)) = \\ &\operatorname{Ind}_{id_{\mathcal{L}}}(H(\cdot,0), B^{\mathcal{L}}(0,R)) = \operatorname{Ind}_{id_{\mathcal{L}}}(\mathcal{O}, B^{\mathcal{L}}(0,R)) = 1,\end{aligned}$$

where $\mathcal{O} : \mathcal{L} \to \mathcal{L}$, $\mathcal{O}(y) = 0$ for any y. Of course the last index is simply the Leray-Schauder one.

We have proved that

$$\operatorname{Ind}_{L_1}(F(0,\cdot), B^{\mathcal{L}}(0,R)) = 1 \neq 0.$$

Step 4. We exam properties of the map $\mathcal{T}(x) := G(\{x\} \times \Omega_F(x))$ (comp. Remark 4.17). Since the respective index is nontrivial, from Lemma 4.10 it follows that $\mathcal{T}$ is defined on the whole space E.

Observe that, if $y \in \Omega_F(x)$ and $y_1 \in \Omega_F(x_1)$, then for any $t \in [0,T]$

$$\begin{aligned}&\int_0^t \|y(s) - y_1(s)\|_E \leq \int_0^t \|K_P\| \left\| f\left(s, x + \int_0^s y(\tau)d\tau, y(s)\right) - f\left(s, x_1 + \int_0^s y_1(\tau)d\tau, y_1(s)\right)\right\|_E ds \leq \\ &\int_0^t \|K_P\| \left(\lambda_1(s)\left\| x + \int_0^s y(\tau)d\tau - x_1 - \int_0^s y_1(\tau)d\tau\right\|_E + \lambda_2(s)\|y(s) - y_1(s)\|_E\right) ds \leq \\ &\int_0^t \lambda_1(s)\|K_P\| \left(\|x - x_1\|_E + \int_0^s \|y(\tau) - y_1(\tau)\|_E d\tau\right) ds + \Lambda_2\|K_P\| \int_0^t \|y(s) - y_1(s)\|_E ds,\end{aligned}$$

what means that

$$\int_0^t \|y(s) - y_1(s)\|_E ds \leq \int_0^t \frac{\lambda_1(s)\|K_P\|}{1 - \Lambda_2\|K_P\|} \left(\|x - x_1\|_E ds + \int_0^s \|y(\tau) - y_1(\tau)\|_E d\tau\right) ds,$$

and hence, by assumption (f_4), also

$$\begin{aligned}&\|x - x_1\|_E + \int_0^t \|y(s) - y_1(s)\|_E ds \leq \\ &\|x - x_1\|_E + \int_0^t \left(\|x - x_1\|_E ds + \int_0^s \|y(\tau) - y_1(\tau)\|_E d\tau\right) ds.\end{aligned}$$

By the Gronwall inequality, we get

$$\|x - x_1\|_E + \int_0^t \|y(s) - y_1(s)\|_E ds \leq \|x - x_1\|_E \cdot e,$$

and consequently

$$\|x + \int_0^T y(s)ds - x_1 - \int_0^T y_1(s)ds\|_E \leq e \cdot \|x - x_1\|_E.$$

If S is a bounded subset of E and $\chi(S) = \delta$, then $\chi(L_2(S)) = \|a\|\delta$, and by the compactness of the map a, also $\chi(a(S)) = 0$. Then, since

$$\mathcal{T}(S) = \bigcup_{x \in S} \left\{ -b\left(x + \int_0^T y(s)ds\right) + a(x) \mid y \in \Omega_F(x) \right\},$$

we know that

$$\chi(\mathcal{T}(S)) \leq \chi\left(\bigcup_{x \in S} \left\{ -b\left(x + \int_0^T y(s)ds\right) \mid y \in \Omega_F(x) \right\}\right).$$

Take an arbitrary $\delta_1 > 0$ and assume that points x_l, where $l = 1, \ldots, n_{\delta_1}$, are elements of $(\delta + \delta_1)$-net in S. For each l take $y_l \in \Omega_F(x_l)$. The previous considerations imply that elements $b(x_l + \int_0^T y_l(s)ds)$ form an $(e \cdot \|b\|(\delta + \delta_1))$-net in the set

$$\bigcup_{x \in S} \left\{ -b\left(x + \int_0^T y(s)ds\right) \mid y \in \Omega_F(x) \right\}.$$

Hence $\chi(\mathcal{T}(S)) \leq e \cdot \|b\|$ and, by assumption (f_6),

$$\chi(\mathcal{T}(S)) < \chi(L_2(S)).$$

It means that the map $\mathcal{T}$ is L_2-condensing, so L_2-fundamentally restrictible with some compact L_2-fundamental for T set containing 0.

