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Asymptotic Stability of a Semigroup Generated by Randomly Connected Poisson Driven Differential Equations.

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Sunto. – *Si considera l'equazione differenziale stocastica del tipo*

$$dX(t) = a(X(t), \xi(t))dt + \int_{\Theta} b(X(t), \theta)\mathcal{N}_p(dt, d\theta) \qquad \text{per} \quad t \geq 0 \tag{1}$$

con condizione iniziale $X(0) = x_0$. Diamo condizioni sufficienti per la stabilità delle soluzioni che generano il semigruppo degli operatori di Markov.

Summary. – *We consider the stochastic differential equation*

$$dX(t) = a(X(t), \xi(t))dt + \int_{\Theta} b(X(t), \theta)\mathcal{N}_p(dt, d\theta) \qquad \text{for} \quad t \geq 0 \tag{1}$$

with the initial condition $X(0) = x_0$. We give sufficient conditions for the asymptotic stability of the semigroup $\{P^t\}_{t \geq 0}$ generated by the stochastic differential equation (1).

0. – Introduction.

We will consider the stochastic differential equation

$$dX(t) = a(X(t), \xi(t))dt + \int_{\Theta} b(X(t), \theta)\mathcal{N}_p(dt, d\theta) \qquad \text{for} \quad t \geq 0 \tag{0.1}$$

with the initial condition

$$X(0) = x_0, \tag{0.2}$$

where $\{X(t)\}_{t \geq 0}$ is a stochastic process with values in the d- dimensional real space $\mathbb{R}^d$ and $\{\xi(t)\}_{t \geq 0}$ is a stochastic process with values in $I = \{1, \ldots, N\}$, describes random switching at random moments t_n. The precise assumptions concerning coefficients $a : \mathbb{R}^d \times I \to \mathbb{R}^d$ and $b : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ and the Poisson random measure $\mathcal{N}_p$ will be formulated in Section 2.

In the case when the coefficient $a : \mathbb{R}^d \times I \to \mathbb{R}^d$ does not depend on the second variable,that is, the differential equations do not randomly switch at jump random moments, we obtain stochastic equation considered by J. Traple [17].

The main aim of our paper is to give sufficient conditions for the asymptotic stability of a semigroup generated by (0.1), (0.2).

This is a problem of great importance in the theory of stochastic processes. It has been considered by many authors (see [1], [2], [3], [16] and references therein). The model under consideration is a particular case for so called piecewise- deterministic Markov processes introduced by M. Davis [2].

However, the technical assumptions in [3] and [16] are difficult to check in our case since they need the limit of the sum of iterations of a kernel of a transition operator and the realization some ergodic properties on compact sets.

It is worth pointing out that in our approach we used purely analitical methods- lower bound technique developed by A. Lasota and J. Yorke [15]. They introduced the class of so called concentrating Markov operators and showed that every operator from this class admits an invariant measure. Further, if a concentrating Markov operator do not increase some distance between two measures, this operator must be asymptotically stable. We develop the method, which allows to verify these assumptions.

Our theorems imply the weak convergence, but the stationary measure may be singular and in this case the convergence can not be strong.

The result of this paper is related to our former papers (see [8], [9]).Recall that in [9] the values of the process $\{X(t)\}_{t\geq 0}$ have been considered only at the «switching» point t_n. In such special case it was proved that the corresponding stochastic process generates a semigroup of Markov operators and that this semigroup has an invariant probability measure.

It was shown in [8] that, under some technical assumptions, there is one-to-one mapping between the invariant probability measures for $\{(X(t), \xi(t))\}_{t\geq 0}$ and the invariant probability measures of the Markov chain given by postjump locations $\{(x_n, \xi(t_n))\}_{n\in\mathbb{N}}$. However this result is not satisfactory for the examination of asymptotic stability of a semigroup generated by randomly connected Poisson driven differential equations. A Markov operator generated by the chain $\{(x_n, \xi(t_n))\}_{n\in\mathbb{N}}$ may be asymptotic stability but a semigroup generated by the Markov processes $\{(X(t), \xi(t))\}_{t\geq 0}$ may not. It is important to stress that the semigroup with continuous time generated by the system of randomly connected Poisson driven differential equations can not be explicitly calculated.

In [11] K. Horbacz, J. Myjak and T. Szarek considered a continuous-time case, but they did not choose jumps ($b = 0$ in (0.1)).

Finally note that the system considered in this paper have been used to description some physical phenomena as wave propagation in random media and turbulence. For instance in [6, 12] there are many examples of problems which generate such equations. Recently equations of type (0.1) have appeared in a model of the growth of a size - structured populations of cells [4].

The organization of the paper is as follows: Section 1 contains notation and definitions. Moreover, for the convenience of the reader, we recall here some known facts concerning asymptotic stability of Markov operators which are crucial in our considerations. In Section 2 we reformulate above problem in more convenient form and we state all necessary hypotheses. Section 3 contains rather technical observations used in proving main results. However, some of them (Proposision 3.1 and Lemma 3.2) seems to be interested in themselves. The main results are contained in last Section.

1. – Preliminaries.

Let (Y, ρ) be a locally compact metric space. Throughout this paper we assume that $B(x, r)$ stands for the closed ball in Y with a center at x and radius r.

By $\mathcal{B}(Y)$ we denote the σ–algebra of Borel subsets of Y and $\mathcal{M}(Y)$ the family of all finite Borel measures (nonnegative, σ–additive) on Y. By $\mathcal{M}_1(Y)$ we denote the subset of $\mathcal{M}(Y)$ such that $\mu(Y) = 1$ for $\mu \in \mathcal{M}_1(Y)$. The elements of $\mathcal{M}_1(Y)$ will be called distributions. Further

$$\mathcal{M}_{sig}(Y) = \{\mu_1 - \mu_2 : \ \mu_1, \mu_2 \in \mathcal{M}(Y)\}.$$

is the space of finite signed measures.

As usually by $B(Y)$ we denote the space of all bounded Borel measurable functions $f : Y \to \mathbb{R}$ and by $C(Y)$ the subspace of all bounded continuous functions with the supremum norm $\| \cdot \|_C$. By $C_0(Y)$ we denote the subspace of $C(Y)$ which contains functions with compact support.

For an $f \in B(Y)$ and $\mu \in \mathcal{M}_{sig}(Y)$ we write

$$\langle f, \mu \rangle = \int_Y f(x)\mu(dx).$$

We say that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$, $\mu_n \in \mathcal{M}_1(Y)$, converges weakly to a measure $\mu \in \mathcal{M}_1(Y)$ if

$$\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle \qquad \text{for} \quad f \in C(Y).$$

In the space $\mathcal{M}_{sig}(Y)$ we introduce the Fortet–Mourier norm [5] , [15] by setting

$$\|\mu\|_\rho = \sup\{\langle f, \mu \rangle : \ f \in \mathcal{F}_\rho\}$$

where

$$\mathcal{F}_\rho = \{f \in C(Y) : \ |f(x)| \le 1 \quad \text{and} \quad |f(x) - f(y)| \le \rho(x, y) \qquad \text{for} \quad x, y \in Y\}.$$

The space $\mathcal{M}_1(Y)$ with the distance $\|\mu_1 - \mu_2\|_\rho$ is a complete metric space and the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\|_\rho = 0 \qquad \text{for} \quad \mu_n, \mu \in \mathcal{M}_1(Y)$$

is equivalent to the condition

$$\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle \qquad \text{for} \quad f \in C(Y).$$

A linear mapping $P : \mathcal{M}_{sig}(Y) \to \mathcal{M}_{sig}(Y)$ is called a Markov operator if $P(\mathcal{M}_1(Y)) \subset \mathcal{M}_1(Y)$. Thus, for every distribution μ the measure $P\mu$ is also a distribution.

A Markov operator P is called a Feller operator, if there is an operator $U : B(Y) \to B(Y)$ (dual to P), such that

$$\langle Uf, \mu \rangle = \langle f, P\mu \rangle \qquad \text{for} \quad f \in B(Y), \quad \mu \in \mathcal{M}_{sig}(Y) \tag{1.1}$$

and

$$Uf \in C(Y) \qquad \text{for} \quad f \in C(Y). \tag{1.2}$$

Setting $\mu = \delta_x$ in (1.1) we obtain

$$Uf(x) = \langle f, P\delta_x \rangle \qquad \text{for} \quad f \in B(Y), \quad x \in Y \tag{1.3}$$

where $\delta_x \in \mathcal{M}_1(Y)$ is a point (Dirac) measure supported at x.

