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Existence and Nonexistence Results for Quasilinear Elliptic Equations Involving the p-Laplacian.

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Sunto. – *L'articolo riguarda lo studio di un'equazione ellittica quasi-lineare con il p-laplaciano, caratterizzata dalla presenza di un termine singolare di tipo Hardy ed una nonlinearità critica. Si dimostrano dapprima risultati di esistenza e non esistenza per l'equazione con un termine singolare concavo. Quindi si passa a studiare il caso critico legato alla disuguaglianza di Hardy, fornendo una descrizione del comportamento delle soluzioni radiali del problema limite e ottenendo risultati di esistenza e molteplicità mediante metodi variazionali e topologici.*

Summary. – *The paper deals with the study of a quasilinear elliptic equation involving the p-laplacian with a Hardy-type singular potential and a critical nonlinearity. Existence and nonexistence results are first proved for the equation with a concave singular term. Then we study the critical case related to Hardy inequality, providing a description of the behavior of radial solutions of the limiting problem and obtaining existence and multiplicity results for perturbed problems through variational and topological arguments.*

1. – Introduction.

In this paper we study the following elliptic problem

$$(1)\qquad \begin{cases} -\Delta_p u = \dfrac{\lambda h(x)}{|x|^p}\,|u|^{q-1}u + g(x)\,|u|^{p^*-1}u, & \text{in } \mathbb{R}^N, \\ u(x) > 0, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases}$$

where $N \geqslant 3$, $\lambda > 0$, $0 < q \leqslant p-1$, $1 < p < N$, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. Here $\mathcal{D}^{1,p}(\mathbb{R}^N)$ denotes the space obtained as the com-

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pletion of the space of smooth functions with compact support with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p}.$$

Notice that the potential $1/|x|^p$ is related to the Hardy-Sobolev inequality. More precisely we have the following result.

LEMMA 1.1 (Hardy-Sobolev inequality). – *Suppose* $1 < p < N$. *Then for all* $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, *we have*

$$\int_{\mathbb{R}^N} |u|^p |x|^{-p} \, dx \leq \Lambda_{N,p}^{-1} \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \qquad \Lambda_{N,p} = \left(\frac{N-p}{p} \right)^p. \tag{2}$$

Moreover $\Lambda_{N,p}^{-1}$ *is optimal and it is not achieved.*

In bounded domains the above problem has been studied in [2], [5], [7], [11], [12], [13], [14] and [18] (see also the references in these papers). In the whole $\mathbb{R}^N$ and for $p = 2$ there are some results in [21] and in [1].

Let us briefly recall the known results for bounded domains and $h = g = 1$, because it will be useful to give some insight to the problem in $\mathbb{R}^N$.

In the case in which $q = p - 1$ and Ω is a starshaped domain with respect to the origin, a Pohozaev type argument proves that there is no positive solution in $W_0^{1,p}(\Omega)$. If $q > p - 1$ and $h(x) \equiv 1$, there is no positive solution even in the stronger sense of entropy solutions (in the case $p = 2$ this notion of solution is equivalent to the distributional one). This nonexistence result is also true in the case $q = p - 1$ and $\lambda > \Lambda_{N,p}$.

Finally if $0 < q < p - 1$, there exists some $\lambda^* > 0$ such that the problem has solution for $\lambda \in (0, \lambda^*]$ and has no solution if $\lambda > \lambda^*$.

This paper is organized as follows. Section 2 is devoted to the study of the case $q < p - 1$; we prove the existence of $\lambda^* > 0$ such that for any $\lambda \leq \lambda^*$ there exists a positive solution. Some results on comparison of solutions and nonexistence for large λ are also obtained.

Section 3 deals with the case $q = p - 1$, $h \equiv g \equiv 1$, and $0 < \lambda < \Lambda_{N,p}$. In this case we prove the existence of a one dimensional manifold of positive solutions. In subsection 3.2 we analyze the behaviour of radial solutions and we get an uniqueness result modulo rescaling. Notice that, in the case $p = 2$, the result obtained by Terracini in [25] gives a complete classification of solutions since moving plane method can be applied in such a case.

Section 4 is devoted to the study of nonexistence and existence for the case $q = p - 1$, $g \equiv 1$, and h satisfying suitable conditions. We will use the *concentration-compactness* principle by P.L. Lions to prove that the Palais-Smale condition holds below some critical threshold, thus obtaining existence results under some condition on h. The same analysis can be carried out if we assume that $h \equiv 1$ and g satisfies some convenient conditions.

In the last section multiplicity of solutions is proved in the case in which $h \equiv 1$ and g satisfies some conditions. Such multiplicity results are obtained by using some variational and topological argument as in [1].

	h	Ω bounded	$\Omega = \mathbb{R}^N$
$q < p - 1$	nonconstant	existence	existence
$q < p - 1$	constant	existence	non existence
$q = p - 1$	constant	non existence in starshaped domains	existence
$q > p - 1$	constant	non existence	non existence

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2. – The concave case related to the p-Laplacian.

Throughout this section we assume that $0 < q < p - 1$ and $g \equiv 1$, namely we deal with the following problem

$$\begin{cases} -\Delta_p u = \dfrac{\lambda h(x)\, u^q}{|x|^p} + u^{p^*-1}, & \text{in } \mathbb{R}^N, \\ u(x) > 0, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{3}$$

where $0 < q < p - 1$ and h is a positive function such that

$$\int_{\mathbb{R}^N} \frac{h^\alpha(x)}{|x|^p}\, dx < \infty \quad \text{where } \alpha = \frac{p}{p-(q+1)}. \tag{\textbf{h}}$$

For simplicity of notation we set

$$\|h\|_{L^\alpha(|x|^{-p}dx)} := \left(\int_{\mathbb{R}^N} \frac{h^\alpha(x)}{|x|^p}\, dx \right)^{\frac{p-(q+1)}{p}}.$$

We will use the following version of the well known Picone's Identity in [19]. For the proof we refer to [2](see also [3]).

THEOREM 2.1. – *If* $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, $u \geq 0$, $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, $-\Delta_p v \geq 0$ *is a bounded Radon measure,* $v \geq 0$ *and not identically zero, then*

$$\int_{\mathbb{R}^N} |\nabla u|^p \geq \int_{\mathbb{R}^N} \frac{u^p}{v^{p-1}}(-\Delta_p v).$$

As an application of Theorem 2.1, we get the following lemma, the proof of which can be obtained as a simple modification of the argument used in [2].

LEMMA 2.2. – *Let* $u, v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *be such that*

$$\begin{cases} -\Delta_p u \geq \dfrac{h(x)\, u^q}{|x|^p}, & in\ \mathbb{R}^N, \\ u > 0\ \ in\ \mathbb{R}^N, & u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{4}$$

$$\begin{cases} -\Delta_p v \leq \dfrac{h(x)\, v^q}{|x|^p} & in\ \mathbb{R}^N, \\ v > 0\ \ in\ \mathbb{R}^N, & v \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{5}$$

where $0 < q < p-1$ *and* h *is a nonnegative function such that* $h \not\equiv 0$. *Then* $u \geq v$ *in* $\mathbb{R}^N$.

As a direct consequence we have the following lemma.

LEMMA 2.3. – *The problem*

$$\begin{cases} -\Delta_p w = \dfrac{h(x)}{|x|^p} w^q & in\ \mathbb{R}^N, \\ w > 0\ \ in\ \mathbb{R}^N, & w \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{6}$$

with $q < p-1$ *and* h *satisfying* (**h**), *has a unique positive solution.*

PROOF. – Existence can be proved by using a classical minimizing argument. To obtain uniqueness one can use Lemma 2.2. ■

Week solutions to problem (3) can be found as critical points of the functional

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \frac{\lambda}{q+1}\int_{\mathbb{R}^N} \frac{h(x)}{|x|^p}|u|^{q+1}\, dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\, dx\,. \tag{7}$$

Using Hölder, Hardy, and Sobolev inequalities we obtain that for some positive constants c and c_1

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^N)} - \frac{\lambda c}{q+1}\|u\|^{q+1}_{\mathcal{D}^{1,p}(\mathbb{R}^N)} - \frac{c_1}{p^*}\|u\|^{p^*}_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \quad \forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N).$$

Therefore we get the existence of $a \in \mathbb{R}^N$, $r_0 > 0$, and $\lambda_1 > 0$ such that for any $\lambda \in [0, \lambda_1]$ there holds

1) $J_\lambda(u)$ is bounded from below in $B_{r_0} \equiv \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} < r_0\}$ and $I = \inf\{J_\lambda(u) \text{ for } u \in B_{r_0}\} < 0$;
2) $J_\lambda(u) \geq a > I$ for $\|u\| = r_0$.

To prove that the minimum is achieved we need the following lemma.

LEMMA 2.4. – *Let $C(N, p, q, h)$ be such that*

$$\frac{1}{N}s^p - \lambda \Lambda_{N,p}^{-\frac{q+1}{p}}\left(\frac{1}{q+1} - \frac{1}{p^*}\right)\|h\|_{L^\alpha(|x|^{-p}dx)}\, s^{q+1} \geq$$

$$-C(N, p, q, h)\,\lambda^{\frac{p}{p-q-1}}, \quad \forall s > 0.$$

Then for any sequence $\{u_n\} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ with

(8) $$J_\lambda(u_n) \to c < c(\lambda) \equiv \frac{1}{N}S^{\frac{N}{p}} - C(N, p, q, h)\,\lambda^{\frac{p}{p-q-1}} \quad \textit{and} \quad J'_\lambda(u_n) \to 0,$$

where S is the Sobolev constant for the p-Laplacian, there exists a subsequence that converges strongly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

PROOF. – We use the following result which can be proved by adapting the argument used in [6] for the Laplacian.

LEMMA 2.5. – *Let $\{u_n\} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a sequence satisfying the hypotheses of Lemma 2.4. Then for any $\eta > 0$ there exists $\varrho > 0$ such that*

$$\int_{|x|>\varrho} |\nabla u_n|^p\, dx < \eta.$$

We come back to the proof of Lemma 2.4. Since $\{u_n\}$ is a Palais-Smale sequence, it is bounded, i.e., $\|u_n\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \leq M$, then up to a subsequence still denoted by $\{u_n\}$,

1. $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$;
2. $u_n \to u_0$ almost everywhere and in $L^\alpha_{\text{loc}}(\mathbb{R}^N)$ for any $\alpha \in [1, p^*)$.