Step 5. At last we prove that there is $r > 0$, such that $\mathcal{T}(\operatorname{bd}(B^E(0,r)) \subset \operatorname{int}(L_2(B^E(0,r)))$, what implies that for $\|x\|_E = r$, if $(1-\mu)L_2(x) \in \mathcal{T}(x)$, then $\mu > 0$.

Let $r_1 > 0$ be such that $a(E) \subset B^E(0, r_1)$. Similarly like in Step 3,

$$\begin{aligned}\|x + \int_0^t y(s)ds\|_E \leq& \|x\|_E + \int_0^t \|y(s)\|_E ds \leq \\ &\leq \|x\|_E + \|K_P\|M + \|K_P\| \int_0^t n(s) \left\| x + \int_0^s y(\tau)d\tau \right\|_E ds,\end{aligned}$$

and, by the Gronwall inequality,

$$\|x + \int_0^t y(s)ds\|_E \leq (\|x\|_E + \|K_P\|M)\exp(\|K_P\|N).$$

It means that, if $v \in T(x)$ and $\|x\|_E = r$, then

$$\|v\|_E \leq \|b\|(r + \|K_P\|M)\exp(\|K_P\|N) + r_1.$$

To get the expected inclusion, it is enough to find $r > 0$ such that

$$\|b\|(R + \|K_P\|M)\exp(\|K_P\|N) + r_1 < \|a\|r \tag{12}$$

But from assumption (f_6) it follows that $\|a\| - \|b\|\exp(\|K_P\|N) > 0$. Therefore, for sufficiently large r the inequality

$$\|b\|\|K_P\|M\exp(\|K_P\|N) + r_1 < r(\|a\| - \|b\|\exp(\|K_P\|N))$$

is true and equivalent to (12). □

EXAMPLE 2. – Consider the following problem

$$\begin{cases} x'(t) = g(t, x(t), x'(t), y(t)), & \text{for a. a. } t \in I = [0, T], \\ x(0) = x(1), \\ y'(t) = f(t, x(t), y(t), y'(t)), & \text{for a. a. } t \in I, \\ 0 \in l(x, y), \end{cases} \tag{13}$$

where $T > 0$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, are Carathéodory functions ([17]) and $l : C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^m) \multimap \mathbb{R}^k$, $1 \leq k \leq m$, is a multivalued map. By $C(I, \mathbb{R}^d)$ we understand the space of continuous function with the standard sup-norm $\|\cdot\|$ and by $C_c(I, \mathbb{R}^d)$ its subspace of constant maps. By a *solution* we mean a pair of absolutely continuous functions $x : I \to \mathbb{R}^n$, $y : I \to \mathbb{R}^m$ satisfying (13).

Comparing it with the example from [14], one can observe that now we admit the situation when functions on the right hand side depend on derivatives. But in a such general situation we can not adapt the previous methods, since the respective maps ([18]) are not compact and do not have suitable topological properties. Moreover, we have to assume that the maps f and g are single-valued. Nevertheless the map l which may be viewed as the system of nonlocal boundary value data is multivalued.

Let f, g be such that

(f_1) there is an integrable function $\gamma \in L^1([0, T], \mathbb{R})$ and $a > 0$ such that $\|f(t, x, y, w)\| \leq \gamma(t) + a\|y\|$ for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and almost all $t \in I$,

([17]) i.e. for almost all $t \in I$, $f(t, \cdot, \cdot, \cdot)$, $g(t, \cdot, \cdot, \cdot)$ are continuous and for all x, y, w from respective spaces $f(\cdot, x, y, w)$ and $g(\cdot, x, y, w)$ are measurable.

([18]) Namely F and G, see further considerations and compare [14].

