Bollettino Unione Matematica Italiana

Giovanni Fiorito

On a recursive formula for the sequence of primes and applications to the twin prime problem

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **9-B** (2006), n.3, p. 667–680.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2006_8_9B_3_667_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2006.

Bollettino U. M. I.
(8) **9**-B (2006), 667-680

On a Recursive Formula for the Sequence of Primes and Applications to the Twin Prime Problem (*).

GIOVANNI FIORITO

Sunto. – *In questo lavoro presentiamo una formula ricorrente per la successione dei numeri primi $\{p_n\}$, che utilizziamo per trovare una condizione necessaria e sufficiente affinché un numero primo p_{n+1} sia uguale a p_n+2. Il precedente risultato viene utilizzato per calcolare la probabilità che p_{n+1} sia uguale a p_n+2. Inoltre proviamo che il limite per n tendente all'infinito della suddetta probabilità è zero. Infine, per ogni numero primo p_n costruiamo una successione i cui termini che appartengono all'intervallo $[p_n^2-2, p_{n+1}^2-2[$ sono i primi termini di due numeri primi gemelli. Questo risultato e alcune sue implicazioni rendono ulteriormente plausibile che l'insieme dei numeri primi gemelli sia infinito.*

Summary. – *In this paper we give a recursive formula for the sequence of primes $\{p_n\}$ and apply it to find a necessary and sufficient condition in order that a prime number p_{n+1} is equal to p_n+2. Applications of previous results are given to evaluate the probability that p_{n+1} is of the form p_n+2; moreover we prove that the limit of this probability is equal to zero as n goes to ∞. Finally, for every prime p_n we construct a sequence whose terms that are in the interval $[p_n^2-2, p_{n+1}^2-2[$ are the first terms of two twin primes. This result and some of its implications make furthermore plausible that the set of twin primes is infinite.*

Introduction.

It is well known there are many open problems about the sequence of primes (see [1], [3], [4], [5], [6], [7]); one of these is the twin prime problem, which consists in finding out if there exist infinitely many primes p such that $p+2$ is also prime (if the numbers p and $p+2$ are both primes, they are called twin primes). In the first part of this paper we give a recursive formula for the sequence of primes $\{p_n\}$, that we think be novel (for other recursive formulas see reference A 17 p. 37 of [3]). In the second part we apply it to find a necessary and sufficient condition in order that a prime number p_{n+1} is equal to

(*) Mathematics Subject Classification: 11A41, 11B25, 40A05, 40A20.
Key words and phrases: primes, recursive formula, twin primes, periodically monotone sequences.

$p_n + 2$. Moreover applications of previous results are given to evaluate the probability that p_{n+1} is of the form $p_n + 2$ and from this we deduce that the limit of this probability is equal to zero when n goes to ∞. Finally, in the third part, for every prime p_n we construct a sequence Σ_{p_n} whose terms that are in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$ are the first terms of two twin primes and moreover we prove a theorem on the mean number of the terms of the sequence Σ_{p_n} that are in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$, which makes furthermore plausible that the set of twin primes is infinite. In the sequel we put as usual

$$\mathbb{N} = \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}.$$

Moreover let us indicate by $\{T_n\}$ the sequence of the first terms of the twin primes, so we have for example

$$T_1 = 3, \quad T_2 = 5, \quad T_3 = 11, \quad T_4 = 17, \quad T_5 = 29.$$

Finally let us denote by $R\left(\frac{p_n}{p_r}\right)$ the remainder of the integral division of p_n by p_r for $r = 1, 2, 3, \dots, n$.

1. – A recursive formula for the sequence of primes.