Using the Concentration Compactness Principle by P. L. Lions (see [15]) we conclude that $\{u_n\}$ satisfies

1. $|\nabla u_n|^p \rightharpoonup d\mu \geqslant |\nabla u_0|^p + \sum_{j\in J} \mu_j \delta_j$.
2. $|u_n|^{p^*} \rightharpoonup d\nu = |u_0|^{p^*} + \sum_{j\in J} \nu_j \delta_j$.
3. $S\nu_j^{\frac{p}{p^*}} \leqslant \mu_j$ for any $j \in J$, where J is an at most countable set.

Then it is not difficult to prove that either $\nu_j = 0$ or $\nu_j = \mu_j$. Therefore, if the singular part is not identically zero, i.e., if $\nu_j \neq 0$, we have that $\nu_j \geqslant S^{\frac{N}{p}}$. In view of hypothesis (**h**) and weak convergence of $\{u_n\}$, Vitali's Convergence Theorem yields

$$\int_{\mathbb{R}^N} \frac{h(x)\,|u_n|^{q+1}}{|x|^p} \to \int_{\mathbb{R}^N} \frac{h(x)\,|u_0|^{q+1}}{|x|^p}\,.$$

If we assume that $\nu_j \neq 0$ for some j, then, for $\varepsilon > 0$, we have

$$c + \varepsilon > J_\lambda(u_n) - \frac{1}{p^*}(J'(u_n), u_n) =$$

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^p - \lambda\left(\frac{1}{q+1} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} \frac{h(x)\,|u_n|^{q+1}}{|x|^p}$$

and, since ε is arbitrary, using the definition of $C(N, p, q, h)$ we obtain that

$$c(\lambda) > c \geqslant \frac{1}{N} S^{\frac{N}{p}} - C(N, p, q, h)\,\lambda^{\frac{p}{p-q-1}}$$

which is a contradiction with the hypothesis on $c(\lambda)$. Then $\nu_j = \mu_j = 0$ for all j and $u_n \to u_0$ strongly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. ■

Notice that for λ small, $c(\lambda) > 0$, therefore since $I < 0$ we get the existence of $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $J_\lambda(u_0) = J_\lambda(|u_0|) = I < 0$. Then problem (3) has at least a positive solution for λ small.

We set

$$\mathcal{A} = \{\lambda > 0 \text{ such that problem (3) } \textit{has a positive solution}\},$$

then using Lemma 2.2 and a monotonicity argument, we can prove easily that $\mathcal{A}$ is an interval and that, for all $\lambda \in \mathcal{A}$, problem (3) has a minimal solution u_λ. We prove now that $\mathcal{A}$ is bounded. More precisely we have the following result.

THEOREM 2.6. – *Let* $\lambda^* = \sup\{\lambda \,|\, \textit{Problem (3) has solution}\}$, *then* $\lambda^* < \infty$.

Theorem 2.6 is a particular case of a result proved in [11]. We formulate here a more general theorem that extends the result in [11] and gives a more precise estimate on λ^*. Namely we consider the problem

$$\begin{cases} -\Delta_p u = \dfrac{\lambda h(x)\, u^q}{|x|^p} + g(x)\, u^{p^*-1} \ \text{ in } \mathbb{R}^N, \\ u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{9}$$

where g is a bounded positive function and q and h are as above. We set

$$\bar{\lambda}^* = \sup\{\lambda \,|\, \text{Problem (9) has solution}\}. \tag{10}$$

If the supports of h and g have nonempty intersection, it was proved in [11] that $\bar{\lambda}^* < \infty$. The following theorem states that the same result holds true in the general case.

THEOREM 2.7. – *Let* $\bar{\lambda}^*$ *be defined in* (10) *then* $\bar{\lambda}^* < \infty$.

PROOF. – When $\operatorname{supp}(h) \cap \operatorname{supp}(g) \neq \emptyset$ the result is known (see for instance [11]). We prove the result in the general case. Without loss of generality we can assume that $\lambda > 1$, if not we are done. Let u_λ be a positive solution to problem (3) with fixed λ. Then $-\Delta_p u_\lambda \geq \lambda |x|^{-p} h(x) u_\lambda^q$. Let v_1 be the unique solution to problem

$$\begin{cases} -\Delta_p v = \dfrac{h(x)}{|x|^p} v^q, \quad x \in \mathbb{R}^N, \\ v > 0, \quad v \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{11}$$

see Lemma 2.3. We set $v_\lambda = \lambda^{\frac{1}{p-(q+1)}} v_1$, then $-\Delta_p v_\lambda \leq \lambda |x|^{-p} h(x) v_\lambda^q$. Since u_λ is a supersolution to problem (3), then from Lemma 2.2 we obtain that $u_\lambda \geq v_\lambda = \lambda^{\frac{1}{p-(q+1)}} v_1$. Consider the following eigenvalue problem

$$\begin{cases} -\Delta_p w = m(p^* - p)\, g(x)\, u_\lambda^{p^*-p} |w|^{p-2} w \quad \text{in } \mathbb{R}^N, \\ w \in \mathcal{D}^{1,p}(\mathbb{R}^N). \end{cases}$$

Let m_1 be the first eigenvalue and w_1 the corresponding normalized eigenfunction. Then we have

$$m_1 = \min_{w \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^p\, dx}{\displaystyle\int_{\mathbb{R}^N} (p^* - p)\, g(x)\, u_\lambda^{p^*-p} |w|^p\, dx}.$$

Since $u_\lambda^{p^*-p} \in L^{\frac{N}{p}}(\mathbb{R}^N)$ and $u_\lambda > 0$, the minimum is achieved. Now by using

Theorem 2.1 we obtain that

$$\int_{\mathbb{R}^N} |\nabla w_1|^p \, dx - \int_{\mathbb{R}^N} \frac{-\Delta_p u_\lambda}{u_\lambda^{p-1}} w_1^p \geqslant 0 \, .$$

Since $-\Delta_p u_\lambda \geqslant g(x)\, u_\lambda^{p^*-1}$ we conclude that

$$\int_{\mathbb{R}^N} |\nabla w_1|^p \, dx - \int_{\mathbb{R}^N} g(x)\, w_1^p u_\lambda^{p^*-p} \geqslant 0 \, .$$

By the definition of w_1 we get

$$\int_{\mathbb{R}^N} |\nabla w_1|^p = m_1 (p^* - p) \int_{\mathbb{R}^N} g(x) w_1^p u_\lambda^{p^*-p}.$$

Therefore we obtain

$$m_1 \geqslant \frac{1}{p^* - p} \, .$$

Using the definition of m_1 we obtain that

$$\frac{1}{p^* - p} \leqslant m_1 \leqslant \inf_{w \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla w|^p \, dx}{(p^* - p) \int_{\mathbb{R}^N} g(x)\, u_\lambda^{p^*-p} \, |w|^p \, dx} \, .$$

Since $u_\lambda \geqslant \lambda^{\frac{1}{p-(q+1)}} v_1$, we have

$$1 \leqslant \inf_{w \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla w|^p \, dx}{\lambda^{\frac{p^*-p}{p-(q+1)}} \int_{\mathbb{R}^N} g(x)\, v_1^{p^*-p} \, |w|^p} \, .$$

So we get

$$\lambda^{\frac{p^*-p}{p-(q+1)}} \leqslant \inf_{w \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla w|^p \, dx}{\int_{\mathbb{R}^N} g(x)\, v_1^{p^*-p} \, |w|^p \, dx} = \overline{m} \, .$$

Then $\lambda^{\frac{p^*-p}{p-(q+1)}} \leqslant \overline{m}$ where $\overline{m}$ is the first eigenvalue to problem

$$\begin{cases} -\Delta_p w = m(g(x)\, v_1^{p^*-p})\,|w|^{p-2} w \ \text{ in } \mathbb{R}^N, \\ w \in \mathcal{D}^{1,p}(\mathbb{R}^N). \end{cases}$$

Then $\overline{\lambda}^* < \overline{m}^{\frac{p-(q+1)}{p^*-p}}$, and the proof is complete. ■

To prove that $\lambda^* \in \mathcal{C}$ the following lemma is in order.

LEMMA 2.8. – *Let u_λ be the minimal solution to problem* (3), *then* $J_\lambda(u_\lambda) < 0$.

PROOF. – Fixed $\lambda_0 \in \mathcal{C}$ and let u_{λ_0} the minimal solution to (3) with $\lambda = \lambda_0$. Let

$$M = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N),\ v_{\lambda_0} \leqslant u \leqslant u_{\lambda_0}\},$$

where v_{λ_0} is the unique positive solution to problem

$$\begin{cases} -\Delta_p w = \lambda_0 \dfrac{h(x)}{|x|^p} w^q & \text{in } \mathbb{R}^N, \\ w > 0, & \text{in } \mathbb{R}^N, \quad w \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases}$$

see Lemma 2.3. Then M is a convex closed set in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Since J_{λ_0} is weakly lower semi continuous, bounded from below, and coercive in M, then we get the existence of $w_0 \in M$ such that $\min_M J_{\lambda_0}(u) = J_{\lambda_0}(w_0)$. Hence for all $v \in M$ we have

$$\int_{\mathbb{R}^N} |\nabla w_0|^{p-2} \nabla w_0 \nabla (v - w_0)\, dx \geqslant \int_{\mathbb{R}^N} \left(\frac{\lambda_0 h(x)\, w_0^q}{|x|^p} + w_0^{p^*-1} \right)(v - w_0), \tag{12}$$

and $v_{\lambda_0} \leqslant w_0 \leqslant u_{\lambda_0}$. We claim that $w_0 = u_{\lambda_0}$. Since $u_{\lambda_0} = \lim_{n\to\infty} u_n$ where u_n is defined by $u_0 = v_{\lambda_0}$ and

$$\begin{cases} -\Delta_p u_{n+1} = \dfrac{\lambda_0 h(x)\, u_n^q}{|x|^p} + u_n^{p^*-1} \ \text{ in } \mathbb{R}^N, \\ u_n > 0 \ \text{ in } \mathbb{R}^N, \quad u_n \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{13}$$

we have just to prove that $u_n \leqslant w_0$ for all n. If $n = 0$ the result is verified by the definition of w_0. Let $v_1 = w_0 + (u_1 - w_0)_+$. Since $v_{\lambda_0} \leqslant u_1 \leqslant u_{\lambda_0}$, then $v_1 \in M$ and by using (12) we obtain that

$$\int_{\mathbb{R}^N} |\nabla w_0|^{p-2} \nabla w_0 \nabla (u_1 - w_0)_+\, dx \geqslant \int_{\mathbb{R}^N} \left(\frac{\lambda_0 h(x)\, w_0^q}{|x|^p} + w_0^{p^*-1} \right)(u_1 - w_0)_+.$$