(f_2) there is $0 < \kappa_1 < 1$, such that $\|f(t,x,y,w_1) - f(t,x,y,w_2)\| \le \kappa_1 \|w_1 - w_2\|$ for any $x \in \mathbb{R}^n$, $y, w_1, w_2 \in \mathbb{R}^m$ and almost all $t \in I$;

(g_1) there is an integrable function $a \in L^1([0,T],\mathbb{R})$ and $b > 0$ such that $\|g(t,x,v,y)\| \le a(t) + b\|x\|$ for any $x, v \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and almost all $t \in I$,
(g_2) there is $0 < \kappa_2 < 1$, such that $\|g(t,x,v_1,y) - g(t,x,v_2,y)\| \le \kappa_2 \|v_1 - v_2\|$ for any $x, v_1, v_2 \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and almost all $t \in I$

Moreover assume that

(l_1) l is a c-admissible map and maps bounded sets onto bounded ones,
(l_2) there is $r \ge 0$ such that for any $x \in C(I,\mathbb{R}^n)$ and any $y \in C(I,\mathbb{R}^m)$, if $0 \in l(x,y)$, then $\|y(t_x)\| \le r$ for some $t_x \in I$,
(l_3) there is $R > 0$ such that for all $y \in C(I,\mathbb{R}^m)$, if $\|y(0)\| \ge R$, then $0 \notin l(0,y)$ and there is a c-admissible pair (p,q) determining l such that

$$\operatorname{Deg}((p',q')(0,\cdot), B_c(0,R), 0) \ne 0 \in \Pi_{m-k},$$

where $B_c(0,R) = \{y \in C_c(I,\mathbb{R}^m) \mid \|y\| \le R\}$ and (p',q') is the restriction of the pair (p,q) to $C(I,\mathbb{R}^n) \times C_c(I,\mathbb{R}^m)$ - see Remark 2.3 (i) ([19]).

Additionally we assume that there is a smooth C^1-function $V : \mathbb{R}^n \to \mathbb{R}$, such that

(V_1) there is $r_1 > 0$ such that for any $x \in \mathbb{R}^n$, if $\|x\| > r_1$, then $\langle \nabla V(x), x\rangle > 0$,
(V_2) there is $r_2 > 0$ such that for any $x, v \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \in I$ if $\|x\| \ge r_2$, then $\langle \nabla V(x), g(t,x,v,y)\rangle \ge 0$,

REMARK 5.3. – Observe that
(i) In fact V is a *guiding function* for g, although V could not be coercitive ([20]). Guiding functions are often used to find periodic trajectories and in others control problems (comp. [17], see also [30]).
(ii) The function $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = \frac{1}{2}\|x\|^2$ satisfies (V_1) ([21]). Condition (V_2) means then that for all $x, v \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $t \in I$, if $\|x\| \ge r_2$, then $\langle x, g(t,x,v,y)\rangle \ge 0$, i.e. $g(t,x,v,y)$ belongs to the half-space $\{z \in \mathbb{R}^n \mid \langle x, z\rangle \ge 0\}$.
(iii) Taking, if necessary, $\max\{\gamma(t), a(t)\}$ and $\max\{a, b, 1\}$, we can assume that $\gamma(t) = a(t) \ge 0$ and $a = b \ge 1$. Moreover, without loss of generality we may assume that in (V_1) and (V_2), $r_1 = r_2$.

THEOREM 5.4. – *Under assumptions* (l_1)-(l_3), (g_1)-(g_2), (f_1)-(f_2) *and* (V_1)-(V_4) *system* (13) *has a solution.*

([19]) By Deg we understand the stable generalized degree defined in [25], see also [14].
([20]) V is a coercitive map if $\lim_{|x|\to\infty} V(x) = +\infty$.
([21]) Such function was considered in e.g. [8] and [30].

Before starting the proof, we write problem (13) in the form which allows us to apply Theorem 4.15.

Let $E_0 = L^1(I, \mathbb{R}^m)$ and

$$E_1 = AC(I, \mathbb{R}^m), \qquad E_1' = L^1(I, \mathbb{R}^m) \times \mathbb{R}^k,$$
$$E_2 = AC_T(\mathbb{R}, \mathbb{R}^n), \qquad E_2' = L^1_T(\mathbb{R}, \mathbb{R}^n),$$

where $AC(I, \mathbb{R}^m)$ and $L^1(I, \mathbb{R}^m)$ are spaces of the absolutely continuous and the integrable maps, respectively, $L^1_T(\mathbb{R}, \mathbb{R}^n)$ is a space of maps from $L^1(I, \mathbb{R}^n)$ extended on $\mathbb{R}$ by T-periodicity with the norm $\|v\|_{E_2'} = \int_0^T \|v(t)\| dt$ and $AC_T(\mathbb{R}, \mathbb{R}^n)$ is a Banach space of absolutely continuous T-periodic maps from $\mathbb{R}$ to $\mathbb{R}^n$ with the norm $\|x\|_{E_2} = \max\left\{ \sup_{t\in[0,T]} \|x(t)\|, \int_0^T \|x'(t)\| dt \right\}$.