A Markov operator $P : \mathcal{M}_{sig}(Y) \to \mathcal{M}_{sig}(Y)$ is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_\rho \le \|\mu_1 - \mu_2\|_\rho \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1(Y). \tag{1.4}$$

A Markov operator is called *asymptotically stable* if there exists a stationary distribution μ_* such that $P\mu_* = \mu_*$ and

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_\rho = 0 \qquad \text{for} \quad \mu \in \mathcal{M}_1(Y). \tag{1.5}$$

We say that a Markov operator P has the *Prohorov property* if for every $\varepsilon > 0$ there is a compact set $F \subset Y$ such that

$$\liminf_{n\to\infty} P^n\mu(F) \ge 1 - \varepsilon \qquad \text{for} \quad \mu \in \mathcal{M}_1(Y) \tag{1.6}$$

A family of Markov operators $\{P^t\}_{t\ge0}$ is called a semigroup if $P^{t+s} = P^t \circ P^s$ for $t, s \in \mathbb{R}_+$ and $P^0 = I$ is the identify operator on $\mathcal{M}_1(Y)$. If operators $P^t, t \ge 0$, are Fellerean , we say that $\{P^t\}_{t\ge0}$ is a Feller semigroup. $\{T^t\}_{t\ge0}$ denotes the semigroup dual to $\{P^t\}_{t\ge0}$, i.e.

$$\langle T^t f, \mu \rangle = \langle f, P^t\mu \rangle \qquad \text{for} \quad f \in C(Y), \mu \in \mathcal{M}_1(Y).$$

The Markov semigroup $\{P^t\}_{t\ge0}$ is called *nonexpansive* if every Markov operator $P^t, t \ge 0$ is nonexpansive.

A measure $\mu \in \mathcal{M}_{fin}(Y)$ is called *stationary* for the Markov semigroup

$\{P^t\}_{t\geq 0}$ if $P^t\mu = \mu$ for $t \geq 0$. The Markov semigroup $\{P^t\}_{t\geq 0}$ is called *asymptotically stable* if there exists a stationary distribution μ_* such that

$$\lim_{t\to\infty} \|P^t\mu - \mu_*\|_\rho = 0 \qquad \mu \in \mathcal{M}_1(Y).$$

A continuous function $V : Y \to [0, +\infty)$ is called a Lyapunov function if

$$\lim_{\rho(x,x_0)\to+\infty} V(x) = +\infty \quad \text{for some} \quad x_0 \in Y.$$

PROPOSITION 1.1. – ([15]) *let P be a Feller operator and U its dual. Assume that there is a Lyapunow function V such that*

$$UV(x) \leq \beta_1 V(x) + \beta_2 \quad \text{for} \quad x \in Y$$

where β_1, β_2 are nonegative constants and $\beta_1 < 1$. Then P admits an invariant distribution and satisfies the Prohorov condition.

PROPOSITION 1.2. – ([15]) *Let P be a nonexpansive Markov operator. If P satisfies a lower bound condition, that is, for every $\varepsilon > 0$ there is a number $\Delta > 0$ such that for every $\mu_1, \mu_2 \in \mathcal{M}_1(Y)$ there exists a Borel measurable set A with $diam_\rho(A) \leq \varepsilon$ and an integer n_0 for which*

$$P^{n_0}\mu_i(A) \geq \Delta \quad \text{for} \quad i = 1, 2. \tag{1.7}$$

Then P satisfies

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_\rho = 0 \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1(Y).$$

PROPOSITION 1.3. – ([15]) *If Y is a locally compact metric space, then the following three properties of an Markov operator P*

(i) P is nonexpansive,
(ii) P has the Prohorov property,
(iii) P satisfies a lower bound condition
imply the asymptotic stability of P.

Finally, following [15] we introduce the class Φ of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is continuous and $\varphi(0) = 0$;

(ii) φ is nondecreasing and concave, i.e.,

$$\sum_{k=1}^{n} a_k\varphi(t_k) \leq \varphi\Big(\sum_{k=1}^{n} a_k t_k\Big), \text{ where } a_k \geq 0, \ \sum_{k=1}^{n} a_k = 1 \tag{1.8}$$

(iii) $\varphi(x) > 0, x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

By Φ_0 we denote the family of all functions satisfying conditions (i) and (ii). Observe that for every $\varphi \in \Phi$ the function $\rho_\varphi = \varphi \circ \rho$ is again a metric on Y. Moreover ρ_φ is equivalent to ρ. For notational convenience we write $\mathcal{F}_\varphi$ and $\|\cdot\|_\varphi$ in the place of $\mathcal{F}_{\rho_\varphi}$ and $\|\cdot\|_{\rho_\varphi}$, respectively.

PROPOSITION 1.4. – ([15]) *Assume that a function $\psi \in \Phi_0$ satisfies the Dini condition*

$$\int_0^\varepsilon \frac{\psi(t)}{t} dt < \infty \quad \text{for some} \quad \varepsilon > 0. \tag{1.9}$$

Let $c \in [0,1)$. Then the equation

$$\psi(t) + \varphi(ct) \le \varphi(t)$$

admits a solution in the class Φ.

2. Randomly connected Poison driven differential equations.

We consider the differential equation with Poisson type perturbations

$$dX(t) = a(X(t), \xi(t))dt + \int_\Theta b(X(t), \theta)\mathcal{N}_p(dt, d\theta) \qquad \text{for} \quad t \ge 0 \tag{2.1}$$

with the initial condition

$$X(0) = x_0. \tag{2.2}$$

In our study of solutions of (2.1), (2.2) we make the following assumptions:

(i) The coefficient $a : \mathbb{R}^d \times I \to \mathbb{R}^d$ is Lipschitzian with respect to the variable in $\mathbb{R}^d$

$$\|a(x,i) - a(y,i)\| \le L_a \|x - y\| \quad \text{for} \quad (x,i),(y,i) \in \mathbb{R}^d \times I$$

where $\|\cdot\|$ denotes the Euclidesian norm in $\mathbb{R}^d$.

(ii) There are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of random variables $\{t_n\}_{n\in\mathbb{N}}$. The variables $s_n = t_n - t_{n-1} (t_0 = 0)$ are nonnegative, independent and equally distributed with the density distribution function $g(t) = \lambda e^{-\lambda t}$ for $t \ge 0$.

(iii) Let $(\Theta, \mathcal{G}, \upsilon)$ be a compact measure space with $\upsilon(\Theta) = 1$. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of random elements with values in Θ. The elements η_n are independent, equally distributed with the distribution υ. The sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{\eta_n\}_{n\in\mathbb{N}}$ are independent.

(iv) Moreover, suppose we are given a probability matrix $[p_{ij}(x)]_{i,j=1}^{N}$ such that

$$p_{ij}(x) \geq 0, \quad \sum_{j=1}^{N} p_{ij}(x) = 1 \qquad \text{for} \quad x \in \mathbb{R}^d \text{ and } \ i,j \in I$$

and a probability vector $(p_1(x), \dots, p_N(x))$ such that

$$p_i(x) \geq 0, \quad \sum_{j=1}^{N} p_j(x) = 1 \quad \text{for} \quad x \in \mathbb{R}^d \text{ and } \ i \in I.$$

For every $i \in I$, denote by $v_i(t) = \Pi_i(t,x)$ the solution of the unperturbed Cauchy problem

$$v_i'(t) = a(v_i(t), i) \qquad \text{and} \quad v_i(0) = x, \qquad x \in \mathbb{R}^d. \tag{2.4}$$

Set

$$q(x,\theta) = x + b(x,\theta) \qquad \text{for} \quad x \in \mathbb{R}^d, \quad \theta \in \Theta.$$

We consider a sequence of random variables $\{x_n\}_{n\in\mathbb{N}}$, $x_n : \Omega \to \mathbb{R}^d$ and a stochastic process $\{\xi(t)\}_{t\geq 0}$, $\xi(t) : \Omega \to I$. We assume that they are related by

$$\xi(t) = \xi(t_n) \qquad \text{for} \quad t_n \leq t < t_{n+1} \quad \text{for} \quad n = 0, 1, \dots, \tag{2.5}$$

$$x_n = q(\Pi_{\xi(t_{n-1})}(t_n - t_{n-1}, x_{n-1}), \eta_n)$$

and

$$\mathbb{P}\{\xi(0) = k | x_0 = x\} = p_k(x),$$

$$\mathbb{P}\{\xi(t_n) = s | x_n = y \quad \text{and} \quad \xi(t_{n-1}) = i\} = p_{is}(y),$$

for $n = 1, \dots$.