Taking $(u_1 - w_0)_+$ as a test function in (13) with $n = 0$ we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - w_0)_+ \, dx = \int_{\mathbb{R}^N} \left(\frac{\lambda_0 h(x) \, v_{\lambda_0}^q}{|x|^p} + v_{\lambda_0}^{p^*-1} \right) (u_1 - w_0)_+ .$$

Then by using the fact that $v_{\lambda_0} \leqslant w_0$ we conclude that

$$\int_{\mathbb{R}^N} \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla w_0|^{p-2} \nabla w_0 \right) \cdot \nabla (u_1 - w_0)_+ \, dx \leqslant 0 . \tag{14}$$

We set $D_p(x, y) = |x|^{p-2}x - |y|^{p-2}y$ where $x, y \in \mathbb{R}^N$, then we have the following inequality (see [22])

$$\langle D_p(x, y), x - y \rangle \geqslant \begin{cases} C_p \, |x - y|^p & \text{if } p \geqslant 2 , \\ C_p \, \dfrac{|x - y|^2}{(|x| + |y|)^{2-p}} & \text{if } p < 2 . \end{cases} \tag{15}$$

Therefore, by (14) and using (15), we conclude that $(u_1 - w_0)_+ = 0$ and then $u_1 \leqslant w_0$. Since the sequence $\{u_n\}$ is increasing, the result follows by an induction argument. Therefore $u_n \leqslant w_0$ and we conclude that $u_{\lambda_0} \leqslant w_0$. Hence $w_0 = u_{\lambda_0}$. Since $J_{\lambda_0}(w_0) \leqslant J_{\lambda_0}(v_{\lambda_0}) < 0$, we conclude that $J_{\lambda_0}(u_{\lambda_0}) < 0$. ∎

We get now the following existence result.

LEMMA 2.9. – $\lambda^* \in \mathcal{A}$.

PROOF. – Let $\{\lambda_n\}$ be an increasing sequence such that $\lambda_n \uparrow \lambda^*$. Denote by u_λ the minimal solution to problem (3). From Lemma 2.8, we know that $J_{\lambda_n}(u_{\lambda_n}) < 0$, which implies $\|u_{\lambda_n}\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \leqslant M$. Since the sequence $\{u_{\lambda_n}\}$ is an increasing sequence, we get the existence of $u_{\lambda^*} = \lim_{n \to \infty} u_{\lambda_n}$ which is a solution to (9) with $\lambda = \lambda^*$. ∎

In the case in which $h \equiv 1$, we have the following nonexistence result.

LEMMA 2.10. – *Let* u_0 *be a solution to the following problem*

$$\begin{cases} -\Delta_p u = \dfrac{\lambda u^q}{|x|^p} + u^{p^*-1} \;\; in \;\; \mathbb{R}^N, \\ u > 0 \;\; in \;\; \mathbb{R}^N, \quad u \in \mathcal{D}^{1,p}_{\mathrm{loc}}(\mathbb{R}^N), \end{cases} \tag{16}$$

where $0 < q < p - 1$, *then* $u_0 \equiv 0$.

PROOF. – For $R \geq 1$, let us consider the problem

$$(17) \qquad \begin{cases} -\Delta_p u = \dfrac{\lambda u^q}{|x|^p} + u^{p^*-1} \text{ in } B_R(0), \\ u > 0 \text{ in } B_R(0), \quad u_{|\partial B_R(0)} = 0 . \end{cases}$$

Let $\lambda_R^* = \max\{\lambda > 0 : \text{problem (17) has a solution}\}$. By a rescaling argument we can prove that $\lambda_R^* = R^{-\frac{p}{p^*-p}(p-q-1)}\lambda_1^*$, hence $\lambda_R^* \to 0$ as $R \to \infty$. Let u_0 be a positive solution to (16), then there exists $R_0 \gg 1$ such that $\lambda_R^* = R^{-\frac{p}{p^*-p}(p-q-1)}\lambda_1^* < \lambda$ for $R \geq R_0$. Since u_0 is a super solution to (17) and v_λ, the solution of

$$(18) \qquad \begin{cases} -\Delta_p v_\lambda = \dfrac{\lambda v_\lambda^q}{|x|^p} \text{ in } B_R(0), \\ v_\lambda > 0 \text{ in } B_R(0), \quad v_{\lambda|\partial B_R(0)} = 0 , \end{cases}$$

is a subsolution of (17) such that $v_\lambda \leq u_0$, then by an iteration argument we can prove that problem (17) has a positive solution w such that $v_\lambda \leq w \leq u_0$ which is a contradiction with the definition of λ_R^*. Hence we conclude. ∎

3. – The critical case related to Hardy inequality.

3.1. *Existence result.*

In this section we will study problem (1) with $h \equiv g \equiv 1$ and $q = p-1$, i.e.

$$(19) \qquad \begin{cases} -\Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + u^{p^*-1}, \quad x \in \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases}$$

where $p^* = \dfrac{pN}{N-p}$ and $0 < \lambda < \left(\dfrac{N-p}{p}\right)^p$. As a consequence of a Pohozaev type identity, one can see that problem (19) does not have nontrivial solution in any bounded starshaped domain with respect to the origin, see Lemma 3.7 of [12]. This motivates the work in $\mathbb{R}^N$.

The case $p = 2$ has been studied in [25], where it is shown that problem (19) (for $p = 2$) has a one dimensional manifold of positive solutions given by $z_\mu(r) = \mu^{\frac{-(N-2)}{2}} z_\lambda\left(\dfrac{r}{\mu}\right)$ where

$$z_\lambda(x) = \frac{c_N}{\left(|x|^{1-\nu_\lambda}(1+|x|^{2\nu_\lambda})\right)^{\frac{N-2}{2}}},$$

$$\nu_\lambda = \left(1 - \frac{4\lambda}{(N-2)^4}\right)^{\frac{1}{2}} \quad \text{and} \quad c_N = \left(N(N-2)\,\nu_\lambda^2\right)^{\frac{N-2}{2}} .$$

We will partially extend the result of [25] to the case of the p-laplacian, namely we will describe the behaviour of all radial positive solutions to equation (19). We set

$$Q_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^p\,dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,dx \tag{20}$$

and

$$K = \left\{ u \in D^{1,p}(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} |u|^{p^*}\,dx = 1 \right\}.$$

Let

$$A(\lambda) = \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_\lambda(u)}{\int_{\mathbb{R}^N} |x|^{-p}|u|^p\,dx}.$$

The first result of this section is the following lemma.

LEMMA 3.1. – *Assume that* $A(\lambda) < 0$, *then problem* (19) *has no positive solution.*

PROOF. – Arguing by contradiction, assume that $A(\lambda) < 0$ and problem (19) has a positive solution u. Then since $A(\lambda) < 0$ there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $Q_\lambda(\phi) < 0$, i.e.

$$\int_{\mathbb{R}^N} |\nabla \phi|^p - \lambda \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p} < 0\,.$$

Since $\phi \in C_0^\infty(\mathbb{R}^N)$, from Theorem 2.1 we obtain that

$$\int_{\mathbb{R}^N} |\nabla \phi|^p \geqslant \int_{\mathbb{R}^N} \frac{-\Delta_p u}{u^{p-1}} |\phi|^p.$$

Therefore we get

$$\int_{\mathbb{R}^N} |\nabla \phi|^p - \lambda \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p} \geqslant \int_{\mathbb{R}^N} u^{p^*-p}|\phi|^p \geqslant 0\,,$$

which yields a contradiction with the choice of ϕ. The proof is thereby complete. ■

LEMMA 3.2. – *Assume that* $A(\lambda) > 0$, *then* $Q_\lambda(u)$ *is an equivalent norm to the norm of the space* $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Set

$$S_\lambda = \inf_{u \in K} Q_\lambda(u). \tag{21}$$

It is easy to see that $S_\lambda > 0$ and $S_\lambda < S$ where S is the best Sobolev constant for the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. We prove now the following existence result.

THEOREM 3.3. – *Assume that* $\lambda \in \left(0, \left(\frac{N-p}{p}\right)^p\right)$, *then there exists* $u_0 \in K$ *such that* $S_\lambda = Q_\lambda(u_0)$. *In particular there exists a positive constant* c *such that* cu_0 *is a positive solution of* (19).

PROOF. – Let $\{u_n\}$ be a minimizing sequence to (21). Since $\lambda \in \left(0, \left(\frac{N-p}{p}\right)^p\right)$ and by classical Hardy inequality, we get that $\{u_n\}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore using the concentration-compactness principle, see [15], we get the existence of a sequence of positive numbers $\{\sigma_n\}$ such that the sequence $\bar{u}_n = \sigma_n^{-\frac{N-p}{N}} u_n\left(\frac{\cdot}{\sigma_n}\right)$ is relatively compact in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. The sequence $\{\bar{u}_n\}_n$ is also a minimizing one. We can get easily that $u_0 = \lim_{n \to \infty} \bar{u}_n \in K$ and $Q_\lambda(u_0) = S_\lambda$.