By μ_1 and μ_2 we denote the Hausdorff measures of noncompactness in spaces E_1' and E_2', respectively.

Observe that $L_1 : E_1 \to E_1'$ given by $L_1(y) = (y', 0)$ is a Fredholm operator with index $i(L_1) = m - k$. Let $F : E_2 \times E_1 \multimap E_1'$ be defined by $F(x, y) = \{\tilde{f}(x, y)\} \times l(x, y)$, where $\tilde{f}(x, y)(t) = f(t, x(t), y(t), y'(t))$ for any $t \in I$. It is clear that the problem

$$\begin{cases} y'(t) = f(t, x(t), y(t), y'(t)), & \text{for a. a. } t \in I, \\ 0 \in l(x, y) \end{cases}$$

is equivalent to

$$L_1(y) \in F(x, y).$$

Consider the continuous linear isomorphism $L_2 : E_2 \to E_2'$, $L_2(x) = x' - x$, and a map $G : E_2 \times E_1 \to E_2'$ given by $G(x, y)(t) = g(t, x(t), x'(t), y(t)) - x(t)$. Like earlier the problem

$$\begin{cases} x'(t) = g(t, x(t), x'(t), y(t)), & \text{for a. a.} t \in I, \\ x(0) = x(T). \end{cases}$$

is equivalent to

$$L_2(x) = G(x, y),$$

and finally problem (13) is equivalent to the following one

$$\begin{cases} L_1(y) \in F(x, y) \\ L_2(x) = G(x, y). \end{cases} \tag{14}$$

It is easy to see that the map F defined above is a c-admissible multivalued map and the map G is a continuous (then c-admissible) single-valued map.

Indeed, observe that $\tilde{f} = f_2 \circ f_1$, where $f_1 : E_2 \times E_1 \to E_2' \times E_0 \times E_0$ is given by $f_1(x, y) = (x, y, y')$ and $f_2 : E_2' \times E_0 \times E_0 \to E_0$ is given by $f_2(x, y, v)(t) = f(t, x(t), y(t), v(t))$. It is obvious that f_1 is continuous and, by following Th. 5.5, so is f_2.

THEOREM 5.5 (comp. [37]). -- *If $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ is a Carathéodory map satisfying the following condition*

$$\|f(t, z)\| \le a(t) + b\|z\|^{p/q},$$

where p, $q < \infty$, $a \in L^q([0, T], \mathbb{R})$ and b is a constant, then the map $N_f : L^p([0, T], \mathbb{R}^n) \to L^q([0, T], \mathbb{R}^m)$ given by

$$N_f(z)(t) := f(t, z(t))$$

is continuous. □

Hence $\tilde{f}$ is a (single-valued) continuous map, what, with assumptions concerning l, ends the proof for F (comp. Rem. 2.3). The same arguments work for the map G.

In further considerations we will need also

THEOREM 5.6 (comp. [4]). – *Any sequence bounded in the space $AC([0, T], \mathbb{R}^n)$ contains a subsequence which is convergent in the space $L^1([0, T], \mathbb{R}^n)$.* □

PROOF OF THEOREM 5.4.

Step 1. We prove that F is L_1-condensing with respect to the second variable.

Let Z be a bounded subset of the space $E_2 \times E_1$ and let Z_1, Z_2 be the projections of Z on spaces E_1, E_2, respectively. Assume that Z_2 is compact and let $\mu_1(L_1(Z_1)) = \varepsilon$.