(v) Let a function $q : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ be a continuous function such that $q(x,\cdot) \in L^1(\upsilon)$ and the inequality

$$\|q(x,\cdot) - q(y,\cdot)\|_{L^1(\upsilon)} \leq L_q \|x - y\| \qquad \text{for all} \quad x, y \in \mathbb{R}^d \tag{2.6}$$

holds for some constant $L_q \geq 0$ and

$$c = \int_{\Theta} \|q(0,\theta)\| \upsilon(d\theta) < \infty. \tag{2.7}$$

Conditions (iii) and (iv) imply that for every measurable set $Z \subset (0, +\infty) \times \Theta$ the variable

$$\mathcal{N}_p(Z) = \#\{i : (t_i, \eta_i) \in Z\}$$

is Poisson distributed. It is called the Poisson random counting measure.

By a solution of (2.1), (2.2) we mean a process $\{X(t)\}_{t\geq 0}$ with values in $\mathbb{R}^d$ such that with probability one the following two conditions are satisfied:

a) The sample paths is a right hand continuous function such that for $t > 0$ the limit $X(t-) = \lim_{s\to t^-} X(s)$ exists and

b) $X(t) = x_0 + \int_0^t a(X(s), \xi(s))ds + \int_0^t \int_\Theta b(X(s-), \theta)\mathcal{N}_p(ds, d\theta) \quad \text{for } t \geq 0.$

Observe that the solution $X(t)$ on each interval $[t_k, t_{k+1})$, $k = 0, 1, \ldots$ satisfies one of the ordinary differential equations

$$v_s'(t) = a(v_s(t), s) \qquad t \in \mathbb{R}_+, s \in I$$

and the initial condition

$$v_s(t_k) = x_k.$$

We give an explicite formula for the solution of (2.1), (2.2). Hence the stochastic process $\{X(t)\}_{t\geq 0}$, $X(t) : \Omega \to \mathbb{R}^d$ given by

$$X(t) = \Pi_{\xi(t_{n-1})}(t - t_{n-1}, x_{n-1}) \qquad \text{for} \quad t_{n-1} < t < t_n \tag{2.8}$$

and

$$X(t_n) = x_n$$

is the solution of (2.1), (2.2).

Let $x \in \mathbb{R}^d$. By $X(t)_x$ denote the solution of problem (2.1), (2.2) with $x_0 = x$.

We are interested in the evolution of distributions coresponding to the stochastic process $\{X(t)\}_{t\geq 0}$. Namely, let μ be the distribution of the initial vector x. For $t > 0$ we denote by $Q^t\mu$ the distribution of $X(t)$, i.e.

$$Q^t\mu(A) = \mathbb{P}(X(t) \in A) = \int_{\mathbb{R}^d} \mathbb{P}(X(t)_x \in A)\mu(dx) \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.9}$$

Simple consideration shows that the process $\{X(t)\}_{t\geq 0}$ is not Markovian. In order to use the handy machinary of Markov operators we must widen the space $\mathbb{R}^d$ and the process $\{X(t)\}_{t\geq 0}$ in such a way that new process become Markovian. In this purpose consider the space $\mathbb{R}^d \times I$ endowed with the metric ρ given by

$$\rho((x,i),(y,j)) = \|x - y\| + \rho_0(i,j) \qquad \text{for} \quad x, y \in \mathbb{R}^d \quad \text{and} \quad i,j \in I$$

where

$$\rho_0(i,j) = \begin{cases} \tilde{c} & \text{for} \quad i \neq j \\ 0 & \text{for} \quad i = j \end{cases}$$

with the constant $\tilde{c}$ suitably choosen.

Now we consider a stochastic process $\{(X(t), \overline{\xi}(t)\}_{t\geq 0}$, $(X(t), \overline{\xi}(t)) : \Omega \to \mathbb{R}^d \times I$.

We choose an initial point $(x, i) \in \mathbb{R}^d \times I$ and define

$$(X(t), \overline{\xi}(t)) = (\Pi_i(t, x), i) \quad \text{for} \quad t \in [0, t_1)$$

and

$$(X(t), \overline{\xi}(t)) = (X(t), \xi(t)) \quad \text{for} \quad t \geq t_1$$

where $X(t), \xi(t)$ are given by (2.5), (2.8).

It is obvious that $(X(t), \overline{\xi}(t))_{t\geq 0}$ constitutes a Markov process on $\mathbb{R}^d \times I$. This process generates a semigroup $\{T^t\}_{t\geq 0}$ defined by

$$T^t f(x, i) = E(f((X(t), \overline{\xi}(t))_{(x,i)})) \qquad \text{for} \quad f \in C(\mathbb{R}^d \times I),$$

where $E(f((X(t), \overline{\xi}(t))_{(x,i)}))$ denotes the mean value of $f((X(t), \overline{\xi}(t))_{(x,i)})$.

It is well known that $\{T^t\}_{t\geq 0}$ is a semigroup of operators from $C(\mathbb{R}^d \times I)$ into itself and for every $t > 0$ the operator T^t is a contraction, i.e. $\|T^t f\|_C \leq \|f\|_C$.

Now we define semigroup operators $\{P^t\}_{t\geq 0}$, $P^t : \mathcal{M}_1(\mathbb{R}^d \times I) \to \mathcal{M}_1(\mathbb{R}^d \times I)$ by

(2.10 $\quad \langle P^t\mu, f\rangle = \langle \mu, T^t f\rangle \quad$ for $\quad f \in C(\mathbb{R}^d \times I), \quad \mu \in \mathcal{M}_1(\mathbb{R}^d \times I) \quad$ and $\quad t \geq 0.$

Setting

$$G_{(x,i)}(t, A) = \text{prob}\{(X(t), \overline{\xi}(t))_{(x,i)} \in A\}$$

we obtain

$$P^t\mu(A) = \int\limits_{\mathbb{R}^d\times I} G_{(x,i)}(t, A) d\mu(x, i) \quad \text{for} \quad \mu \in \mathcal{M}_1(\mathbb{R}^d \times I), A \in \mathcal{B}(\mathbb{R}^d \times I) \text{ and } \ t \geq 0.$$

The semigroup $\{P^t\}_{t\geq 0}$ is called the Markov semigroup generated by the problem (2.1), (2.2).

Moreover using (2.10) the semigroup $\{P^t\}_{t\geq 0}$ can be easily extended to the vector space $\mathcal{M}_{sig}(\mathbb{R}^d \times I)$.

To study the asymptotic stability of the semigroup $\{P^t\}_{t\geq 0}$ we assume that the solutions $\Pi_j : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ of the equations (2.4) and the transition probabilities $p_{ij} : \mathbb{R}^d \to [0, 1]$ satisfy the following conditions:

$$\sum_{j=1}^{N} |p_{ij}(x) - p_{ij}(y)| \leq \psi(\|x - y\|) \quad \text{for} \quad x, y \in \mathbb{R}^d, \quad i \in I \tag{2.11}$$

and

$$\sum_{j=1}^{N} p_{ij}(y)\|\Pi_j(t, x) - \Pi_j(t, y)\| \leq Le^{at}\|x - y\| \quad \text{for} \quad x, y \in \mathbb{R}^d, \ i \in I, \tag{2.12}$$

where $L \geq 1$, a is a real constant and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, is nondecreasing and concave function satisfying condition (1.9). Finally, we assume that

$$\sigma = \inf \left\{ p_{ij}(x) \ : \ i,j \in I, \ x \in \mathbb{R}^d \right\} > 0. \tag{2.13}$$

3. – Auxillary results.

In this section we give rather technical observations used in proving main results. However, some of them (Proposition 3.1 and Lemma 3.2) seems to be of interest in themselves. Let $\Pi_i : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $i \in I$ be the solutions of the equations (2.4). We start with some notation useful in the sequel. For $n \in \mathbb{N}$ consider the function $\overline{\Pi}_n : \mathbb{R}^d \times I^n \times \mathbb{R}_+^n \times \Theta^n \to \mathbb{R}^d$ defined by the reccurent formula

$$\overline{\Pi}_1(x, i, s_1, \theta_1) = q\big(\Pi_i(s_1, x), \theta_1\big);$$

$$\begin{aligned} &\overline{\Pi}_n(x, i, i_1, \dots, i_{n-1}, s_1, \dots, s_n, \theta_1, \dots, \theta_n) \\ &= q\big(\Pi_{i_{n-1}}(s_n, \overline{\Pi}_{n-1}(x, i, i_1, \dots, i_{n-2}, s_1, \dots, s_{n-1}, \theta_1, \dots, \theta_{n-1})), \theta_n\big). \end{aligned} \tag{3.1}$$