Moreover u_0 satisfies the following Euler-Lagrange equation

$$-\Delta_p u - \lambda \frac{u^{p-1}}{|x|^p} = S_\lambda u^{p^*-1}. \tag{22}$$

If we set $v = cu_0$ where $c = S_\lambda^{\frac{1}{p^*-p}}$ then v is a solution of (19). ■

Now we have the following result concerning the regularity of solutions to (19).

REMARK 3.4. – *Let* u *be any solution of* (19), *then* $u \in C^{1,\alpha}(\mathbb{R}^N - \{0\})$.

PROOF. – Let u_0 be any solution. For $0 < \varepsilon < R$, we set $\Omega = B(R) \backslash B(\varepsilon)$ where $B(\varepsilon)$ (resp. $B(R)$) is the ball in $\mathbb{R}^N$ of center 0 and radius ε (resp. R). Since $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, then $u_0 \in W^{1/p',p}(\partial B(\varepsilon))$ and $u_0 \in W^{1/p',p}(\partial B(R))$. Since u_0 is a solution to problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + u^{p^*-1}, \quad x \in \Omega \\ u|_{\partial B(R)} = u_0|_{\partial B(R)}, \\ u|_{\partial B(R(\varepsilon))} = u_0|_{\partial B(R(\varepsilon))}, \\ u > 0 \text{ in } \Omega, \quad u \in W^{1,p}(\Omega), \end{cases} \tag{23}$$

from [24] we get that $u_0 \in C^{1,\alpha}(\Omega)$. Since ε and R are arbitrary, we obtain the desired result. ■

It is easy to check that all dilations of u_0 of the form $\sigma^{-\frac{N-p}{N}} u_0\left(\frac{\cdot}{\sigma}\right)$ where $\sigma > 0$ are also solutions of the minimizing problem (21). Therefore we get a family of solutions to problem (19). Moreover we have the following characterization of minimizers in problem (21).

LEMMA 3.5. – *All minimizers of* (21) *are radial.*

PROOF. – Since if $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a minimizer of S_λ (i.e $K(u_0) = 1$ and $Q(u_0) = S_\lambda$) then the decreasing rearrangement u_0^* of u_0 given by

$$u_0^*(x) = \inf\{t > 0 : |\{y \in \mathbb{R}^N : u(y) > t\}| \leq \omega_N |x|^N\}$$

where ω_N denotes the volume of the standard unit N-sphere (see [20]), is also a minimizer, so it satisfies the same Euler-Lagrange equation i.e

$$-\Delta_p u_0^* - \lambda \frac{(u_0^*)^{p-1}}{|x|^p} = S_\lambda (u_0^*)^{p^*-1}. \tag{24}$$

Notice that by the classical result by Polya-Szegö (see [20]) we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_0|^p \, dx \geq \int_{\mathbb{R}^N} |\nabla u_0^*|^p \, dx .$$

Since u_0^* is a solution to (24) we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_0^*|^p \, dx = \int_{\mathbb{R}^N} \left(\lambda \frac{(u_0^*)^p}{|x|^p} + S_\lambda (u_0^*)^{p^*}\right) dx \geq$$

$$\int_{\mathbb{R}^N} \left(\lambda \frac{|u_0|^p}{|x|^p} + S_\lambda |u_0|^{p^*}\right) dx = \int_{\mathbb{R}^N} |\nabla u_0|^p \, dx .$$

Hence we conclude that $\int_{\mathbb{R}^N} |\nabla u_0^*|^p \, dx = \int_{\mathbb{R}^N} |\nabla u_0|^p \, dx$. Notice that u_0^* is strictly increasing, then $|\{\nabla u_0^* = 0\}| = 0$. Then from [8], there exists $x_0 \in \mathbb{R}^N$ such that $u_0(\cdot) = u_0^*(\cdot + x_0)$. Since equation (22) is not invariant by translation we obtain that $x_0 = 0$ and the result follows. ■

3.2 *The behavior of the radial solutions.*

We study now the asymptotic behavior of all radial solutions of the problem (19).

Let $u(r)$ be a radial positive solution of (19), then

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}\left(\lambda \frac{u^{p-1}}{r^p} + u^{p^*-1}\right) = 0\,. \tag{25}$$

We set

$$t = \log r, \quad y(t) = r^{\delta} u(r) \quad \text{and} \quad z(t) = r^{(1+\delta)(p-1)}|u'(r)|^{p-2}u'(r), \tag{26}$$

where $\delta = \dfrac{N-p}{p}$.

Then using the equation (25) we obtain the following system in y and z

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{N-p}{p} y + |z|^{\frac{2-p}{p-1}} z\,, \\ \dfrac{dz}{dt} = -\dfrac{N-p}{p} z - |y|^{p^*-2} y - \lambda |y|^{p-2} y\,. \end{cases} \tag{27}$$

Notice that by a direct calculus we obtain easily that y satisfies the following nonlinear equation

$$(p-1)|\delta y - y'|^{p-2}\{\delta y' - y''\} + \delta|\delta y - y'|^{p-2}\{\delta y - y'\} - \lambda y^{p-1} - y^{p^*-1} = 0\,. \tag{28}$$

By the initial equation of u we conclude that $r^{N-1}|u'(r)|^{p-2}u'(r)$ is a strictly decreasing function, then it has a limit as $r \to 0$.

Since $\nabla u \in L^p(\mathbb{R}^N)$, such a limit must be 0, hence $r^{N-1}|u'(r)|^{p-2}u'(r) < 0$ and then $u'(r) < 0$, which yields $z < 0$.

The stationary points of the system are $P_1 = (0, 0)$ and $P_2 = (y_0, z_0)$ where

$$y_0 = \left\{\left(\frac{N-p}{p}\right)^p - \lambda\right\}^{\frac{N-p}{p^2}} \quad \text{and} \quad z_0 = -\left(\frac{N-p}{p}\right)^{p-1} y_0^{p-1}.$$

The complete integral of the system is given by

$$V(y, z) \equiv \frac{1}{p^*}|y|^{p^*} + \frac{\lambda}{p}|y|^p + \frac{p-1}{p}|z|^{\frac{p}{p-1}} + \frac{N-p}{p} yz. \tag{29}$$

We set $V(t) = V(y(t), z(t))$. Since $\dfrac{\partial V(t)}{\partial t} = 0$ for all $t \in \mathbb{R}$, we get that

$$V(t) = V(y(t), z(t)) = K_0 \tag{30}$$

for some real constant K_0.

LEMMA 3.6. – *y and z are bounded.*

PROOF. – By Young inequality, (29), and (30), we obtain that

$$\frac{1}{p^*}|y|^{p^*} + \frac{\lambda}{p}|y|^p - \frac{|\delta y|^p}{p} \leqslant K_0,$$

from which we can conclude that y is bounded in $\mathbb{R}$. Again by Young inequality we have that for any $\varepsilon > 0$ there exists C_ε such that

$$|y(t)\, z(t)| \leqslant \varepsilon |z(t)|^{\frac{p}{p-1}} + C_\varepsilon |y(t)|^p.$$

Hence from (30) and (29) we have

$$K_0 \geqslant \frac{p-1}{p}|z(t)|^{\frac{p}{p-1}} - \delta\varepsilon |z(t)|^{\frac{p}{p-1}} - \delta C_\varepsilon |y(t)|^p.$$

Therefore, taking ε small enough, from the boundedness of $y(t)$ we deduce that z is also bounded. ■

The following lemma states that $K_0 = 0$.

LEMMA 3.7. – *For any $t \in \mathbb{R}^N$*

$$(y(t), z(t)) \in \left\{(y, z) \in \mathbb{R}^2 \colon V(y, z) = 0\right\}.$$

PROOF. – Let us define the following even function

$$\phi(s) = K_0 + \frac{\delta^p - \lambda}{p}|s|^p - \frac{1}{p^*}|s|^{p^*}. \tag{31}$$

It is easy to obtain that ϕ is strictly increasing in $[0, s_0]$ and strictly decreasing in $[s_0, \infty)$ where $s_0 = (\delta\, p - \lambda)^\delta$ and $\phi(s_0) = K_0 + K_1$ where $K_1 = \frac{1}{N}(\delta^p - \lambda)^{N/p}$. Since $\phi(y(t)) \geqslant 0$ we obtain that $K_0 \geqslant -K_1$. We have four cases

1. $K_0 = -K_1$;
2. $-K_1 < K_0 < 0$;
3. $K_0 > 0$;
4. $K_0 = 0$.

In the first case the maximum of ϕ is zero but since $\phi(y(t)) \geqslant 0$ we obtain that $y(t) = s_0$ and $u(r) = \frac{s_0}{r^\delta} \notin \mathcal{D}^{1,p}(\mathbb{R}^N)$. In the second case, i.e. $-K_1 < K_0 < 0$, let s_1 be the first zero of ϕ, then s_1 is strictly positive and $y(t) \geqslant s_1$ for all $t \in \mathbb{R}$, hence $u \notin \mathcal{D}^{1,p}(\mathbb{R}^N)$. In order to exclude the third case let us observe that if

$K_0 > 0$, then ϕ vanishes only at a positive value b. If $\bar{t}$ is a critical point of y, i.e. $y'(\bar{t}) = 0$, then from (27) and the negativity of z, we obtain that

$$\delta y(\bar{t}) = |z(\bar{t})|^{\frac{1}{p-1}}. \tag{32}$$

From (29), (30), and (32), it follows that $\phi(y(\bar{t})) = 0$. Hence $y(\bar{t}) = b$. Hence all the stationary points of y must stay on the same level $b > 0$. From this fact and the integrability condition on u, it follows that y must be strictly increasing for $t \leqslant -R$ for some large $R > 0$. In particular there exists $\lim_{t \to -\infty} y(t)$ and by integrability of u such limit must be 0. Since $y(t) \to 0$ as $t \to -\infty$ and $z(t)$ is bounded, from (29) and (30), we deduce that there exists $\ell = \lim_{t \to -\infty} z(t)$ and

$$K_0 = \frac{p-1}{p} |\ell|^{\frac{p}{p-1}}.$$

On the other hand from the second equation in (27), we infer that ℓ must be 0, which is not possible if $K_0 > 0$. Hence the only possible case is case 4, i.e. $K_0 = 0$. The conclusion follows from $K_0 = 0$ and (30). ■