Take any $\varepsilon_1 > 0$ and denote by $\{(z_i, 0)\}_{i=1}^{n_\varepsilon}$ a finite $(\varepsilon + \varepsilon_1)$-net in $L_1(Z_1)$. Consider the family of sets

$$\Gamma_i = \{w \in E_0 \| w(t) = f(t, x(t), y(t), z_i(t)),\ (x, y) \in Z\} \quad i = 1, \ldots n_\varepsilon.$$

Any set Γ_i is relatively compact. Indeed, if $(w_j)_{j=1}^\infty$ is a sequence in Γ_i, then there is a sequence $((x_j, y_j))_{j=1}^\infty$ in Z, such that $w_j(t) = f(t, x_j(t), y_j(t), z_i(t))$ for almost all $t \in [0, T]$. Moreover, $(x_j)_{j=1}^\infty$ is contained in a compact set Z_2, therefore it has a subsequence converging to $x_0 \in E_2$ and, since $(y_j)_{j=1}^\infty$ is bounded in E_1, by Th. 5.6, it has a subsequence converging in E_0 to y_0. It means that some subsequence of $((x_j, y_j))_{j=1}^\infty$ converges in $E_2' \times E_0$ to (x_0, y_0). Assumption (f_1) and Th. 5.5 imply that the map $\widehat{f_i} : E_2' \times E_0 \to E_0$ given by $\widehat{f_i}(x, y) = f(\cdot, x(\cdot), y(\cdot), z_i(\cdot))$ is continuous. Hence the respective subsequence of $(w_j)_{j=1}^\infty$ converges (in E_0) to $w_0 = f(\cdot, x_0(\cdot), y_0(\cdot), z_i(\cdot))$, and Γ_i is relatively compact.

Therefore $\Gamma = \bigcup_{i-1}^{n_\varepsilon} \Gamma_i$ is also a relatively compact set and so is $\Gamma \times l(Z)$, because $l(Z)$ is a bounded subset of $\mathbb{R}^k$.

Let $(w, a) \in F(Z)$, that is for some $(x, y) \in Z$, $a \in l(x, y)$ and $w(t) = f(t, x(t), y(t), y'(t))$ for almost all $t \in [0, T]$. Take $i \in \mathbb{N}$, $1 \le i \le n_\varepsilon$ such that $\|(y', 0) - (z_i, 0)\|_{E_1'} < \varepsilon + \varepsilon_1$ and put $\widehat{w} = f(\cdot, x(\,\cdot\,), y(\,\cdot\,), z_i(\,\cdot\,)) \in \Gamma_i$. Assumption (f_2) implies

$$\|w - \widehat{w}\|_{E_0} = \int_0^T \|w(t) - \widehat{w}(t)\| dt = \int_0^T \|f(t, x(t), y(t), y'(t)) - f(t, x(t), y(t), z_i(t))\| dt \le$$

$$\int_0^T \kappa_1 \|y'(t) - z_i(t)\| dt = \kappa_1 \int_0^T \|y'(t) - z_i(t)\| dt = \kappa_1 \|y' - z_i\|_{E_0} < \kappa_1(\varepsilon + \varepsilon_1),$$

what means that

$$\|(w, a) - (\widehat{w}, a)\|_{E_1'} < \kappa_1(\varepsilon + \varepsilon_1).$$

We have just proved that elements of $\Gamma \times l(Z)$ compose a $\kappa_1(\varepsilon + \varepsilon_1)$-net of $F(Z)$. Therefore

$$\mu_1(F(Z)) \le \kappa_1 \mu_1(L_1(Z)),$$

and F satisfies assumption 4.5.

Step 2. Let $X = \{(x, y) \in E_2 \times E_1 \mid \|y\|_{E_1} \le M\}$, where

$$M = \max\ (R, (r + \|\gamma\|_{L^1}) \exp(aT), \|\gamma\|_{L^1} + Ta(r + \|\gamma\|_{L^1}) \exp(aT)\ ) + 1.$$

Obviously X satisfies assumptions 4.6 and 4.8. We have to check assumptions 4.7 and 4.9. Observe that, if $(x, y) \in C(F)$, then, in particular, $0 \in l(x, y)$ and $y' = f(\cdot, x(\,\cdot\,), y(\,\cdot\,), y'(\,\cdot\,))$. Hence (l_2) and (f_1) imply that for any $t \in [0, T]$,

$$\|y(t)\| \le \|y(t_x)\| + \int_{t_x}^t \|f(s, x(s), y(s), y'(s))\| ds \le r + \int_{t_x}^t \gamma(s) ds + a \int_{t_x}^t \|y(s)\| ds,$$