Next consider the multiple transition probabilities $\mathcal{P}_n : \mathbb{R}^d \times I^{n+1} \times \mathbb{R}_+^n \times \Theta^n \to [0,1]$ given by

$$\begin{aligned} &\mathcal{P}_n(x, i, i_1, \dots, i_n, s_1, \dots, s_n, \theta_1, \dots, \theta_n) \\ &= p_{ii_1}(\overline{\Pi}_1(x, i, s_1, \theta_1)) \cdot \ldots \cdot p_{i_{n-1}i_n}(\overline{\Pi}_n(x, i, i_1, \dots, i_{n-1}, s_1, \dots, s_n, \theta_1, \dots, \theta_n)) \end{aligned} \tag{3.2}$$

with $i_0 = i$ and the functions $q_n : \mathbb{R}^d \times \Theta^n \to \mathbb{R}^d$ given by

$$\begin{aligned} &q_0(x) = x, \quad q_1(x, \theta_1) = q(x, \theta_1) \\ &q_n(x, \theta_1, \dots, \theta_{n-1}, \theta_n) = q(q_{n-1}(x, \theta_1, \dots, \theta_{n-1}), \theta_n). \end{aligned} \tag{3.3}$$

REMARK 3.1. – *Observe that for every* $n \in \mathbb{N}$, $s_1, \dots, s_n \in \mathbb{R}_+, x \in \mathbb{R}^d$, $i, i_1, \dots, i_{n+1} \in I$ *and* $\theta_1, \dots, \theta_{n+1} \in \Theta$ *we have*

$$\begin{aligned} &\mathcal{P}_{n+1}(x, i, i_1, \dots, i_{n+1}, s_1, \dots, s_{n+1}, \theta_1, \dots, \theta_{n+1}) \\ &= \mathcal{P}_n(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, \dots, i_{n+1}, s_2, \dots, s_{n+1}, \theta_2, \dots, \theta_{n+1}) \\ &\cdot p_{ii_1}(\overline{\Pi}_1(x, i, s_1, \theta_1)) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\overline{\Pi}_{n+1}(x, i, i_1, \dots, i_n, s_1, \dots, s_{n+1}, \theta_1, \dots, \theta_{n+1}) \\ &= \overline{\Pi}_n(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, \dots, i_n, s_2, \dots, s_{n+1}, \theta_2, \dots, \theta_{n+1}). \end{aligned} \tag{3.5}$$

Finally, given a function $f : \mathbb{R}^d \times I \to \mathbb{R}$ we consider the function $f_n : \mathbb{R}^d \times I \times \mathbb{R}_+^{n+1} \to \mathbb{R}$ given by

$$
\begin{aligned}
&f_n(x, i, s_1, \ldots, s_{n+1}) \\
(3.6)\qquad &= \underbrace{\int_\Theta \ldots \int_\Theta}_{n} \sum_{i_1,\ldots,i_n=1}^{N} f(\Pi_{i_n}(s_{n+1}, \overline{\Pi}_n(x, i, i_1, \ldots, i_{n-1}, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)), i_n) \\
&\cdot \mathcal{P}_n(x, i, i_1, \ldots, i_n, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)\upsilon(d\theta_1)\ldots\upsilon(d\theta_n).
\end{aligned}
$$

By Remark 3.1 we have

$$
\begin{aligned}
&f_{n+1}(x, i, s_1, \ldots, s_{n+2}) \\
(3.7)\qquad &= \int_\Theta \sum_{i_1=1}^{N} p_{ii_1}\big(\overline{\Pi}_1(x, i, s_1, \theta_1)\big) \cdot f_n\big(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, s_2, \ldots, s_{n+2}\big)\upsilon(d\theta_1).
\end{aligned}
$$

For the convenience of the reader sometimes we will write $f_n(x, i, \boldsymbol{s})$ instead of $f_n(x, i, s_1, \ldots, s_n)$ and analogously $\overline{\Pi}_n(x, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta})$ instead of $\overline{\Pi}_n(x, i_1, \ldots, i_n, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$ where $\boldsymbol{s} = (s_1, \ldots, s_n)$, $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_n)$ and $\boldsymbol{i} = (i_1, \ldots, i_n)$.

LEMMA 3.1. – *Assume that conditions* (2.6), (2.11) - (2.13) *are satisfied. Let* $\varphi \in \Phi$ *and* $f \in \mathcal{F}_\varphi(\mathbb{R}^d \times I)$. *Then for every* $n \in \mathbb{N}, i \in I, x, y \in \mathbb{R}^d$ *and* $s_1, \ldots, s_{n+1} \in \mathbb{R}_+$ *we have*

$$
\begin{aligned}
&|f_n(x, i, s_1, \ldots, s_{n+1}) - f_n(y, i, s_1, \ldots, s_{n+1})| \\
(3.8)\qquad &\le \varphi\left(L_q^n \frac{L^{n+1}}{\sigma} e^{a(s_1+\ldots+s_{n+1})}\|x-y\|\right) + \sum_{k=1}^{n} \psi\left(\frac{(L_q L)^k}{\sigma} e^{a(s_1+\ldots+s_k)}\|x-y\|\right).
\end{aligned}
$$

PROOF. – The proof of Lemma 3.1 is by induction on n and follows from Remark 3.1. ■

We are going to show that the semigroup $\{P^t\}_{t\ge0}$ given by (2.10) is asymptotically stable. The proof will be long and technical. First, we show that the semigroup $\{P^t\}_{t\ge0}$ is nonexpansive. Second, we prove that for some $t_* > 0$ the Markov operator P^{t_*} has the Prohorov property. Next, we show that the operator P^{t_*} satisfies a lower bound condition. Finally, from the above three facts we draw a conclusion that the semigroup $\{P^t\}_{t\ge0}$ is asymptotically stable.

PROPOSITION 3.1. – *Assume that the solutions Π_i of the unperturbed system* (2.4) *satisfy* (2.12) . *Assume, moreover that the conditions* (2.6), (2.11), (2.13) *are satisfied. If in addition the constants a, L, L_q and λ satisfy the inequality*

$$
(3.9)\qquad LL_q + \frac{a}{\lambda} < 1.
$$

Then there exists $t_* > 0$ such that for every $t \ge t_*$ the operators P^t given by (2.10) *are nonexpansive with respect to a metric ρ_φ.*

PROOF. – By virtue of (3.9), we can choose $t_* > 0$ such that

$$r_0 = \frac{L}{\sigma} e^{-(\lambda - a - \lambda L L_q)t_*} < 1.$$

Moreover, let $\overline{\psi} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\overline{\psi}(t) = \begin{cases} \lambda t_* \psi\left(\dfrac{LL_q}{\sigma} e^{-\lambda(1-LL_q)t_*} t\right) & \text{if} \quad a < 0 \\ \lambda t_* \psi\left(\dfrac{1}{\sigma} e^{at_*} t\right) & \text{if} \quad a \geq 0. \end{cases}$$

Since $\overline{\psi} \in \Phi_0$ and satisfies the hypotheses of Proposition 1.4, then there is $\varphi \in \Phi$ such that

$$\overline{\psi}(t) + \varphi(r_0 t) \leq \varphi(t) \qquad \text{for} \quad t \geq 0.$$

Now choose $\tilde{c} > 0$ such that $\varphi(\tilde{c}) \geq 2$ and consider the metric ρ_0 with such $\tilde{c}$.

Denote by $\|\cdot\|_\varphi$ the Fortet–Mourier norm in $\mathcal{M}_1(\mathbb{R}^d \times I)$ given by

$$\|\mu_1 - \mu_2\|_\varphi = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}_\varphi\}$$

where $\mathcal{F}_\varphi$ is the set of functions such that $|f| \leq 1$ and

$$|f(x,i) - f(y,j)| \leq \rho_\varphi((x,i),(y,j)) = \varphi(\rho((x,i),(y,j)))$$

for $\quad x, y \in \mathbb{R}^d, \ i,j \in I$.