LEMMA 3.8. – *There exists $t_0 \in \mathbb{R}$ such that $y(t)$ is strictly increasing for $t < t_0$ and strictly decreasing for $t > t_0$. Moreover*

$$\max_{t \in \mathbb{R}^N} y(t) = y(t_0) = \left[\frac{N}{N-p} (\delta^p - \lambda) \right]^{1/(p^* - p)}. \tag{33}$$

PROOF. – In view of the integrability condition on u and since y is a strictly positive function, to conclude it is enough to show that y has only one critical point. Arguing as above, it is possible to show that if $y'(\bar{t}) = 0$ then $\phi(y(\bar{t})) = 0$, where the function ϕ is defined in (31). Since $K_0 = 0$, ϕ has only two zeros, which are $s = 0$ and $s = b$, where

$$b = \left[\frac{N}{N-p} (\delta^p - \lambda) \right]^{1/(p^* - p)}.$$

Since y is strictly positive, we deduce that $y(\bar{t}) = b$. Hence all the critical points of y must stay on the same level $b > 0$. As a consequence, if y has two distinct critical points $t_1 < t_2$, it must be $y(t) = b$ for any $t_1 \leqslant t \leqslant t_2$, hence $y'(t) = 0$ for all $t \in [t_1, t_2]$. Therefore, using (27) we conclude that $z(t) = -(\delta b)^{p-1}$ for all $t \in [t_1, t_2]$ and then $z'(t) = 0$ for all $t \in (t_1, t_2)$. Now in view of Lemma 3.7 and from (27) we obtain that $z'(t) = -\dfrac{p^* - p}{p^*} y^{p^*-1}(t) < 0$ for all $t \in (t_1, t_2)$ a contradiction with the fact that $z'(t) = 0$ in (t_1, t_2).

Hence we conclude that y has only a critical point t_0, which must be a global maximum point in view of the integrability of u and the positivity of y. Moreover $\max_{\mathbb{R}^N} y = y(t_0) = b$. ■

Since the system (27) is autonomous, then modulo translation we can assume that $t_0 = 0$. Using (28) we get

$$|\delta y - y'|^{p-2}\{\delta y - y'\} = \mathrm{e}^{-\delta t}\int_{-\infty}^{t} \mathrm{e}^{\delta s}(\lambda y^{p-1}(s) + y^{p^*-1}(s))\, ds\,. \tag{34}$$

Hence we conclude that $\delta y - y' > 0$. The following result gives the exact behavior of y as $t \to \pm\infty$.

LEMMA 3.9. – *Suppose that y is a positive solution of* (28) *such that y is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$, then there exist positive constants c_1, c_2, such that*

$$\lim_{t\to-\infty} \mathrm{e}^{(l_1-\delta)t} y(t) = y(0)\, c_1 > 0 \tag{35}$$

$$\lim_{t\to\infty} \mathrm{e}^{(l_2-\delta)t} y(t) = y(0) c_2 > 0 \tag{36}$$

where l_1, l_2 are the zeros of the function $\xi(s) = (p-1)\, s^p - (N-p)\, s^{p-1} + \lambda$ such that $0 < l_1 < l_2$.

PROOF. – It is easy to see that $l_1 < \delta < l_2$. Let us now prove (35). Using (27) we obtain that

$$\frac{d}{dt}(\mathrm{e}^{-(\delta-l_1)t} y(t)) = \mathrm{e}^{-(\delta-l_1)t} y(t)\left(l_1 - \frac{|z(t)|^{\frac{1}{p-1}}}{y(t)}\right).$$

Therefore we get

$$\mathrm{e}^{-(\delta-l_1)t} y(t) = y(0)\, \mathrm{e}^{-\int_t^0 (l_1 - y(s)^{-1}|z(s)|^{1/(p-1)})ds}. \tag{37}$$

We set $H(s) = \dfrac{|z(s)|^{\frac{1}{p-1}}}{y(s)}$. We claim that

(38) $\quad H$ is an increasing function from $(-\infty, 0]$ to $(l_1, \delta]$.

To prove the claim, we first show that $H'(s) > 0$ for all $s < 0$. Indeed, assume by contradiction that there exists $s_0 < 0$ such that $H'(s_0) \leqslant 0$. Since

$$H'(s) = \frac{-\dfrac{1}{p-1} y(s) z'(s)\,|z(s)|^{\frac{2-p}{p-1}} - |z(s)|^{\frac{1}{p-1}}\, y'(s)}{y^2(s)}$$

from $H'(s_0) \leqslant 0$, (27), and (29), it follows that $\left(\dfrac{1}{p} - \dfrac{1}{p^*}\right) y^{p^*}(s_0) \leqslant 0$ which yields a contradiction with the positivity of y. Therefore $H' > 0$ and then H is a strictly increasing function. Using (27) and the fact that $y'(0) = 0$, we find that $H(0) = (N-p)/p$. From (29) we conclude that $\lim_{s \to -\infty} H(s) = l_1$. The claim is thereby proved.

From (37) and (38) we conclude that $e^{-(\delta - l_1)t} y(t)$ is a decreasing function, therefore there exists $\lim_{t \to -\infty} e^{-(\delta - l_1)t} y(t)$ and

$$\alpha \equiv \lim_{t \to -\infty} e^{-(\delta - l_1)t} y(t) = y(0)\, e^{-\int_{-\infty}^{0} (H(s) - l_1)\, ds} > 0 .$$

Hence to prove (35) it is enough to show that $\alpha < +\infty$. To this aim let us note that from a direct computation

$$H'(s) = -\frac{p}{(p-1)(N-p)} H(s)^{2-p} \xi(H(s))$$

where ξ is given by $\xi(s) = (p-1)\, s^p - (N-p)\, s^{p-1} + \lambda$. Thus performing the change of variable $\sigma = H(s)$, we have $d\sigma = H'(s) ds \equiv \varrho(\sigma)\, ds$ where $\varrho(\sigma) = -\frac{p}{(p-1)(N-p)} \sigma^{2-p} \xi(\sigma)$. We can write $\varrho(\sigma) = (\sigma - l_1)(\sigma - l_2)\, g(\sigma)$ where g is a negative function such that $|g(\sigma)| \geqslant \text{const} > 0$ for $\sigma \in [l_1, (N-p)/p]$. Therefore we obtain

$$\alpha = \lim_{t \to -\infty} e^{-(\delta - l_1)t} y(t) = y(0)\, e^{-\int_{-\infty}^{0} (H(s) - l_1)\, ds} = y(0)\, e^{-\int_{l_1}^{\delta} [(\sigma - l_2) g(\sigma)]^{-1} d\sigma} .$$

Since $l_2 > \delta$ and $|g(\sigma)| \geqslant c_1$ if $\sigma \in [l_1, (N-p)/p]$, we conclude that $\int_{l_1}^{\delta} \frac{1}{(\sigma - l_2)\, g(\sigma)} d\sigma < +\infty$, hence $\alpha < +\infty$. The proof of (36) can be done observing that $\lim_{t^{+\infty}} H(t) = l_2$ and using the same argument. ■

In the following corollary we translate the results above to energy solutions u of equation (25), namely to radial solutions of (19) in the energy space $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

COROLLARY 3.10. – *Let u be a positive energy solution to* (25), *then there exist positive constants C_1 and C_2 such that*

$$\lim_{r \to 0} r^{l_1} u(r) = C_1 > 0 , \tag{39}$$

$$\lim_{r \to \infty} r^{l_2} u(r) = C_2 > 0 \tag{40}$$

and

$$\lim_{r \to 0} r^{l_1 + 1} |u'(r)| = C_1 l_1 > 0 \quad \textit{and} \quad \lim_{r \to +\infty} r^{l_2 + 1} |u'(r)| = C_2 l_2 > 0 . \tag{41}$$

PROOF. – (39) and (40) follow from (35), (36), and (26), while (41) follows from (26) and the fact that $\lim_{t \to -\infty} H(t) = l_1$ and $\lim_{t \to +\infty} H(t) = l_2$. ■

Notice that since $\lim_{s \to -\infty} H(s) = l_1$ and $\lim_{t \to -\infty} e^{(l_1 - \delta)t} y(t) = y(0)\, c_1$, we obtain that

$$\lim_{t \to -\infty} e^{(l_1 - \delta)t} |z(t)|^{\frac{1}{p-1}} = c_1 y(0)\, l_1 > 0 \,, \tag{42}$$

and since $\lim_{s \to +\infty} H(s) = l_2$

$$\lim_{t \to +\infty} e^{(l_2 - \delta)t} |z(t)|^{\frac{1}{p-1}} = c_2 y(0) l_2 > 0 \,. \tag{43}$$

The uniqueness in the case of bounded solutions to quasilinear equations could be seen in [10]. We state and prove now the uniqueness result for energy positive solutions to problem (25), that requires a different approach based on the previous analysis.

THEOREM 3.11. – *Let $u_1(r)$ and $u_2(r)$ be two positive energy solutions to equation* (19). *Let us denote by $(y_1(t), z_1(t))$ and $(y_2(t), z_2(t))$ the solutions to system* (27) *corresponding to u_1 and u_2 respectively. Assume that*

$$\max_{t \in (-\infty,\, \infty)} y_1(t) = y_1(0) = \left(\frac{N}{N-p} \right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}} \,.$$

If $y_2(0) = y_1(0)$, then $(y_1(t), z_1(t)) = (y_2(t), z_2(t))$ and hence $u_1 = u_2$.

Before proving the above uniqueness result, we state the main consequence of Theorem 3.11.

THEOREM 3.12. – *Let $u_1(r)$ be the fixed energy solution to* (19) *such that, if $(y_1(t), z_1(t))$ is the solution to system* (27) *corresponding to u_1, then*

$$\max_{t \in (-\infty,\, \infty)} y_1(t) = y(0) = \left(\frac{N}{N-p} \right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}} \,.$$

Then for any other solution v there exists $\mu_0 > 0$ such that $v(r) = \mu_0^{-(N-p)/p} u_1(r/\mu_0)$.