THEOREM 1.1. – *The sequence of primes* $\{p_n\}$ *is given by the following recursive formula*:

$$\text{(1.1)} \qquad \begin{cases} p_1 = 2, \\ p_{n+1} = p_n + \min\left\{\mathbb{N} - \bigcup_{r=1}^{n}\left\{p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0\right\}\right\} \quad \forall n \geq 1. \end{cases}$$

PROOF. – The proof is elementary. Indeed, by observing that

$$p_n + p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \ (r = 1, 2, \dots, n)$$

represents all the multiple numbers of p_r, that are greater than p_n, it follows that the set

$$\mathbb{N} - \bigcup_{r=1}^{n}\left\{p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0\right\}$$

is not empty and the number

$$p = p_n + \min\left\{\mathbb{N} - \bigcup_{r=1}^{n}\left\{p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0\right\}\right\}$$

is not divisible for $p_1, p_2, \dots, p_n$. Now, let us observe that a prime number q

such that

$$p_n < q < p$$

cannot be exists. Indeed, on the contrary, it should be

$$q - p_n \in \mathbb{N} - \bigcup_{r=1}^{n} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\}$$

and then

$$q - p_n \geqslant \min \left\{ \mathbb{N} - \bigcup_{r=1}^{n} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\}.$$

On the other hand we also have

$$q - p_n < p - p_n = \min \left\{ \mathbb{N} - \bigcup_{r=1}^{n} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\},$$

and this is a contraddiction.

From the previous considerations it follows easily that p is the least prime greater than p_n, and so we have $p_{n+1} = p$. ■

REMARK 1.1. – For the Bertrand's postulate (see [4], theorem 418 p. 343) we have for $n \geqslant 2$

$$\min \left\{ \mathbb{N} - \bigcup_{r=1}^{n} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\} < p_n$$

and therefore we get

$$\min \left\{ \mathbb{N} - \bigcup_{r=1}^{n} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\} = \min \left\{ \mathbb{N} - \bigcup_{r=1}^{n-1} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\}.$$

REMARK 1.2. – For computing the number

$$d_n = \min \left\{ \mathbb{N} - \bigcup_{r=1}^{n-1} \left\{ p_r - R\left(\frac{p_n}{p_r}\right) + kp_r, \quad \forall k \in \mathbb{N}_0 \right\} \right\}$$

for $n \geqslant 2$, it is useful to observe that d_n is even and then to proceed as follows. If

$$p_r - R\left(\frac{p_n}{p_r}\right) \neq 2 \quad \textit{for} \quad r = 1, 2, 3, \ldots, n-1$$

then $d_n = 2$, otherwise $d_n \geqslant 4$. If $d_n \geqslant 4$ and if the equations

$$p_r - R\left(\frac{p_n}{p_r}\right) + kp_r = 4 \quad (r = 1, 2, 3, \dots, n-1)$$

have no integral solutions k then $d_n = 4$, otherwise $d_n \geqslant 6$, and so on.

REMARK 1.3. – For computing the number $R\left(\frac{p_n}{p_r}\right)$ for $r = 2, 3, \dots, n-1$ it is useful to observe that for $r = 2, 3, \dots, n-2$ and $n \geqslant 4$ the following recursive formula holds

$$R\left(\frac{p_n}{p_r}\right) = R\left(\frac{R\left(\frac{p_{n-1}}{p_r}\right) + d_{n-1}}{p_r}\right) \tag{1.2}$$

and for $r = n-1$ we have obviously $R\left(\frac{p_n}{p_r}\right) = d_{n-1}$.

REMARK 1.4. – Taking into account the Remark 1.3, the formula 1.1 can be employed to construct easily tables of primes.

2. – Some consequences of recursive formula.

The following theorems follow from theorem 1.1 and Remark 1.2.

THEOREM 2.1. – *We have for* $n \geqslant 3$

$$p_{n+1} = p_n + 2$$

(*and therefore* p_n *and* p_{n+1} *are twin primes*) *if and only if it results*

$$p_r - R\left(\frac{p_n}{p_r}\right) \neq 2$$

for $r \geqslant 1$ *and such that* $p_r \leqslant \sqrt{p_n + 2}$.