so, by the Gronwall inequality,

$$\|y(t)\| \le (r + \|\gamma\|_{L^1}) \exp(aT),$$

and hence

$$\|y'\|_{E_0} = \int_0^T \|y'(t)\| dt = \int_0^T \|f(t, x(t), y(t), y'(t))\| dt \le \int_0^T \gamma(t) dt + \int_0^T a \|y(t)\| dt \le$$

$$\|\gamma\|_{L^1} + Ta(r + \|\gamma\|_{L^1}) \exp(aT).$$

Therefore

$$\|y\|_{E_1} = \max\left\{ \sup_{t\in[0,T]} \|y(t)\|, \|y'\|_{E_0} \right\} < M,$$

that is $(x, y) \in \operatorname{int} X$ and 4.7 is satisfied.

Consider the map $H : X(0) \times [0,1] \multimap E_1'$ given by $H(y,\lambda) = \{\lambda\tilde{f}(0,y)\}\times l(0,y)$. Observe that, if $L_1(y) \in H(y,\lambda)$ for some $\lambda \in [0,1]$, then $l(y,0) = 0$ and $y' = \lambda f(\cdot, x(\,\cdot\,), y(\,\cdot\,), y'(\,\cdot\,))$. One can check in the same way as above that then $y \in \operatorname{int} X(0)$.

Since $F(0,\cdot)\|_{X(0)}$ is L_1-condensing, there is a compact set K, being L_1-fundamental for $F(0,\cdot)$ and containing the compact set $\operatorname{cl}(\{0\} \times l(\{0\} \times X(0)))$ ([22]).

Let $\overline{f} : X(0) \to K \cap (E_0 \times \{0\})$ be a compact extension of the map $\tilde{f}(0,\cdot)\|_{L_1^{-1}(K)\cap X(0)}$. Observe that $X(0) \ni y \mapsto \overline{F}(y) := (\{\overline{f}(y)\} \times l(0,y)) \subset K$ is a compact map K-associated to $F(0,\cdot)\|_{X(0)}$. Therefore there is a (L_1, K)-homotopy between $F(0,\cdot)\|_{X(0)}$ and $\overline{F}$ (see Th. 3.18) and, by Th. 3.21 (*iv*),

$$\operatorname{Ind}_{L_1}(F(0,\cdot), X(0)) = \operatorname{Ind}_{L_1}(\overline{F}, X(0)).$$

Consider a map $\overline{H} : X(0) \times [0,1] \to E_1'$ given by $\overline{H}(y,\lambda) := \{\lambda\hat{f}(y)\} \times l(0,y)$. It is a compact (then L_1-fundamentally restrictible) homotopy between $\overline{H}(\cdot,1) = \overline{F}$ and $\overline{H}(\cdot,0) = (\{0\} \times l(0,\cdot))$. Moreover, if for some λ one knows that $L_1(y) \in \overline{H}(\lambda,y)$, then $L_1(y) \in K$ and $y \in L_1^{-1}(K) \cap X(0)$. It follows that $\overline{f}(y) = \tilde{f}(0,y)$, that is $\overline{H}(y,\lambda) = H(y,\lambda)$. Therefore $L_1(y) \in H(y,\lambda)$, and consequently $y \in \operatorname{int} X(0)$.

Once more Th. 3.21 (*iv*) implies

$$\operatorname{Ind}_{L_1}(F(0,\cdot), X(0)) = \operatorname{Ind}_{L_1}((\{0\} \times l(0,\cdot)), X(0)),$$

while Th. 3.21 (*vii*) and assumption (l_3) imply

$$\operatorname{Ind}_{L_1}((\{0\} \times l(0,\cdot)), X(0)) = \operatorname{Deg}(-l(0,\cdot), X(0) \cap C_c([0,T], \mathbb{R}^m)) \neq 0, \tag{15}$$

and 4.9 is satisfied.

Step 3. Observe that L_2 is a linear isomorphism, so, in particular, a Fredholm operator. We are going to prove that G is an L_2-condensing map.

Let Z be a bounded subset of E_2 and $Z' = \{x' \mid x \in Z\}$. Obviously Z' is contained in E_2' and Z is relatively compact in E_2' (see Th. 5.6), i.e. $\mu_2(Z) = 0$. Since a measure of noncompactness is a seminorm, also $\mu_2(\{-x \mid x \in Z\}) = 0$ and

$$\mu_2(L_2(Z)) = \mu_2(\{x' - x \mid x \in Z\}) \le \mu_2(Z') + \mu_2(\{-x \mid x \in Z\}) = \mu_2(Z'),$$

$$\mu_2(Z') \le \mu_2(L_2(Z)) + \mu_2(Z),$$

what implies that $\mu_2(L_2(Z)) = \mu_2(Z')$.