PROOF. – Let $(y_2(t), z_2(t))$ be the solution to system (27) corresponding to v. From Lemma 3.8, there exists $t_0 \in (-\infty,\, \infty)$ such that

$$\max_{t \in (-\infty,\, \infty)} y_2(t) = y(t_0) = \left(\frac{N}{N-p} \right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}} \,.$$

We set $\overline{y}_2(t) = y(t - t_0)$ and $\overline{z}_2(t) = z_2(t - t_0)$. Notice that

$$\max_{t \in (-\infty,\, \infty)} \overline{y}_2(t) = \overline{y}_2(0) = \left(\frac{N}{N-p} \right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}} .$$

Using the fact that the system (27) is autonomous we obtain that $(\overline{y}_2, \overline{z}_2)$ is also a solution to (27). Since $\overline{y}_2(0) = y_1(0)$, from Theorem 3.11 we obtain that $(\overline{y}_2(t), \overline{z}_2(t)) = (y_1(t), z_1(t))$. Hence from (26) we conclude that

$$u_1(r) = \frac{1}{e^{\delta t_0}} v\left(\frac{r}{e^{t_0}} \right).$$

Therefore we conclude that $v(r) = \mu_0^{-\delta} u_1(r/\mu_0)$ where $\mu = e^{-t_0}$. ∎

PROOF OF THEOREM 3.11. – Let u_1, u_2 be two solutions to problem (19) and let $(y_1(t), z_1(t))$, $(y_2(t), z_2(t))$ be the solutions to system (27) corresponding to u_1 and u_2 respectively such that

$$\max_{t \in (-\infty,\, \infty)} y_1(t) = y_1(0) = \left(\frac{N}{N-p} \right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}} .$$

Assume that $y_2(0) = y_1(0)$. From Lemma 3.8 we know that y_2 has a unique maximum point t_0 at which $y_2(t_0) = (N(\delta^p - \lambda)/(N-p))^{1/(p^*-p)}$. Since $y_2(0) = y_1(0) = (N(\delta^p - \lambda)/(N-p))^{1/(p^*-p)}$ we conclude that $t_0 = 0$. Hence $y_2'(0) = 0$. From (27) we get

$$e^{-\delta t} y(t) = y(0) - \int_0^t e^{-\delta\sigma} |z(\sigma)|^{\frac{1}{p-1}} d\sigma .$$

Hence we obtain that

$$|y_1(t) - y_2(t)| \leq e^{\delta t} \int_0^t e^{-\delta\sigma} \left| \, |z_1(\sigma)|^{\frac{1}{p-1}} - |z_2(\sigma)|^{\frac{1}{p-1}} \right| d\sigma .$$

Since from (27) we have that $z_1(0) = z_2(0) = -(\delta y_1(0))^{p-1}$, we get the existence of $\sigma_1 > 0$ such that for all $\sigma \in [0, \sigma_1]$ we have

$$\left| \, |z_1(\sigma)|^{\frac{1}{p-1}} - |z_2(\sigma)|^{\frac{1}{p-1}} \right| \leq C(\sigma_1) |z_1(\sigma) - z_2(\sigma)| .$$

Therefore we conclude that

$$|y_1(t) - y_2(t)| \leq e^{\delta t} C(\sigma_1) \int_0^t e^{-\delta\sigma} |z_1(\sigma) - z_2(\sigma)| d\sigma .$$

Now from (27) we obtain that

$$e^{\delta\sigma} z_i(\sigma) = z_1(0) - \int_0^\sigma e^{\delta s}[\lambda y_i^{p-1}(s) + y_i^{p^*-1}(s)]\, ds\, .$$

Hence

$$|z_1(\sigma) - z_2(\sigma)| \leq \lambda e^{-\delta\sigma} \int_0^\sigma e^{\delta s} |y_1^{p-1}(s) - y_2^{p-1}(s)| ds +$$

$$e^{-\delta\sigma} \int_0^\sigma e^{\delta s} |y_1^{p^*-1}(s) - y_2^{p^*-1}(s)| ds\, .$$

As above, we can prove the existence of $s_1 > 0$ such that for $s \in [0, s_1]$ we have

$$|y_1^{p-1}(s) - y_2^{p-1}(s)| \leq C_1(s_1) |y_1(s) - y_2(s)|$$

and

$$|y_1^{p^*-1}(s) - y_2^{p^*-1}(s)| \leq C_2(s_1) |y_1(s) - y_2(s)|\, .$$

Hence

$$|z_1(\sigma) - z_2(\sigma)| \leq \widetilde{C}(s_1)(\lambda + 1)\, e^{-\delta\sigma} \int_0^\sigma e^{\delta s} |y_1(s) - y_2(s)| ds\, .$$

Therefore, if $0 \leq t \leq \min\{\sigma_1, s_1\}$ we obtain that

$$|y_1(t) - y_2(t)| \leq e^{\delta t} C \int_0^t e^{-2\delta\sigma} \left\{ \int_0^\sigma e^{\delta s} |y_1(s) - y_2(s)| ds \right\} d\sigma$$

where $C = C(\sigma_1)\, \widetilde{C}(s_1)(\lambda + 1)$, and hence

$$|y_1(t) - y_2(t)| \leq C e^{\delta t} \int_0^t e^{\delta\sigma} |y_1(\sigma) - y_2(\sigma)| \left\{ \int_\sigma^t e^{-2\delta s} ds \right\} d\sigma\, .$$

Consequently we obtain

$$e^{-\delta t} |y_1(t) - y_2(t)| \leq C_2 \int_0^t e^{-\delta\sigma} |y_1(\sigma) - y_2(\sigma)| d\sigma\, .$$

Therefore, using Gronwall Lemma we conclude that $y_1(t) = y_2(t)$ for $t \in [0, \min\{\sigma_1, s_1\}]$ and then $u_1(r) = u_2(r)$ in $[1, r_0]$ where $r_0 > 1$. To prove the identity for all $r \geq 0$ it is enough to iterate the above argument. ■

We can resume in the next statement the main results obtained in this section.

THEOREM 3.13. – *All positive radial solutions of* (19) *are*

$$u(\cdot) = \sigma^{-\frac{N-p}{p}} u_0\left(\frac{\cdot}{\sigma}\right)$$

where u_0 *is the unique solution of* (19) *such that* $u_0(1) = y(0) = \left(\frac{N}{N-p}\right)^{\frac{1}{p^*-p}} (\delta^p - \lambda)^{\frac{1}{p^*-p}}$. *Moreover there exist constants* $C_1,\ C_2 > 0$ *such that*

$$0 < C_1 \leqslant \frac{u_0(x)}{\left(|x|^{l_1/\delta} + |x|^{l_2/\delta}\right)^{-\delta}} \leqslant C_2 .$$

4. – Existence result for perturbed problems.

4.1. *Perturbation in the linear term.*

In this section we will prove some existence and nonexistence results in the case $q = p-1$, extending to the p-laplacian operator the analogous results obtained in [1] for $p = 2$. Let us start by considering the case of a perturbed coefficient of the Hardy-type potential, i.e. we deal with the following problem

$$(44)\qquad \begin{cases} -\Delta_p u = \dfrac{\lambda + h(x)}{|x|^p} u^{p-1} + u^{p^*-1}, & x \in \mathbb{R}^N, \\ u > 0 \ \text{ in } \mathbb{R}^N, \ \text{ and } u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases}$$

where $N \geqslant 3$ and $p^* = \dfrac{pN}{N-p}$. Hypotheses on h will be given below.

4.2. *Nonexistence results.*

The following nonexistence results show how in this kind of problems both the size and the shape of the perturbation are important. We set

$$(45)\qquad \begin{cases} Q(u) = \displaystyle\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} |u|^p\, dx , \\ \mathcal{K} = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \,\middle|\, \displaystyle\int_{\mathbb{R}^N} |u|^{p^*} dx = 1 \right\}, \end{cases}$$

and consider $I_1 = \inf\limits_{u \in \mathcal{K}} Q(u)$.

LEMMA 4.1. – *Problem* (44) *has no positive solution in each one of the following cases*:

(1) *if* $\lambda + h(x) \geq 0$ *in some ball* $B_\delta(0)$ *and* $I_1 < 0$;

(2) *if* h *is a differentiable function such that* $\langle h'(x), x\rangle$ *has a fixed sign.*

PROOF. – Let us first prove nonexistence under hypothesis (1). Suppose that $I_1 < 0$ and let u be a positive solution to (44). By classical regularity results for elliptic equations we obtain that $u \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$. On the other hand, since $\lambda + h(x) \geq 0$ in $B_\delta(0)$, we obtain that $-\Delta_p u \geq 0$ in the distributional sense in the ball $B_\delta(0)$. Therefore, as $u \geq 0$ and $u \neq 0$, by the strong maximum principle we obtain that $u(x) \geq c > 0$ in some ball $B_{\tilde\delta}(0) \subset\subset B_\delta(0)$.

Let $\phi_n \in C_0^\infty(\mathbb{R}^N)$, $\phi_n \geq 0$, $\|\phi_n\|_{p^*} = 1$, be a minimizing sequence of I_1. Using Theorem 2.1 we obtain that

$$\int_{\mathbb{R}^N} |\nabla \phi_n|^p \, dx \geq \int_{\mathbb{R}^N} \frac{-\Delta_p u}{u^{p-1}} |\phi_n|^p \, dx \, .$$

Hence

$$\int_{\mathbb{R}^N} |\nabla \phi_n|^p \, dx \geq \int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} \phi_n^p + \int_{\mathbb{R}^N} \phi_n^p u^{p^*-p}.$$

On the other hand, $I_1 < 0$ implies that there exists an integer n_0 such that if $n \geq n_0$

$$\int_{\mathbb{R}^N} |\nabla \phi_n|^p - \int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} \phi_n^p < 0 \, .$$

As a consequence $\int_{\mathbb{R}^N} \phi_n^p u^{p^*-p} < 0$ for $n \geq n_0$, which is a contradiction with the hypothesis $u > 0$.