PROOF. – From theorem 1.1 and Remark 1.2 it follows that

$$p_{n+1} = p_n + 2$$

if and only if it results

$$p_r - R\left(\frac{p_n}{p_r}\right) \neq 2$$

for $r = 1, 2, 3, \ldots, n-1$. Now let us observe that the condition

$$p_r - R\left(\frac{p_n}{p_r}\right) \neq 2$$

is equivalent to say that $p_n + 2$ is not divisible by p_r. Indeed if

$$p_r - R\left(\frac{p_n}{p_r}\right) = 2$$

we have

$$p_n = qp_r + p_r - 2\,,$$

where q is the quotient of p_n by p_r. From the previous relation it follows

$$p_n + 2 = (q+1)\,p_r,$$

and therefore $p_n + 2$ is divisible by p_r. If

$$R\left(\frac{p_n}{p_r}\right) \neq p_r - 2$$

let us distinguish two cases. If

$$R\left(\frac{p_n}{p_r}\right) < p_r - 2$$

then it follows

$$p_n = qp_r + R\left(\frac{p_n}{p_r}\right)$$

and then

$$p_n + 2 = qp_r + R\left(\frac{p_n}{p_r}\right) + 2\,,$$

and this proves that $p_n + 2$ is not divisible by p_r. If

$$R\left(\frac{p_n}{p_r}\right) = p_r - 1$$

then it follows

$$p_n = qp_r + p_r - 1$$

and then

$$p_n + 2 = (q+1)p_r + 1$$

and this proves again that $p_n + 2$ is not divisible by p_r. From the previous observation the thesis follows easily. ■

If we observe that

$$R\left(\frac{p_n}{p_r}\right) \in \{1, 2, 3, \dots, p_r - 1\}$$

we can evaluate easily the probability that

$$p_{n+1} = p_n + 2$$

for large n. Indeed the following theorem holds.

THEOREM 2.2. – *The probability*

$$P(p_{n+1} = p_n + 2)$$

for large n is given by the formula

$$P(p_{n+1} = p_n + 2) = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{9}{10} \cdots \frac{p_r - 2}{p_r - 1},$$

where p_r is the greatest prime such that $p_r \leqslant \sqrt{p_n + 2}$.

From previous theorem we get also the following result.

THEOREM 2.3. – *The formula*

$$\lim_{n \to \infty} P(p_{n+1} = p_n + 2) = 0$$

holds.

PROOF. – From the theorem 2.2 we get

$$P(p_{n+1} = p_n + 2) = \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{6}\right) \cdots \left(1 - \frac{1}{p_r - 1}\right),$$

where p_r is the greatest prime such that $p_r \leqslant \sqrt{p_n + 2}$. Therefore

$$\lim_{n \to \infty} P(p_{n+1} = p_n + 2) = \prod_{n=2}^{\infty} \left(1 - \frac{1}{p_n - 1}\right).$$

But the infinite product is divergent to zero because the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

is divergent (to ∞) (see [4], p. 16 Th. 19), and the thesis follows. ■

Let us observe that the previous theorems are a theoretic explanation of the fact that twin primes become rarer and rarer as n becomes very large. Moreover these results match (but they are of different type) with other theoretic explanations (see for instance [4] p. 412 and [6] p. 133 ex. 9.1.15 and p. 143 ex. 9.3.12).

3. – The twin prime problem.

By using the theorem 2.1, for every prime $p_n \geqslant 3$ it is possible to construct a sequence Σ_{p_n} of natural numbers that contains among its terms all terms of the sequence $\{T_m\}$ for $T_m > p_n$, moreover all the terms of the sequence Σ_{p_n} that are in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$ are terms of the sequence $\{T_n\}$. Let us proceed to construct Σ_3.