([22]) The proof is the same as in Example 3.6, but instead of a point one have to take this compact set.

Similarly, if $Z_g := \{v \in E_2' \,|\, v(t) = g(t, x(t), x'(t), y(t));\ (x, y) \in X, x \in Z\}$, then

$$\mu_2(G(X \cap Z) \times E_2)) \le \mu_2(Z_g) + \mu_2(\{-x \mid x \in Z\}) = \mu_2(Z_g).$$

Let $\mu_2(Z') = d$ and $\varepsilon > 0$. Denote by $z_1, \dots z_{n_\varepsilon}$ elements of a finite $(d+\varepsilon)$-net in Z' and consider the family of sets $\Gamma_i := \{g(\cdot, x(\,\cdot\,), z_i(\,\cdot\,), y(\,\cdot\,)) \mid (x, y) \in (Z \times E_1) \cap X\}$, where $i = 1, \dots, n_\varepsilon$.

If a sequence (v_k) is contained in Γ_i, then there are two sequences: (x_k) in Z and (y_k) in E_1 such that $v_k(t) = g(t, x_k(t), z_i(t), y_k(t))$ for $t \in [0, T]$ and $\|y_k\|_{E_1} \le M$. They are bounded in spaces E_2 and E_1, respectively, so contain subsequences convergent in E_2' and E_0 (comp. Theorem 5.6). But since the map $E_2' \times E_0 \ni (x, y) \mapsto g(\cdot, x(\,\cdot\,), z_i(\,\cdot\,), y(\,\cdot\,)) \in E_2'$ is continuous, the respective subsequence of $\{v_k\}$ converges in E_2', what implies that Γ_i is relatively compact and consequently so is $\Gamma := \bigcup_{i=1}^{n_\varepsilon} \Gamma_i$.

Assume that $v \in Z_g$, that is $v(t) = g(t, x(t), x'(t), y(t))$ for some $x \in Z$ and $y \in E_1$. Since then $x' \in Z'$, there is z_i such that $\|x' - z_i\|_{E_2'} \le d + \varepsilon$. Let $\overline{v} := g(\cdot, x(\,\cdot\,), z_i(\,\cdot\,), y(\,\cdot\,)) \in \Gamma_i$. Observe that

$$\|v - \overline{v}\|_{E_2'} = \int_0^T \|g(t, x(t), x'(t), y(t)) - g(t, x(t), z_i(t), y(t))\| dt \le \int_0^T \kappa_2 \|x'(t) - z_i(t)\| dt = \kappa_2 \|x' - z_i\|_{E_2'}.$$

It means that elements of the (relatively compact) set Γ form a $\kappa_2(d+\varepsilon)$-net of Z_g and

$$\mu_2(G(X \cap Z) \times E_2)) \le \mu_2(Z_g) \le \kappa_2 \mu_2(Z') = \kappa_2 \mu_2(L_2(Z)).$$

We have just proved that G is L_2-condensing.

Step 4. At last we have to check that if $(1-\mu)L_2(x) \in T(x) := G(\{x\} \times \Omega_F(x))$ for $\|x\|_{E_2} = R_1$, where $R_1 = e^{bt}(r_1 + r + 2\int_0^T a(s)ds)$, then $\mu > 0$ (comp. Remark 4.13). In order to show it, suppose that there is $\mu \le 0$ such that $(1-\mu)L_2(x) \in T(x)$ and prove at first that then $\min_{t \in I} \|x(t)\| > r_1$.