Let us now prove (2). Testing the equation with the Pohozaev multiplier, we obtain that any positive solution u to (44) satisfies the following identity

$$\int_{\mathbb{R}^N} \frac{\langle h'(x), x\rangle}{|x|^p} |u|^p \, dx = 0 \, ,$$

which is not possible if $\langle h'(x), x\rangle$ has a fixed sign and $u \not\equiv 0$. ■

COROLLARY 4.2. – *Assume either*

i) $\lambda > \Lambda_{N,p}$ *and* $h \geq 0$, *or*

ii) $\lambda > \Lambda_{N,p}$ *and* $1 \leq \dfrac{\lambda}{\Lambda_{N,p}\|h\|_\infty}$,

then problem (44) *has no positive solution.*

4.3. *The local Palais-Smale condition: existence results.*

Existence results will be obtained through a variational approach. More precisely we look for critical points of the associated functional

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \frac{1}{p}\int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p}|u|^p\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\,dx\,. \tag{46}$$

We suppose that h verifies the following hypotheses

$(h0)$ $\lambda + h(0) > 0$,

$(h1)$ $h \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$(h2)$ for some $c_0 > 0$, $\lambda + \|h\|_\infty \leq \Lambda_{N,p} - c_0$.

Solutions to equation (44) can be found as critical points of J in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. The following theorem yields a local Palais-Smale condition for J.

THEOREM 4.3. – *Suppose that h satisfies $(h0)$, $(h1)$, and $(h2)$ and denote $h(\infty) \equiv \limsup_{|x|\to\infty} h(x)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence for J, namely*

$$J(u_n) \to c < \infty \quad\text{and}\quad J'(u_n) \to 0\,.$$

If

$$c < c^* = \frac{1}{N}\min\left\{S^{N/p}_{(\lambda + h(0))},\, S^{N/p}_{(\lambda + h(\infty))}\right\}$$

where $S^{N/p}_{(\lambda + h(0))}$ and $S^{N/p}_{(\lambda + h(\infty))}$ are defined in (21), *then $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.*

PROOF. – Let $\{u_n\}_n$ be a Palais-Smale sequence for J, then according to $(h1)-(h2)$, $\{u_n\}_n$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore, up to a subsequence, $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, $u_n \to u_0$ a.e., and $u_n \to u_0$ in $L^\alpha_{loc}(\mathbb{R}^N)$, $\alpha \in [1, p^*)$. Hence, by the *Concentration Compactness Principle* by P. L. Lions (see [15] and [16]), there exists a subsequence still denoted by $\{u_n\}_n$ and an at most countable set $\mathcal{J}$ such that

1. $|\nabla u_n|^p \rightharpoonup d\mu \geq |\nabla u_0|^p + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} + \mu_0 \delta_0$,
2 $|u_n|^{p^*} \rightharpoonup d\nu = |u_0|^{p^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} + \nu_0 \delta_0$,
3. $S\nu_j^{\frac{p}{p^*}} \leq \mu_j$ for all $j \in \mathcal{J} \cup \{0\}$,
4. $\dfrac{u_n^p}{|x|^p} \rightharpoonup d\gamma = \dfrac{u_0^p}{|x|^p} + \gamma_0 \delta_0$,
5. $\Lambda_N \gamma_0 \leq \mu_0$.

To study the concentration at infinity of the sequence, we also need to introduce the following quantities

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*}dx, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p dx$$

and

$$\gamma_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{|u_n|^p}{|x|^p} dx .$$

We claim that $\mathfrak{J}$ is finite and that for any $j \in \mathfrak{J}$ either $\nu_j = 0$ or $\nu_j \geqslant S^{N/2}$. We follow closely the arguments in [6] (see also [1]). Let $\varepsilon > 0$ and let ϕ be a smooth cut-off function centered at x_j such that $0 \leqslant \phi(x) \leqslant 1$,

$$\phi(x) = \begin{cases} 1, & \text{if } |x-x_j| \leqslant \varepsilon/2 , \\ 0, & \text{if } |x-x_j| \geqslant \varepsilon , \end{cases}$$

and $|\nabla\phi| \leqslant 4/\varepsilon$. Testing $J'(u_n)$ with $u_n\phi$ we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle J'(u_n), u_n\phi\rangle \\ &= \lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p\phi + \int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi - \int_{\mathbb{R}^N} \frac{\lambda+h(x)}{|x|^p}|u_n|^p\phi - \int_{\mathbb{R}^N}\phi|u_n|^{p^*}\right). \end{aligned}$$

From 1), 2) and 4) and since $0 \notin \operatorname{supp}(\phi)$ we find that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p\phi = \int_{\mathbb{R}^N}\phi\, d\mu, \quad \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\phi = \int_{\mathbb{R}^N}\phi\, d\nu ,$$

and

$$\lim_{n\to\infty} \int_{B_\varepsilon(x_j)} \frac{\lambda+h(x)}{|x|^p}|u_n|^p\phi = \int_{B_\varepsilon(x_j)} \frac{\lambda+h(x)}{|x|^p}|u_0|^p\phi .$$

Taking limits as $\varepsilon\to 0$ we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n||\nabla u_n|^{p-1}|\nabla\phi| \to 0 .$$

Hence

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n), u_n\phi\rangle \geqslant \mu_j - \nu_j .$$

By 3) we have that $S\nu_j^{\frac{p}{p^*}} \leqslant \mu_j$, then we obtain that either $\nu_j = 0$ or $\nu_j \geqslant S^{N/p}$, which implies that $\mathfrak{J}$ is finite. The claim is proved.

Let us now study the possibility of concentration at $x=0$ and at ∞. Let ψ be a regular function such that $0 \leqslant \psi(x) \leqslant 1$,

$$\psi(x) = \begin{cases} 1, & \text{if } |x| > R+1 \\ 0, & \text{if } |x| < R, \end{cases}$$

and $|\nabla\psi| \leqslant 4/R$. From (21) we obtain that

$$\frac{\int_{\mathbb{R}^N} |\nabla(u_n \psi)|^p \, dx - (\lambda + h(\infty)) \int_{\mathbb{R}^N} \frac{|\psi u_n|^p}{|x|^p} \, dx}{\left(\int_{\mathbb{R}^N} |\psi u_n|^{p^*} \right)^{p/p^*}} \geqslant S_{(\lambda + h(\infty))}. \tag{47}$$

Hence

$$\int_{\mathbb{R}^N} |\nabla(u_n \psi)|^p \, dx - (\lambda + h(\infty)) \int_{\mathbb{R}^N} \frac{|\psi u_n|^p}{|x|^p} \, dx \geqslant S_{(\lambda + h(\infty))} \left(\int_{\mathbb{R}^N} |\psi u_n|^{p^*} \right)^{p/p^*}.$$

Therefore we conclude that

$$\int_{\mathbb{R}^N} |\psi \nabla u_n + u_n \nabla\psi|^p \, dx \geqslant (\lambda + h(\infty)) \int_{\mathbb{R}^N} \frac{|\psi u_n|^p}{|x|^p} \, dx + S_{(\lambda + h(\infty))} \left(\int_{\mathbb{R}^N} |\psi u_n|^{p^*} \right)^{p/p^*}. \tag{48}$$

We claim that

$$\lim_{R \to \infty} \limsup_{n \to \infty} \left\{ \int_{\mathbb{R}^N} |\psi \nabla u_n + u_n \nabla\psi|^p \, dx - \int_{\mathbb{R}^N} \psi^p |\nabla u_n|^p \, dx \right\} = 0. \tag{49}$$

Indeed from the following elementary inequality

$$\big| |X+Y|^p - |X|^p \big| \leqslant C(|X|^{p-1}|Y| + |Y|^p) \text{ for all } X, Y \in \mathbb{R}^N,$$

it follows that

$$\int_{\mathbb{R}^N} \big| |\psi \nabla u_n + u_n \nabla\psi|^p - \psi^p |\nabla u_n|^p \big| \, dx \leqslant C \int_{\mathbb{R}^N} (|\psi \nabla u_n|^{p-1} |u_n \nabla\psi| + |u_n \nabla\psi|^p) \, dx.$$

From Hölder inequality we obtain

$$\int_{\mathbb{R}^N} |u_n|\,|\psi\nabla u_n|^{p-1}\,|\nabla\psi|\,dx \leqslant$$

$$\Bigg(\int_{R<|x|<R+1} |u_n|^p\,|\nabla\psi|^p\,dx\Bigg)^{\frac{1}{p}}\Bigg(\int_{R<|x|<R+1} |\nabla u_n|^p\,dx\Bigg)^{\frac{p-1}{p}}.$$

Hence

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|\,\psi^{p-1}\,|\nabla u_n|^{p-1}\,|\nabla\psi|\,dx$$

$$\leqslant C\Bigg(\int_{R<|x|<R+1} |u_0|^p\,|\nabla\psi|^p\,dx\Bigg)^{\frac{1}{p}}$$

$$\leqslant C\Bigg(\int_{R<|x|<R+1} |u_0|^{p^*}\,dx\Bigg)^{\frac{p}{p^*}}\Bigg(\int_{R<|x|<R+1} |\nabla\psi|^N\,dx\Bigg)^{\frac{p}{N}}$$

$$\leqslant \overline{C}\Bigg(\int_{R<|x|<R+1} |u_0|^{p^*}\,dx\Bigg)^{p/p^*}.$$

Therefore we conclude that

$$\lim_{R\to\infty}\ \limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|\,\psi^{p-1}\,|\nabla u_n|^{p-1}\,|\nabla\psi|\,dx \leqslant$$

$$\overline{C}\lim_{R\to\infty}\Bigg(\int_{R<|x|<R+1} |u_0|^{p^*}\,dx\Bigg)^{p/p^*} = 0\,.$$

Using the same argument we can prove that

$$\lim_{R\to\infty}\ \limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^p\,|\nabla\psi|^p\,dx = 0\,.$$

The claim is thereby proved. From (48) and (49), we deduce that

$$\mu_\infty - (\lambda + h(\infty))\,\gamma_\infty \geqslant S_{(\lambda + h(\infty))}\,\nu_\infty^{p/p^*}. \tag{50}$$