If p_n ($p_n \geqslant 5$) is such that $p_{n+1} = p_n + 2$, then necessarely it is a term of the sequence

$$\{5 + 6k\}_{k \in \mathbb{N}_0}. \tag{3.1}$$

Indeed this is obvious for $p_n = 5$; if $p_n > 5$, for the theorem 2.1, we have

$$R\left(\frac{p_n}{3}\right) = 2\,,$$

and being also $R\left(\frac{5}{3}\right) = 2$, it follows that $p_n - 5$ is multiple of 3; but $p_n - 5$ must be even and therefore we have

$$p_n - 5 = 6k$$

for some $k \in \mathbb{N}$. Therefore Σ_3 is the sequence

$$\{5 + 6k\}_{k \in \mathbb{N}_0},$$

which is an arithmetic progression with difference 6. Let us observe that all the terms of Σ_3 that are in the interval $[3^2 - 2, 5^2 - 2[$ are terms of $\{T_n\}$, namely 11 and 17.

Let us also observe that from 3.1 it follows that the difference

$$T_n - T_m \quad \forall n, m \ (n > m)$$

is multiple of 6.

Now, for finding Σ_5, let us consider the sequence $\{5 + 6k\}_{k \in \mathbb{N}_0}$ and search k such that

$$R\left(\frac{5 + 6k}{5}\right) \neq 0 \quad \text{and} \quad R\left(\frac{5 + 6k}{5}\right) \neq 3\,. \tag{3.2}$$

Computing

$$R\left(\frac{5+6k}{5}\right)$$

for $k = 0, 1, 2, 3, 4$, we obtain respectively the numbers $0, 1, 2, 3, 4$; therefore, by observing that the sequence

$$\left\{R\left(\frac{5+6k}{5}\right)\right\}$$

is periodic with period 5, it follows that the condition 3.2 is verified if

$$k = 1 + 5h \quad \text{or} \quad k = 2 + 5h \ \text{or} \quad k = 4 + 5h \quad (h \in \mathbb{N}_0).$$

In this way we get the 3 sequences

$$\{11 + 30k\}_{k \in \mathbb{N}_0}, \quad \{17 + 30k\}_{k \in \mathbb{N}_0}, \quad \{29 + 30k\}_{k \in \mathbb{N}_0} \tag{3.3}$$

(which are arithmetic progressions with difference 30). So we obtain that Σ_5 is the periodically monotone sequence ([1]) whose principal terms are

$$11, \quad 17, \quad 29,$$

whose period is 3 and whose monotony constant is 30. Let us observe that all the terms of Σ_5 that are in the interval $[5^2 - 2, 7^2 - 2[$ are terms of $\{T_n\}$, namely 29 and 41.

Let us also observe that from 3.3 it follows that all terms of the sequence $\{T_n\}$ for $T_n \geqslant 11$ have as unity digit always one of the numbers $1, 7, 9$.

Now, for finding Σ_7, let us search k such that

$$\begin{aligned}
&R\left(\frac{11+30k}{7}\right) \neq 0, \quad R\left(\frac{11+30k}{7}\right) \neq 5.\\
&R\left(\frac{17+30k}{7}\right) \neq 0, \quad R\left(\frac{17+30k}{7}\right) \neq 5.\\
&R\left(\frac{29+30k}{7}\right) \neq 0, \quad R\left(\frac{29+30k}{7}\right) \neq 5.
\end{aligned} \tag{3.4}$$

([1]) A sequence $\{x_n\}$ in $\mathbb{R}$ is called periodically monotone if there exist a natural number q and a real number k such that

$$x_{n+q} = x_n + k \qquad \forall n \in \mathbb{N} \tag{*}$$

The lowest natural number q for which (*) holds is called period. the constant k is called monotony constant. The terms $x_1, x_2, \dots, x_q$ are called principal terms of $\{x_n\}$. The periodically monotone sequences generalize the periodic sequences and the arithmetic progressions (see [2]).