We will prove that T^{t_*} is a nonexpansive operator with respect to the norm ρ_φ. For $n \in \mathbb{N} \cup \{0\}$ we set

$$\Omega_n = \Omega_n(t_*) = \{\omega \in \Omega : t_n(\omega) \leq t_* \quad \text{and} \quad t_{n+1}(\omega) > t_*\}. \tag{3.10}$$

Obviously $\mathbb{P}(\cup_{n=0}^{+\infty} \Omega_n(t_*)) = 1$. Let $f : \mathbb{R}^d \times I \to \mathbb{R}$ be a bounded continuous function and let $x \in \mathbb{R}^d$ and $i \in I$ be given. Since

$$s_n = t_n - t_{n-1} \quad \text{for} \quad n = 1, \ldots,$$

setting $\boldsymbol{s}_n = (s_1, \ldots, s_n)$ simple calculations shows that

$$Ef((X(t), \overline{\xi}(t))_{(x,i)}) = e^{-\lambda t_*} f(\Pi_i(t_*, x), i) + \sum_{n=1}^{\infty} \int_{\Omega_n} f_n(x, i, \boldsymbol{s}_n(\omega), t_* - t_n(\omega)) \mathbb{P}(d\omega). \tag{3.11}$$

Fix an $f \in \mathcal{F}_\varphi$. Evidently $|T^{t_*} f| \leq 1$, so we have to prove that

$$|T^{t_*} f(x,i) - T^{t_*} f(y,j)| \leq \rho_\varphi((x,i),(y,j)) \qquad \text{for} \quad x, y \in \mathbb{R}^d \text{ and } \ i,j \in I. \tag{3.12}$$

Since by assumption $\rho_0(i,j) = \tilde{c}$ for $i \neq j$ and $\varphi(\tilde{c}) \geq 2$, then for $i \neq j$ the condition

(3.12) is satisfied. However for $i = j$, we have

$$|T^{t_*}f(x,i) - T^{t_*}f(y,i)| \le E(|f((X(t_*),\overline{\xi}(t_*))_{(x,i)}) - f((X(t_*),\overline{\xi}(t_*))_{(y,i)})|)$$

$$\le e^{-\lambda t_*}|f(\Pi_i(t_*,x),i) - f(\Pi_i(t_*,y),i)|$$

$$+\sum_{n=1}^{+\infty}\int_{\Omega_n}|f_n(x,i,\boldsymbol{s}_n(\omega),t_*-t_n(\omega)) - f_n(y,i,\boldsymbol{s}_n(\omega),t_*-t_n(\omega))|\mathbb{P}(d\omega).$$

From this and (3.8) we obtain

$$|T^{t_*}f(x,i) - T^{t_*}f(y,i)| \le e^{-\lambda t_*}\varphi(\|\Pi_i(t_*,x) - \Pi_i(t_*,y)\|)$$

$$+\sum_{n=1}^{+\infty}\int_{\Omega_n}\left[\varphi\left(\frac{L_q^nL^{n+1}}{\sigma}e^{at_*}\|x-y\|\right)+\sum_{j=1}^{n}\psi(\frac{(L_qL)^j}{\sigma}e^{at_j(\omega)}\|x-y\|)\right]\mathbb{P}(d\omega).$$

In the case when $a < 0$ without loss of generality we can assume that $LL_q \ge 1$. Then we obtain

$$|T^{t_*}f(x,i) - T^{t_*}f(y,i)| \le e^{-\lambda t_*}\left[\varphi\left(\frac{L}{\sigma}e^{at_*}\|x-y\|\right)\right.$$

$$\left.+\sum_{n=1}^{+\infty}\frac{(\lambda t_*)^n}{n!}\left(\varphi\left(\frac{L_q^nL^{n+1}}{\sigma}e^{at_*}\|x-y\|\right)+\sum_{j=1}^{n}\psi\left(\frac{(L_qL)^j}{\sigma}\|x-y\|\right)\right)\right]$$

$$\le \varphi\left(\frac{L}{\sigma}e^{-(\lambda-a-\lambda L_qL)t_*}\|x-y\|\right)+e^{-\lambda t_*}\sum_{n=1}^{+\infty}\frac{(\lambda t_*)^n}{(n-1)!}\psi\left(\frac{(L_qL)^n}{\sigma}\|x-y\|\right)$$

$$\le \varphi(r_0\|x-y\|) + \overline{\psi}(\|x-y\|).$$

Suppose now that $a \ge 0$, then $LL_q < 1$ and we have

$$|T^{t_*}f(x,i) - T^{t_*}f(y,i)| \le \varphi(r_0\|x-y\|) + e^{-\lambda t_*}\sum_{n=1}^{+\infty}\frac{(\lambda t_*)^n}{(n-1)!}\psi\left(\frac{e^{at_*}}{\sigma}\|x-y\|\right)$$

$$\le \varphi(r_0\|x-y\|) + \overline{\psi}(\|x-y\|).$$

From the last inequalities and the choice of φ it follows that

$$|T^{t_*}f(x,i) - T^{t_*}f(y,i)| \le \varphi(\|x-y\|).$$

Consequently, for every $f \in \mathcal{F}_\varphi(\mathbb{R}^d \times I)$ and $t \ge t_*$ we have

$$|T^tf(x,i) - T^tf(y,i)| \le \varphi(\|x-y\|),$$

which implies that operator P^t is nonexpansive in the norm $\|\cdot\|_\varphi$.This completes the proof. ■

Denote by υ^n the measure on Θ^n generated by υ (i.e. $\upsilon^n = \upsilon\otimes\cdots\otimes\upsilon$).

LEMMA 3.2. – *Assume that there exists* $\gamma > 0$ *such that*

(3.13 $$\|\Pi_i(t,x) - x\| \le \gamma t \quad \text{for} \quad i \in I, t > 0 \quad \text{and} \quad x \in \mathbb{R}^d$$

and in addition the function q satisfies the conditions (2.6)-(2.7) *then for every* $n \in \mathbb{N}$

(3.14) $$\int_{\Theta^n} \|\overline{\Pi}_n(0, \boldsymbol{i}, s_1, \ldots, s_n, \boldsymbol{\theta})\| \nu^n(d\boldsymbol{\theta}) \le \overline{L}_q^n \gamma(s_1 + \ldots + s_n) + nc\overline{L}_q^{n-1}$$

for $s_1, \ldots, s_n \in \mathbb{R}_+, \boldsymbol{i} \in I^n$ *and* $\overline{L}_q = max\{1, L_q\}$.

PROOF. – The technical proof is left to the reader. ■

4. – Invariant measure and stability.

The remainder of this section is devoted to the proof of the weakly convergence of the family $\{Q^t\}_{t\ge 0}$ generated by the stochastic differential equation (2.1) (2.2).

First, we show that the semigroup $\{P^t\}_{t\ge 0}$ given by (2.10) is asymptotically stable.

THEOREM 4.1. – *Let the assumptions of Proposition* 3.1 *hold. Assume, moreover that the conditions* (2.7) *and* (3.13) *are satisfied. Let* $t_* > 0$ *be such that*

$$\frac{L}{\sigma} e^{-(\lambda - a - \lambda L L_q)t_*} < 1.$$

Then the Markov operator P^{t_*} *given by* (2.10) *admits an invariant measure and satisfies the Prohorov condition.*

PROOF. – Denote $\overline{P} = P^{t_*}$ and $\overline{T} = T^{t_*}$. Set

$$V(x,i) = \|x\| \quad \text{for} \quad x \in \mathbb{R}^d \quad \text{and} \quad i \in I.$$

To finish the proof it is enough to show that there exist constants $\beta_1, \beta_2 \in \mathbb{R}_+$, $\beta_1 < 1$ such that

$$\overline{T}V(x,i) \le \beta_1 V(x,i) + \beta_2 \quad \text{for} \quad x \in \mathbb{R}^d, \quad i \in I.$$

According to (3.11) we have

$$\overline{T}V(x,i) = e^{-\lambda t_*} V(\Pi_i(t_*, x), i) + \sum_{n=1}^{+\infty} \int_{\Omega_n} V_n(x, i, \boldsymbol{s}_n(\omega), t_* - t_n(\omega)) \mathbb{P}(d\omega)$$

where V_n is the function corresponding to $f = V$ according to formula (3.6),

$s_n = t_n - t_{n-1}$ and $\boldsymbol{s}_n = (s_1, \ldots, s_n)$. Thus

$$\overline{T}V(x,i) \le e^{-\lambda t_*}\|\Pi_i(t_*,x) - \Pi_i(t_*,0)\| + e^{-\lambda t_*}\|\Pi_i(t_*,0)\|$$
$$+\sum_{n=1}^{+\infty}\int_{\Omega_n}\int_{\Theta^n}\sum_{\boldsymbol{i}\in I^{n-1}, i_n\in I}\|\Pi_{i_n}(t_* - t_n(\omega), \overline{\Pi}_n(x,i,\boldsymbol{i},\boldsymbol{s}_n(\omega),\boldsymbol{\theta}))$$
$$-\Pi_{i_n}(t_* - t_n(\omega), \overline{\Pi}_n(0,i,\boldsymbol{i},\boldsymbol{s}_n(\omega),\boldsymbol{\theta}))\| \cdot \mathcal{P}_n(x,i,\boldsymbol{i},i_n,\boldsymbol{s}_n(\omega),\boldsymbol{\theta})))\upsilon^n(d\boldsymbol{\theta})\mathbb{P}(d\omega)$$
$$+\sum_{n=1}^{+\infty}\int_{\Omega_n}\int_{\Theta^n}\sum_{\boldsymbol{i}\in I^{n-1}, i_n\in I}\|\Pi_{i_n}(t_* - t_n(\omega), \overline{\Pi}_n(0,i,\boldsymbol{i},\boldsymbol{s}_n(\omega),\boldsymbol{\theta}))\|$$
$$\cdot\mathcal{P}_n(x,i,\boldsymbol{i},i_n,\boldsymbol{s}_n(\omega),\boldsymbol{\theta})))\upsilon^n(d\boldsymbol{\theta})\mathbb{P}(d\omega).$$