Observe that, if $\|x(t_1)\| \le r_1$ for some $t_1 \in I$, then for $\lambda := \frac{1}{1-\mu} \in (0,1)$

$$x'(t) = \lambda g(t, x(t), x'(t), y(t)) + (1-\lambda)x(t)$$

and, consequently,

$$x(t) = x(t_1) + \int_{t_1}^{t} (\lambda g(s, x(s), x'(s), y(s)) + (1-\lambda)x(s)) ds.$$

By (g_1), since we can assume that $b > 1$ (see Remark 5.3) we get for all $t \in I$

$$
\begin{aligned}
\|x(t)\| \le \|x(t_1)\| \quad & + \int_{t_1}^{t} (\lambda(a(s) + b\|x(s)\|) + (1-\lambda)\|x(s)\|)ds \\
& = \|x(t_1)\| + \int_{t_1}^{t} (\lambda a(s) + (1 + b\lambda - \lambda)\|x(s)\|)ds \\
& \le \|x(t_1)\| + \int_{t_1}^{t} (a(s) + b\|x(s)\|)ds.
\end{aligned}
\tag{16}
$$

Similarly, if $y \in \Omega_F(x)$, then $y'(t) = f(t, x(t)y(t), y'(t))$ and $0 \in l(x,y)$. By (l_2) and (f_1) (see also Remark 5.3) there is $t_x \in I$ such that $\|y(t_x)\| < r$ and, since $y(t) = y(t_x) + \int_{t_x}^{t} f(s, x(s), y(s), y'(s))ds$,

$$
\|y(t) \le \|y(t_x)\| + \int_{t_x}^{t} (a(s) + b\|y(s)\|)ds. \tag{17}
$$

Hence, by (16) and (17), for all $t \in I$,

$$
\begin{aligned}
\|x(t)\| + \|y(t)\| \le \|x(t_1)\| + \|y(t_x)\| + \int_{t_1}^{t} (a(s) + b\|x(s)\|)ds + \int_{t_x}^{t} (a(s) + b\|y(s)\|)ds \le \\
(r_1 + r) + 2\int_{0}^{T} a(s)ds + \int_{t_1}^{t} b\|x(s)\|ds + \int_{t_x}^{t} b\|y(s)\|ds.
\end{aligned}
$$

By Lemma 5.7 given below, for $A = r_1 + r + 2\int_0^T a(s)ds$, $p(t) = \|x(t)\|$, $q(t) = \|y(t)\|$,

$$
R_1 \le \max_{t \in I}\{\|x(t)\| + \|y(t)\|\} \le e^{bt}(r_1 + r + 2\int_0^T a(s)ds) - 1 = R_1 - 1.
$$

The contradiction establishes $\min_{t\in I} \|x(t)\| > r_1$.

Now, since $x(0) = x(T)$, $\|x(t)\| > r_1$ for all $t \in I$ and $\lambda \in (0,1)$, by (V_1) and (V_2) (see Remark 5.3 (iii)), we have

$$
\begin{aligned}
0 = V(x(T)) - V(x(0)) = \int_0^T \langle \nabla V(x(t)), x'(t) \rangle dt = \\
\lambda \int_0^T \langle \nabla V(x(t)), g(t, x(t), x'(t), y(t)) \rangle dt + (1-\lambda) \int_0^T \langle \nabla V(x(t)), x(t) \rangle dt > 0.
\end{aligned}
$$

This contradiction implies that $\mu > 0$ and then concludes the proof. □

LEMMA 5.7. – *Let $p, q \in C(I, [0,\infty))$, $A \in \mathbb{R}$, $b \geq 0$ and $t_1, t_2 \in I$. If for all $t \in I$*

$$p(t) + q(t) \leq A + \int_{t_1}^{t} bp(s)ds + \int_{t_2}^{t} bq(s)ds,$$

then, for all $t \in I$,

$$p(t) + q(t) \leq Ae^{bt} - 1.$$

PROOF. – Let $h(t) = A + 1 + \int_{t_1}^{t} bp(s)ds + \int_{t_2}^{t} bq(s)ds$. Then for $t \in I$, $p(t) + q(t) \leq h(t) - 1$ and

$$h'(t) = bp(t) + bq(t) = b(p(t) + q(t)) \leq bh(t).$$

Hence, for any $t \in I$,

$$\frac{d}{dt}\left[h(t) \cdot e^{-bt}\right] = h'(t)e^{-bt} - bh(t)e^{-bt} = e^{-bt}(h'(t) - bh(t)) \leq 0$$

and consequently

$$h(t) \cdot e^{-bt} \leq h(0) \cdot e^{-b \cdot 0} = h(0) = A + 1 + \int_{t_1}^{0} bp(s)ds + \int_{t_2}^{0} bq(s)ds \leq A.$$

Finally, $h(t) \leq Ae^{bt}$, what ends the proof. $\square$

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*Pervenuta in Redazione
il 30 aprile 2005*