Computing all the remainders for $k = 0, 1, 2, 3, 4, 5, 6$ and taking into account that the sequences

$$\left\{R\left(\frac{11+30k}{7}\right)\right\}, \quad \left\{R\left(\frac{17+30k}{7}\right)\right\}, \quad \left\{R\left(\frac{29+30k}{7}\right)\right\}$$

are periodic with period 7, we obtain the following 15 sequences (which are arithmetic progressions with difference $2\cdot 3\cdot 5\cdot 7 = 210$ and $k \in \mathbb{N}_0$):

$$\{11+210k\}, \quad \{17+210k\}, \quad \{29+210k\},$$

$$\{41+210k\}, \quad \{59+210k\}, \quad \{71+210k\},$$

$$\{101+210k\}, \quad \{107+210k\}, \quad \{137+210k\},$$

$$\{149+210k\}, \quad \{167+210k\}, \quad \{179+210k\},$$

$$\{191+210k\}, \quad \{197+210k\}, \quad \{209+210k\}.$$

So we obtain that Σ_7 is the periodically monotone sequence whose principal terms are

$$11, 17, 29, 41, 59, 71, 101, 107, 137, 149, 167, 179, 191, 197, 209,$$

whose period is 15 and whose monotony constant is $210 = 2\cdot 3\cdot 5\cdot 7$.

Let us observe that all the terms of Σ_7 that are in the interval $[7^2-2, 11^2-2[$ are terms of $\{T_n\}$, namely 59, 71, 101 and 107.

The reasoning can be iterated so that the following theorem holds.

THEOREM 3.1. – *For every prime number p_n $(n \geqslant 2)$ there exists a periodically monotone sequence Σ_{p_n} with period*

$$1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 15\cdot \ldots \cdot (p_n - 2),$$

whose monotony constant is

$$2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n$$

and such that every term p of Σ_{p_n} satisfies the conditions

$$R\left(\frac{p}{p_r}\right) \neq 0 \quad \text{and} \quad R\left(\frac{p}{p_r}\right) \neq p_r - 2$$

$\forall\ r = 1, 2, 3, 4, 5, \ldots, n.$

REMARK 3.1. – The principal terms of the sequence Σ_{p_n} are obtained taking the terms of the sequence $\Sigma_{p_{n-1}}$ that are in the interval

$$]p_n, 2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n[$$

and deleting those terms that are of the form

$$p_n \cdot p \quad \text{or} \quad p_n \cdot p - 2\,,$$

where p is prime greater than or equal to p_n or p is composite with prime factors greater than or equal to p_n. Consequentely the principal terms of the sequence Σ_{p_n} are distributed in the interval $]p_n, 2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n[$.

The following definitions will be used in the sequel.

DEFINITION 3.1. – The number

$$\varrho_n = \frac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n}{1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 15\cdot \ldots \cdot (p_n - 2)}\,,$$

that is the quotient of the monotony constant by the period of the sequence Σ_{p_n}, is called mean distance between two consecutive terms of Σ_{p_n}.

DEFINITION 3.2. – Let (a, b) an interval $(a \geqslant p_n)$. The number

$$\frac{b-a}{\varrho_n}$$

is called mean number of the terms of the sequence Σ_{p_n} that are in the interval (a, b).

We have also the following theorems

THEOREM 3.2. – *For every prime number p_n $(n \geqslant 2)$ all the terms of the sequence Σ_{p_n} that are in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$ are terms of the sequence $\{T_n\}$. Moreover if a term of the sequence $\{T_n\}$ is in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$, then it is a term of the sequence Σ_{p_n}.*

PROOF. – Let p be an arbitrary term of the sequence Σ_{p_n} such that

$$p \in [p_n^2 - 2, p_{n+1}^2 - 2[\,.$$

For the theorem 3.1 p is not divisible by $p_1, p_2, p_3, \ldots, p_n$ and then p is prime. Moreover, again for the theorem 3.1 and for the observation contained in the proof of theorem 2.1, also $p+2$ is not divisible by $p_1, p_2, p_3, \ldots, p_n$ and therefore $p+2$ is prime too. The second part of the statement is obvious. ■

THEOREM 3.3. – *The set $\{T_n, \forall n \in \mathbb{N}\}$ is infinite (and therefore there are infinitely many twin primes) if and only if there exists a subsequence*

$$\{[p_{n_k}^2 - 2, p_{n_k+1}^2 - 2[\}$$

such that every interval $[p_{n_k}^2 - 2, p_{n_k+1}^2 - 2[$ contains at least a term of the sequence $\Sigma_{p_{n_k}}$.