By an argument similar to that of proving Proposition 3.1, using (2.6), (2.12) and (2.13) one can show that

$$\begin{aligned}&\int_{\Omega_n}\int_{\Theta^n}\sum_{\boldsymbol{i}\in I^{n-1}, i_n\in I}\|\Pi_{i_n}(t_* - t_n(\omega).\overline{\Pi}_n(x,i,\boldsymbol{i},\boldsymbol{s}_n(\omega),\boldsymbol{\theta}))\\ &-\Pi_{i_n}(t_* - t_n(\omega), \overline{\Pi}_n(0,i,\boldsymbol{i},\boldsymbol{s}_n(\omega),\boldsymbol{\theta})),\|\\ &\cdot\mathcal{P}_n(x,i,\boldsymbol{i},i_n,\boldsymbol{s}_n(\omega),\boldsymbol{\theta})))\upsilon^n(d\boldsymbol{\theta})\mathbb{P}(d\omega) \le \frac{(L_qL)^nL}{\sigma}e^{at_*}\|x\|.\end{aligned} \tag{4.1}$$

By (2.7), (3.13), (4.1) and Lemma 3.2 we have

$$\overline{T}V(x,i) \le e^{-(\lambda-a)t_*}\tfrac{L}{\sigma}\|x\| + e^{-\lambda t_*}\gamma t_* + \sum_{n=1}^{\infty}\int_{\Omega_n}\frac{(LL_q)^nL}{\sigma}e^{at_*}\|x\|\mathbb{P}(d\omega)$$
$$+\sum_{n=1}^{\infty}\int_{\Omega_n}\left(\gamma(t_* - t_n(\omega)) + \gamma\overline{L}_q^n(t_n(\omega)) + nc\overline{L}_q^{n-1}\right)\mathbb{P}(d\omega).$$

Thus

$$\overline{T}V(x,i) \le \frac{L}{\sigma}e^{-(\lambda-a-\lambda L_qL)t_*}\|x\| + t_*e^{-\lambda(1-\overline{L}_q)t_*}(\gamma+\lambda c).$$

Setting $\beta_1 = \frac{L}{\sigma}e^{-(\lambda-a-\lambda L_qL)t_*}$ and $\beta_2 = t_*e^{-\lambda(1-\overline{L}_q)t_*}(\gamma+\lambda c)$ finishes the proof. ■

Having this we may formulate the main result of this paper.

THEOREM 4.2. – *Assume that the conditions* (2.6)-(2.7) *and* (2.11)-(2.13) *are satisfied. Assume, moreover that the solutions Π_i of the unterpurbed system* (2.4) *satisfy* (3.13). *If in addition the constants a, L, L_q and λ satisfy* (3.9) *then the semigroup $\{P^t\}_{t\ge0}$ given by* (2.10) *is asymptotically stable.*

PROOF. – By assumption (3.9) we may choose $t_* > 0$ such that

$$\frac{L}{\sigma} e^{-(\lambda - a - \lambda L_q L)t_*} < 1. \tag{4.2}$$

By Proposition 3.1 and Theorem 4.1, the operator P^{t_*} is nonexpansive with respect to the metric ρ_φ and has the Prohorov property. Now we show that P^{t_*} satisfies a lower bound condition. Set $\overline{P} = P^{t_*}$ and $\overline{T} = T^{t_*}$.

By the Prohorov property there exists a compact set $F \subset \mathbb{R}^d \times I$ such that for every $\mu \in \mathcal{M}_1(\mathbb{R}^d \times I)$ there exists an integer $n_1 = n_1(\mu)$ for which

$$\overline{P}^n \mu(F) \geq \frac{1}{2} \qquad \text{for} \quad n \geq n_1.$$

Define

$$F_Y = \{x \in \mathbb{R}^d : (x, i) \in F \quad \text{for some} \quad i \in I\}.$$

We consider two cases : $a < 0$ and $a \geq 0$.

Case 1. Suppose first that $a < 0$. We may choose t_* such that (4.2) is satisfied and

$$r_0 = \frac{2L_q L}{\sigma} e^{at_*} < 1. \tag{4.3}$$

Fix $\varepsilon_1 > 0$. We can find $\varepsilon > 0$ such that $\varphi(\varepsilon) \leq \varepsilon_1$. Let $m \geq 2$ be such that

$$r_0^m diam_{\varphi_\rho} F < \frac{\varepsilon}{3}. \tag{4.4}$$

Fix $i_1, \ldots, i_m \in I$, set $\boldsymbol{i} = (i_1, \ldots, i_{m-1}) \in I^{m-1}$ and $\boldsymbol{t}_* = \underbrace{(t_*, \ldots, t_*)}_{m-1} \in \mathbb{R}_+^{m-1}$. Define

$$\begin{aligned} \Theta_0 = \{\theta \in \Theta : \|q(\Pi_j(t,x),\theta) - q(\Pi_j(t,y),\theta)\| \leq 2L_q \|\Pi_j(t,x) - \Pi_j(t,y)\| \\ \text{for} \quad (t,x),(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad j \in I\}, \end{aligned} \tag{4.5}$$

by assumption (2.6) it is evident that

$$\upsilon(\Theta_0) \geq \frac{1}{2}.$$

Since Θ is compact and the functions q and Π_i are continuous there is $\theta_{l_*} \in \Theta_0$ and a neighbourhood $\mathcal{B}(\theta_{l_*})$ of θ_{l_*} such that $\Theta_{l_*} = \mathcal{B}(\theta_{l_*}) \cap \Theta_0 \neq \emptyset$, $\upsilon(\Theta_{l_*}) > 0$ and

$$\|\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_1), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta})) - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*}))\| \leq \frac{\varepsilon}{9} \tag{4.6}$$

for $x \in F_Y$, $\theta_1 \in \Theta_{l_*}, \boldsymbol{\theta} \in \Theta_{l_*}^{m-1}$ and $\boldsymbol{\theta}_{l_*} = \underbrace{(\theta_{l_*}, \ldots, \theta_{l_*})}_{m-1}$.

Since F^2 is a compact set, there is a finite covering

$$(O(z_1,j_1) \times O(y_1,k_1)) \cup \ldots \cup \big(O(z_q,j_q) \times O(y_q,k_q)\big) \supset F^2 \tag{4.7}$$

where $(z_l,j_l),(y_l,k_l) \in F$ and

$$\begin{aligned} O(z_l,j_l) = \Big\{ x : \| &\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*})) \\ &-\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_l,\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*}))\| \le \frac{\varepsilon}{9} \Big\} \times \{j_l\} \end{aligned} \tag{4.8}$$

for $l = 1, \ldots, q$. Let $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}^d \times I)$ be given. By the Prohorov condition there is an integer $\overline{n} = \overline{n}(\mu_1,\mu_2)$ such that

$$\overline{P}^n \mu_k(F) \ge \frac{1}{2} \quad \text{for} \quad n \ge \overline{n}, \quad k = 1,2.$$

Set $\overline{\mu}_k = \overline{P}^{\overline{n}} \mu_k$. Then $(\overline{\mu}_1 \times \overline{\mu}_2)(F^2) \ge \frac{1}{4}$ and according to (4.7) there is an integer $r = r(\mu_1,\mu_2)$ such that

$$\overline{\mu}_1((O(z_r,j_r)) \ge \frac{1}{4q}, \quad \overline{\mu}_2((O(y_r,k_r)) \ge \frac{1}{4q}.$$

Write for simplicity $z_{l_1} = z_r, z_{l_2} = y_r, j_1 = j_r, j_2 = k_r$ and $O_1 = O(z_{l_1},j_1), O_2 = O(z_{l_2},j_2)$.