PROOF. – Let the set $\{T_n, \forall n \in \mathbb{N}\}$ be infinite. Then we define p_{n_1} a prime such that

$$11 \in [p_{n_1}^2 - 2,\ p_{n_1+1}^2 - 2[\,.$$

For the theorem 3.2 we have that 11 is a term of the sequence $\Sigma_{p_{n_1}}$. Now we take $T_{k_1} > p_{n_1+1}^2 - 2$ and denote by p_{n_2} $(n_2 > n_1)$ a prime such that

$$T_{k_1} \in [p_{n_2}^2 - 2,\ p_{n_2+1}^2 - 2[\,;$$

For the theorem 3.2 we have that T_{k_1} is a term of the sequence $\Sigma_{p_{n_2}}$. Similarly we take $T_{k_2} > p_{n_2+1}^2 - 2$ and denote by p_{n_3} $(n_3 > n_2)$ a prime such that

$$T_{k_2} \in [p_{n_3}^2 - 2,\ p_{n_3+1}^2 - 2[\,;$$

Again for the theorem 3.2 we have that T_{k_2} is a term of the sequence $\Sigma_{p_{n_3}}$. Because the reasoning can be iterated we have proved the «only if» part of the statement. The «if» part of the statement is obvious. ■

THEOREM 3.4. – *The mean number σ_{p_n} of the terms of the sequence Σ_{p_n} that are in the interval $[p_n^2 - 2,\ p_{n+1}^2 - 2[$ satisfies the condition*

$$\sigma_{p_n} \geqslant 19 \qquad \forall\, p_n \geqslant 661\,.$$

PROOF. – For the definitions 3.1 and 3.2 we have

$$\sigma_{p_n} = \frac{p_{n+1}^2 - p_n^2}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n}{1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 15\cdot \ldots \cdot (p_n - 2)}}\,.$$

Now it results

$$\sigma_{p_n} \geqslant \frac{4p_n}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n}{1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 15\cdot \ldots \cdot (p_n - 2)}}\,.$$

But, putting

$$S_n = \frac{4p_n}{\dfrac{2\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot p_n}{1\cdot 3\cdot 5\cdot 9\cdot 11\cdot 15\cdot \ldots \cdot (p_n - 2)}}\,,$$

we have

$$\frac{S_{n+1}}{S_n} = \frac{p_{n+1} - 2}{p_n} \geqslant 1\,,$$

hence the sequence $\{S_n\}$ is not-decreasing. Because it results $S_n \geqslant 19$ for $p_n = 661$, the thesis follows. ■

The previous theorems make plausible that the following proposition is true (but we cannot prove it) and therefore the set $\{T_n, \forall n \in \mathbb{N}\}$ is infinite.

PROPOSITION 3.1. – *The number of the terms of the sequence $\{T_n\}$ that are less than $p_{n+1}^2 - 2$ is approximatively given by*

$$\tau_n = 2 + \sum_{k=2}^{n} \frac{p_{k+1}^2 - p_k^2}{\dfrac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_k}{1 \cdot 3 \cdot 5 \cdot 9 \cdot 11 \cdot 15 \cdot \ldots \cdot (p_k - 2)}}$$

and we have

$$\lim_{n \to \infty} \tau_n = +\infty .$$

In the following table are listed the values of σ_{p_n}, ϱ_n and μ_n in some intervals $[p_n^2 - 2, p_{n+1}^2 - 2[$, where σ_{p_n}, ϱ_n have been defined previously and μ_n denotes the number of the terms of the sequence $\{T_n\}$ that are in the interval $[p_n^2 - 2, p_{n+1}^2 - 2[$.