Moreover, from the definition of $\overline{\Pi}_m$ and conditions (2.6), (2.12), (2.13) and (4.5) it follows that

$$\begin{aligned} &\|\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_1},\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*})) - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_2},\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*}))\| \\ &\qquad \le \left(\frac{2L_q L}{\sigma} e^{at_*} \right)^m \|z_{l_1} - z_{l_2}\| \le r_0^m diam_{\varphi_\rho} F \le \frac{\varepsilon}{3}. \end{aligned} \tag{4.9}$$

Now define $A = A_1 \cup A_2$ where

$$A_k = B\Big(\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_k},\theta_{l_*}), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}_{l_*})), \frac{\varepsilon}{3}\Big) \times \{i_m\}.$$

Namely

$$diam_{\rho_\varphi}(A) = diam_{\varphi\cdot\rho}(A) \le \varphi(diam_\rho(A)) \le \varphi(\varepsilon) \le \varepsilon_1.$$

By continuity Π_i, q and (3.5) there exists $\eta > 0$ such that

$$\begin{aligned} &\|\Pi_{i_m}(\delta_{m+1}, \overline{\Pi}_m(x,i,\boldsymbol{i},\boldsymbol{\delta},\theta_1,\boldsymbol{\theta})) - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_1), \boldsymbol{i}, \boldsymbol{t}_*, \boldsymbol{\theta}))\| \\ &= \|\Pi_{i_m}(\delta_{m+1}, \overline{\Pi}_m(x,i,\boldsymbol{i},\boldsymbol{\delta},\theta_1,\boldsymbol{\theta})) - \Pi_{i_m}(t_*, \overline{\Pi}_m(x,i,\boldsymbol{i},0,\boldsymbol{t}_*,\theta_1,\boldsymbol{\theta}))\| \le \frac{\varepsilon}{9} \end{aligned} \tag{4.10}$$

for arbitrary $(x,i) \in F, \boldsymbol{\delta} = (\delta_1, \ldots, \delta_m), \delta_1 \in (0,\eta), \delta_2, \ldots, \delta_{m+1} \in (t_* - \eta, t_* + \eta)$ and $\theta_1 \in \Theta_{l_*}, \boldsymbol{\theta} \in \Theta_{l_*}^{m-1}$.

Set

$$\Omega_* = \{\omega \in \Omega : s_1(\omega) \le \eta, s_2(\omega), \dots, s_m(\omega) \in (t_* - \frac{\eta}{m-1}, t_*), s_1(\omega) + \dots + s_{m+1}(\omega) > mt_*\}$$

where $s_i(\omega) = t_i(\omega) - t_{i-1}(\omega)$.

Let $n_0 = \overline{n} + m$ and $\boldsymbol{s}_m = (s_1, \dots, s_m)$. We have

$$\begin{aligned}
&\overline{P}^{n_0}\mu_k(A) = \int_{\mathbb{R}^d \times I} \overline{T}^m 1_A(x,i) d\overline{\mu}_k(x,i) \ge \int_{\mathbb{R}^d \times I} \mathbb{E}(1_{A_k}(X(mt_*), \overline{\xi}(mt_*))_{(x,i)})) d\overline{\mu}_k(x,i) \\
&\ge \int_{O_k}\int_{\Omega_*}\int_{\Theta^m_{l_*}} 1_{A_k}(\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \\
&\mathcal{P}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta}) \upsilon^m(d\boldsymbol{\theta}) \mathbb{P}(d\omega) d\overline{\mu}_k(x,i) \\
&\ge \sigma^m \int_{O_k}\int_{\Omega_*}\int_{\Theta^m_{l_*}} 1_{A_k}(\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \\
&\upsilon^m(d\boldsymbol{\theta}) \mathbb{P}(d\omega) d\overline{\mu}_k(x,i).
\end{aligned} \tag{4.11}$$

By (4.6), (4.8) and (4.10) we obtain

$$\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \in A_k$$

for arbitrary $\omega \in \Omega_*, (x,i) \in O_k$ and $\boldsymbol{\theta} \in \Theta^m_{l_*}$. Thus

$$\overline{P}^{n_0}\mu_k(A) \ge \frac{\sigma^m}{4q} \upsilon(\Theta_{l_*})^m \mathbb{P}(\Omega_*).$$

As a consequence the operator P^{t_*} is asymptotically stable in the metric space $(\mathbb{R}^d \times I, \varphi \circ \rho)$.

Case II.

Suppose now that $a \ge 0$. In this case condition (3.9) implies that $L_q < 1$. Let $m \in \mathbb{N}$ be such that

$$L_q^m \cdot diam_{\varphi_\rho} F < \frac{\varepsilon}{2}. \tag{4.12}$$

By continuity and compactness conditions there exists $\delta \in (0, \frac{\varepsilon}{16\gamma m})$ such that

$$\|\overline{\Pi}_m(x, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta})\| < \frac{\varepsilon}{32} \tag{4.13}$$

for every $\boldsymbol{i} \in I^m, \boldsymbol{s} \in (0, \delta]^m$, $\boldsymbol{\theta} \in \Theta^m$, $x \in F_Y$.

Given $\tilde{\boldsymbol{\theta}} \in \Theta^m$ we define

$$\boldsymbol{V}(\tilde{\boldsymbol{\theta}}) = \Big\{\boldsymbol{\theta} \in \Theta^m : \|q_m(x, \boldsymbol{\theta}) - q_m(x, \tilde{\boldsymbol{\theta}}\| < \frac{\varepsilon}{32} \ \text{ for every } \ x \in F_Y\Big\}. \tag{4.14}$$

Clearly $\boldsymbol{V}(\tilde{\boldsymbol{\theta}})$ is an open neighborhood of $\tilde{\boldsymbol{\theta}}$. Since Θ^m is compact, there exists a finite family $\boldsymbol{V}_j = \boldsymbol{V}(\boldsymbol{\theta}_j), j = 1, \ldots, \overline{m}$ such that $\Theta^m = \cup_{j=1}^{\overline{m}} \boldsymbol{V}_j$. Set

$$J = \{j \in \{1, \ldots, \overline{m}\} : \upsilon^m(\boldsymbol{V}_j) > 0\}$$

and

$$\vartheta = \min_{j \in J} \upsilon^m(\boldsymbol{V}_j). \tag{4.15}$$

Clearly $\vartheta > 0$. Given $x \in \mathbb{R}^d$ define

$$O(x) = \Big\{z \in \mathbb{R}^d : \|q_m(x, \boldsymbol{\theta}_j) - q_m(z, \boldsymbol{\theta}_j)\| < \frac{\varepsilon}{32} \quad \text{for} \quad j \in J\Big\}. \tag{4.16}$$

Clearly $O(x)$ is an open neighborhood of x. Let $z_1, \ldots, z_{m_0} \in F_Y$ be such that $F \subset G$ where G is given

$$G = \bigcup_{l=1}^{m_0} (O(z_l) \times I).$$

Let $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}^d \times I)$ be arbitrary. Analogously as in Case I there exist $\overline{n} = \overline{n}(\mu_1, \mu_2), l_1, l_2 \in \{1, \ldots, m_0\}, j_1, j_2 \in I$ such that

$$\overline{P}^{\overline{n}} \mu_k(O(z_{l_k} \times \{j_k\}) \geq \frac{1}{2m_0 N}.$$

From condition (2.6) it follows that there exists a subset $\boldsymbol{\Theta}_0$ of Θ^m such that $\upsilon^m(\boldsymbol{\Theta}_0) > 0$ and

$$\|q_m(z_{l_1}, \boldsymbol{\theta}) - q_m(z_{l_2}, \boldsymbol{\theta})\| \leq L_q^m \|z_{l_1} - z_{l_2}\| \quad \text{for every} \quad \boldsymbol{\theta} \in \boldsymbol{\Theta}_0. \tag{4.17}$$

Since $\boldsymbol{\Theta}_0$ is of positive measure, there exists $j_0 \in J$ such that $\boldsymbol{\Theta}_0 \cap \boldsymbol{V}_0 \neq \emptyset$, where $\boldsymbol{V}_0 = \boldsymbol{V}(\boldsymbol{\theta}_{j_0})$. Choose $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}_0 \cap \boldsymbol{V}_0$, $i_m \in I$ and define

$$A_k = B\Big(q_m(z_{l_k}, \boldsymbol{\theta}_0), \frac{\varepsilon}{4}\Big) \times \{i_m\} \quad \text{for} \quad k = 1, 2$$

and $A = A_1 \cup A_2$.

From (4.12) and (4.17) it follows that $diam_{\varphi_\rho} A < \varepsilon$.

For $\boldsymbol{\theta} \in \boldsymbol{V}_0$, $\boldsymbol{i} \in I^{m-1}$, $\boldsymbol{s} \in [0, \delta]^m$ and $x \in O(z_{l_k})$ by virtue of (4.13) - (4.16) we have

$$\begin{aligned} \|\overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| &\leq \|\overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta})\| \\ &+ \|q_m(x, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta}_{j_0})\| + \|q_m(x, \boldsymbol{\theta}_{j_0}) - q_m(z_{l_k}, \boldsymbol{\theta}_{j_0})\| \\ &+ \|q_m(z_{l_k}, \boldsymbol{\theta}_{j_0}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| < \frac{\varepsilon}{8} \end{aligned} \tag{4.18}$$

and by (3.14) we obtain for every $t > 0$

$$\|\Pi_{i_m}(t, \overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta})) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| \le \gamma t + \|\overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}, \boldsymbol{\theta}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| \le \gamma t + \frac{\varepsilon}{8}. \tag{4.19}$$

Fix $\bar{t}$ such that

$$m\delta < m\bar{t} < m\delta + \frac{\varepsilon}{16\gamma}$$

and set

$$\Omega_* = \{\omega \in \Omega : s_i(\omega) \le \delta \quad \text{for} \quad i = 1, \dots, m \quad \text{and} \quad s_1(\omega) + \dots + s_{m+1}(\omega) > m\bar{t}\}$$

where $s_i(\omega) = t_i(\omega) - t_{i-1}(\omega)$.

Let $n_0 = \overline{n} + m$ and $\overline{\mu}_k = \overline{P}^{\overline{n}} \mu_k$. Fix $i_1, \dots, i_{m-1} \in I$, set $\boldsymbol{i} = (i_1, \dots, i_{m-1})$ and $\boldsymbol{s}_m = (s_1, \dots, s_m)$. We have

$$\overline{P}^{n_0} \mu_k(A) = \int_{\mathbb{R}^d \times I} \overline{T}^m 1_A(x, i) d\overline{\mu}_k(x, i) \ge \int_{\mathbb{R}^d \times I} \mathbb{E}(1_{A_k}(X(m\bar{t}), \bar{\xi}(m\bar{t}))_{(x,i)})) d\overline{\mu}_k(x, i)$$

$$\begin{aligned} &\ge \int_{O(z_{l_k}) \times j_k} \int_{\Omega_*} \int_{\boldsymbol{V}_0} 1_{A_k}(\Pi_{i_m}(m\bar{t} - t_m(\omega), \overline{\Pi}_m(x, i, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \\ &\mathcal{P}_m(x, i, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta}) \upsilon^m(d\boldsymbol{\theta}) \mathbb{P}(d\omega) d\overline{\mu}_k(x, i) \\ &\ge \sigma^m \int_{O(z_{l_k}) \times j_k} \int_{\Omega_*} \int_{\boldsymbol{V}_0} 1_{A_k}(\Pi_{i_m}(m\bar{t} - t_m(\omega), \overline{\Pi}_m(x, i, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \\ &\upsilon^m(d\boldsymbol{\theta}) \mathbb{P}(d\omega) d\overline{\mu}_k(x, i). \end{aligned} \tag{4.20}$$

By (4.19) we obtain

$$(\Pi_{i_m}(m\bar{t} - t_m(\omega), \overline{\Pi}_m(x, j_k, \boldsymbol{i}, \boldsymbol{s}_m(\omega), \boldsymbol{\theta})), i_m) \in A_k$$

for arbitrary $\omega \in \Omega_*, x \in O(z_{l_k})$ and $\boldsymbol{\theta} \in \boldsymbol{V}_0$. Thus

$$\overline{P}^{n_0} \mu_k(A) \ge \frac{\sigma^m}{2m_0 q N} \vartheta \mathbb{P}(\Omega_*).$$

The proof in Case II is completed.

Now, let μ_* be the invariant distribution of P^{t_*}. Then for $t \in \mathbb{R}_+$ we have

$$P^{t_*}(P^t \mu_*) = P^t(P^{t_*} \mu_*) = P^t \mu_*.$$

Since μ_* is unique, it follows that $P^t \mu_* = \mu_*$. On the other hand using non-expansiveness of $\{P^t\}_{t \ge 0}$ we obtain

$$\lim_{t \to +\infty} \|P^t \mu - \mu_*\|_\varphi = \lim_{t \to +\infty} \|P^t \mu - P^t \mu_*\|_\varphi \le \lim_{n \to +\infty} \|(P^{t_*})^n \mu - \mu_*\|_\varphi = 0$$

for $\mu \in \mathcal{M}_1(\mathbb{R}^d \times I)$. However the metrics ρ and $\varphi \circ \rho$ define the same space of continuous functions $C(\mathbb{R}^d \times I)$ and the weak convergence of a sequence of measures in the space $(\mathbb{R}^d \times I, \rho)$ and $(\mathbb{R}^d \times I, \varphi \circ \rho)$ is the same. This proves that $\{P^t\}_{t \geq 0}$ generated by the equation (2.1), (2.2) is also asymptotically stable in $(\mathbb{R}^d \times I, \rho)$. ∎

Now we turn to the family $\{Q^t\}_{t \geq 0}$ of distributions generated by the stochastic differential equation (2.1), (2.2) on the starting space $(\mathbb{R}^d, \| \cdot \|)$.

Since weak convergence of a joint distribution implies weak convergence of the marginals, by Theorem 4.2 we obtain

THEOREM 4.3. – *Let the assumptions of Theorem* 4.2 *hold. Then there exists a distribution* $\overline{\mu}_* \in \mathcal{M}_1(\mathbb{R}^d)$ *such that the family of operators* $\{Q^t\}_{t \geq 0}$ *given by* (2.9) *is weakly convergent to* $\overline{\mu}_*$.

REMARK 4.1. – Under assumptions (2.6), (2.7) and (2.11) - (2.13) the condition (3.9) is optimal. Consider the following example. Let $a(x, i) = 0$ and $b(x, i) = -2x$ for $x \in \mathbb{R}^d$ and $k \in I = \{1\}$. Then $L = 1, L_q = 1$ and $a = 0$, so the equality in condition (3.9) holds. For arbitrary initial x_0 we obtain $x_n = -x_{n-1}$ for every $n \in \mathbb{N}$. By (2.8) we obtain

$$X(t) = x_{n-1} \quad \text{for} \quad t_{n-1} \leq t < t_n.$$

Thus this process is not asymptotically stable.

REFERENCES

[1] O.L.V. COSTA, *Stationary distributions for piecewise-deterministic Markov processes*, J. Appl. Prob., **27** (1990), 60-73.

[2] M.H.A. DAVIS, *Piecewise-deterministic Markov processes : A general class of non-diffusion stochastic models*, J. R. Statist. Soc. B (1984, **46**), 353-388.

[3] M.H.A. DAVIS, *Markov Models and Optimization*, Chapman and Hall, London (1993).

[4] O. DIEKMANN - H.J.A.M. HEIJMANS - H.R. THIEME, *On the stability of the cell size distribution*, J. Math. Biol., **19** (1984), 227-248.

[5] R. FORTET - B. MOURIER, *Convergence de la ŕepartition empirique vers la répartition théorétique*, Ann. Sci. École. Norm. Sup., 70 (1953), 267-285.

[6] U. FRISCH, *Wave propagation in random media, Probabilistic Methods in Applied Mathematics ed. A.T. Bharucha - Reid*, Academic Press 1968.

[7] K. HORBACZ, *Randomly connected dynamical systems - asymptotic stability*, Ann. Polon. Math., **68.1** (1998), 31-50.

[8] K. HORBACZ, *Invariant measures related with randomly connected Poisson driven diferential equations*, Ann. Polon. Math., **79.1** (2002), 31-44.

[9] K. HORBACZ, *Randomly connected differential equations with Poisson type perturbations*, Nonlinear Studies, **9.1** (2002), 81-98.

[10] K. HORBACZ, *Random dynamical systems with jumps*, J. Appl. Prob., **41** (2004), 890-910.
[11] K. HORBACZ, J. MYJAK - T. SZAREK, *Stability of random dynamical system on Banach spaces*, (to appear).
[12] J.B. KELLER, *Stochastic equations and wave propagation in random media*, Proc. Symp. Appl. Math., **16** (1964), 1456-1470.
[13] A. LASOTA - M.C. MACKEY, *Chaos, Fractals and Noise - Stochastic Aspect of Dynamics*, Springer-Verlag New York (1994).
[14] A. LASOTA - J. TRAPLE, *Invariant measures related with Poisson driven stochastic differential equation*, Stoch. Proc. and Their Appl. **106.1** (2003), 81-93.
[15] A. LASOTA - J.A. YORKE, *Lower bound technique for Markov operators and iterated function systems*, Random and Computational Dynamics, **2** (1994), 41-77.
[16] S. MEYN and R TWEEDIE, *Markov Chains and Stochastic Stability*, Springer-Verlag Berlin 1993.
[17] J. TRAPLE, *Markov semigroup generated by Poisson driven differential equations*, Bull. Pol. Ac. Math., **44** (1996), 161-182.

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