$[p_n^2 - 2,\quad p_{n+1}^2 - 2[$	σ_{p_n}	ϱ_n	μ_n
$[3^2 - 2,\quad 5^2 - 2[$	2.6	6	2
$[5^2 - 2,\quad 7^2 - 2[$	2.4	10	2
$[7^2 - 2,\quad 11^2 - 2[$	5.1	14	4
$[11^2 - 2,\quad 13^2 - 2[$	2.8	17.1	2
$[13^2 - 2,\quad 17^2 - 2[$	5.2	20.2	7
$[17^2 - 2,\quad 19^2 - 2[$	3.1	22.9	2
$[19^2 - 2,\quad 23^2 - 2[$	6.6	25.6	4
$[23^2 - 2,\quad 29^2 - 2[$	11	28.0	7
$[29^2 - 2,\quad 31^2 - 2[$	4	30	2
$[31^2 - 2,\quad 37^2 - 2[$	12.7	32.2	10
$[37^2 - 2,\quad 41^2 - 2[$	9.2	34	7
$[41^2 - 2,\quad 43^2 - 2[$	4.7	35.8	3
$[43^2 - 2,\quad 47^2 - 2[$	9.7	37.5	11

$[47^2-2,$	$53^2-2[$	15.3	39.2	12
$[53^2-2,$	$59^2-2[$	16.5	40.7	11
$[59^2-2,$	$61^2-2[$	5.7	42	5
$[61^2-2,$	$67^2-2[$	17.6	43.6	19
$[89^2-2,$	$97^2-2[$	29	51	21
$[151^2-2,$	$157^2-2[$	21.9	84.2	20
$[283^2-2,$	$293^2-2[$	54	106.5	68
$[421^2-2,$	$431^2-2[$	71	119.8	90
$[661^2-2,$	$673^2-2[$	115.5	138.6	108
$[953^2-2,$	$967^2-2[$	175.7	153	201
$[1361^2-2,$	$1367^2-2[$	97	168.8	111
$[1709^2-2,$	$1721^2-2[$	228.7	179.9	239
$[2027^2-2,$	$2029^2-2[$	43	187.8	42
$[2411^2-2,$	$2417^2-2[$	147.3	196.6	156
$[2903^2-2,$	$2909^2-2[$	169.3	206	175
$[3203^2-2,$	$3209^2-2[$	182.8	210.5	217
$[3449^2-2,$	$3457^2-2[$	258	214.2	279
$[3659^2-2,$	$3671^2-2[$	403.9	217.7	438
$[3803^2-2,$	$3821^2-2[$	624.2	219.8	667
$[4093^2-2,$	$4099^2-2[$	219.6	223.8	212

REFERENCES

[1] R. CRANDALL - C. POMERANCE, *Prime Numbers A Computational Perspective*, Springer-Verlag New York (2001).
[2] G. FIORITO, *On Properties of Periodically Monotone Sequences*, Applied Mathematics and Computation, **72** (1995), 259-275.
[3] R. K. GUY, *Unsolved problems in Number Theory*, Springer-Verlag New York (1994).

[4] G. HARDY - E. WRIGHT, *An Introduction to the Theory of Numbers*, Clarendon Press Oxford (1954).
[5] K. IRELAND - M. ROSEN, *A Classical Introduction to Modern Number Theory*, Springer-Verlag New York (1981).
[6] M. R. MURTY, *Problems in Analytic Number Theory*, Springer-Verlag (1999).
[7] M. B. NATHANSON, *Elementary Methods in Number Theory*, Springer-Verlag (1999).

Dipartimento di Matematica e Informatica Università di Catania
viale Andrea Doria 6, Italy; e-mail: fiorito@dmi.unict.it

Pervenuta in Redazione
l'8 gennaio 2004