Since $\lim_{R\to\infty}\ \lim_{n\to\infty}\langle J'(u_n),\, u_n\psi\rangle = 0$, we obtain that $\mu_\infty - (\lambda + h(\infty))\,\gamma_\infty \leqslant \nu_\infty$. Therefore we conclude that either $\nu_\infty = 0$ or $\nu_\infty \geqslant S_{(\lambda + h(\infty))}^{\frac{N}{p}}$. The same holds for the concentration at $x_0 = 0$, namely that either

$$\nu_0 = 0 \quad \text{or} \quad \nu_0 \geqslant S_{(\lambda + h(0))}^{\frac{N}{p}}.$$

As a conclusion we obtain

$$c = J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle + o(1)$$

$$= \frac{1}{N}\int_{\mathbb{R}^N} |u_n|^{p^*} dx + o(1) = \frac{1}{N}\left\{ \int_{\mathbb{R}^N} |u_0|^{p^*} dx + \nu_0 + \nu_\infty + \sum_{j \in \mathcal{J}} \nu_j \right\}.$$

If we assume the existence of $j \in \mathcal{J} \cup \{0, \infty\}$ such that $\nu_j \neq 0$, then we obtain that $c \geq c^*$, a contradiction with the hypothesis. Hence, up to a subsequence, $u_n \to u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. ■

To find solutions through the Mountain Pass Theorem, we need to find some path in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ along which the maximum of $J(\gamma(t))$ is strictly below c^*. To this aim, we set $H = \max\{h(0), h(\infty)\}$ and consider $\{w_\mu\}$ the one parameter family of minimizers to problem (21) where λ is replaced by $\lambda + H$. The following theorem provides a sufficient condition for the minimax level to stay below the critical threshold c^*.

THEOREM 4.4. – *Suppose that* $(h1)$ *and* $(h2)$ *hold. Assume the existence of* $\mu_0 > 0$ *such that*

$$\int_{\mathbb{R}^N} h(x)\frac{w_{\mu_0}^p(x)}{|x|^p}\,dx > H\int_{\mathbb{R}^N} \frac{w_{\mu_0}^p(x)}{|x|^p}\,dx\,, \tag{51}$$

then (44) *has at least a positive solution.*

PROOF. – Let μ_0 be as in the hypothesis, then if we set

$f(t) = J(tw_{\mu_0}) =$

$$\frac{t^p}{p}\left(\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^p\,dx - \int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} w_{\mu_0}^p\,dx\right) - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} |w_{\mu_0}|^{p^*}dx, \quad t \geq 0$$

we can see easily that f achieves its maximum at some $t_0 > 0$ and that there exists some $\varrho > 0$ such that $J(tw_{\mu_0}) < 0$ if $\|tw_{\mu_0}\| \geq \varrho$. A simple calculation yields

$$t_0 = \left[\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^p\,dx - \int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} w_{\mu_0}^p\,dx}{\displaystyle\int_{\mathbb{R}^N} |w_{\mu_0}|^{p^*}dx}\right]^{(N-p)/p^2}$$

and

$$J(t_0 w_{\mu_0}) = \max_{t \geqslant 0} J(tw_{\mu_0}) = \frac{1}{N}\left[\frac{\int\limits_{\mathbb{R}^N} |\nabla w_{\mu_0}|^p\,dx - \int\limits_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p} w_{\mu_0}^p\,dx}{\left(\int\limits_{\mathbb{R}^N} |w_{\mu_0}|^{p^*}\,dx\right)^{p/p^*}}\right]^{N/p}.$$

Using (51) we obtain that

$$J(t_0 w_{\mu_0}) < \frac{1}{N}\left[\frac{\int\limits_{\mathbb{R}^N} |\nabla w_{\mu_0}|^p\,dx - (\lambda + H)\int\limits_{\mathbb{R}^N} \frac{w_{\mu_0}^p}{|x|^p}\,dx}{\left(\int\limits_{\mathbb{R}^N} |w_{\mu_0}|^{p^*}\,dx\right)^{p/p^*}}\right]^{N/p} = \frac{1}{N} S_{(\lambda+H)}^{\frac{N}{p}} \leqslant c^*.$$

We set

$$\Gamma = \{\gamma \in C([0,1], \mathcal{D}^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0 \ \text{ and } \ J(\gamma(1)) < 0\}.$$

Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Since $J(t_0 w_{\mu_0}) < c^*$, then we get a mountain pass critical point u_0. Then we have just to prove that we can choose $u_0 \geqslant 0$. Consider the Nehari manifold

$$M \equiv \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : u \neq 0 \ \text{ and } \ \langle J'(u), u\rangle = 0\}$$

$$= \left\{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : u \neq 0 \ \text{ and } \int\limits_{\mathbb{R}^N} |\nabla u|^p\,dx = \int\limits_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^p}|u|^p\,dx + \int\limits_{\mathbb{R}^N} |u|^{p^*}\,dx\right\}.$$

Notice that u_0, $|u_0| \in M$. Since u_0 is a mountain pass solution to problem (44), then one can prove easily that $c \equiv J(u_0) = \min\limits_{u \in M} J(u)$ (see [27]). Hence $J(|u_0|) = \min\limits_{u \in M} J(u)$ and then $|u_0|$ is a critical point of J. Therefore by using the strong maximum principle by J. L. Vázquez, see [26], we conclude that $u_0 > 0$. ∎

REMARK 4.5. – *It is immediate to see that hypothesis* (51) *is satisfied for example in the case in which* $h(0) = h(\infty) = \min\limits_{x \in \mathbb{R}^N} h(x)$ *and* $h \not\equiv \text{const}$.

4.4. *Perturbation in the nonlinear term.*

In this section we deal with problem (19) with a perturbed coefficient of the nonlinear term, namely we study the following problem

$$\begin{cases} -\Delta_p u = \dfrac{\lambda}{|x|^p} u^{p-1} + k(x) u^{p^*-1}, & x \in \mathbb{R}^N, \\ u > 0 \ \text{ in } \mathbb{R}^N, \ \text{ and } u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{52}$$

where $N \geqslant 3$, $0 < \lambda < \Lambda_{N,p}$ and k is a positive function.

4.5. *Existence.*

Assume that k verifies the following hypothesis

$$(K0) \qquad k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \quad \text{and} \quad \|k\|_\infty > \max\{k(0), k(\infty)\},$$

where $k(\infty) \equiv \limsup_{|x| \to \infty} k(x)$. Let

$$J_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} k(x) |u|^{p^*} dx,$$

then critical points of J_λ are solutions to equation (52). Arguing as in Subsection 4.3, we can prove that Palais-Smale condition is satisfied below some level as stated in the following lemma.

LEMMA 4.6. – *Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence for J_λ, namely*

$$J_\lambda(u_n) \to c < \infty \quad \text{and} \quad J'_\lambda(u_n) \to 0.$$

If

$$c < \tilde{c}(\lambda) = \frac{1}{N} \min\left\{ S^{\frac{N}{p}} \|k\|_\infty^{-\frac{N-p}{p}}, S_\lambda^{\frac{N}{p}} (k(0))^{-\frac{N-p}{p}}, S_\lambda^{\frac{N}{p}} (k(\infty))^{-\frac{N-p}{p}} \right\}$$

then $\{u_n\}_{n \in \mathbb{N}}$ has a converging subsequence.

Since the proof is similar to the proof of Theorem 4.3, we omit it. If k is a radial positive function, we can prove the following improved Palais-Smale condition.

LEMMA 4.7. – *Define*

$$\tilde{c}_1(\lambda) = \frac{1}{N} S_\lambda^{\frac{N}{p}} \min\left\{ (k(0))^{-\frac{N-p}{p}}, (k(\infty))^{-\frac{N-p}{p}} \right\}.$$

If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *is a Palais-Smale sequence for* J_λ, *namely* $J_\lambda(u_n) \to c$, $J'_\lambda(u_n) \to 0$, *and* $c < \tilde{c}_1$, *then* $\{u_n\}_{n\in\mathbb{N}}$ *has a converging subsequence.*

We define

$$b(\lambda) \equiv \begin{cases} +\infty & \text{if } k(0) = k(\infty) = 0 \\ \dfrac{1}{N} S_\lambda^{N/p} \min\left\{k(0)^{-\frac{N-p}{p}}, k(\infty)^{-\frac{N-p}{p}}\right\} & \text{otherwise.} \end{cases}$$

LEMMA 4.8. – *If* $(K0)$ *holds, there exists* $\varepsilon_0 > 0$ *such that* $\frac{1}{N} S^{N/p} \|k\|_\infty^{-(N-p)/p} \leq b(\lambda)$ *for all* $\lambda \leq \varepsilon_0$ *and*

$$\tilde{c}(\lambda) = \tilde{c} \equiv \frac{1}{N} S^{N/p} \|k\|_\infty^{-\frac{N-p}{p}} \tag{53}$$

for any $0 < \lambda \leq \varepsilon_0$.

PROOF. – From $(K0)$ and by the fact that $S_\lambda \to S$ as $\lambda \to 0$, it follows that if λ is sufficiently small then $\frac{1}{N} S^{N/p} \|k\|_\infty^{-\frac{N-p}{p}} \leq b(\lambda)$ and hence the result follows. ∎

As a consequence we obtain the following existence result.

THEOREM 4.9. – *Let k be a positive function such that* $(K0)$ *is satisfied. Assume that there exists* $\mu_0 > 0$ *such that*

$$\int_{\mathbb{R}^N} k(x)\, w_{\mu_0}^{p^*}(x)\, dx > \max\{k(0), k(\infty)\} \int_{\mathbb{R}^N} w_{\mu_0}^{p^*}(x)\, dx, \tag{54}$$

where w_{μ_0} *is a solution to problem*

$$\begin{cases} -\Delta_p w = \dfrac{\lambda}{|x|^p} w^{p-1} + w^{p^*-1}, & x \in \mathbb{R}^N, \\ w > 0 \ \ in \ \mathbb{R}^N, \ \text{ and } w \in \mathcal{D}^{1,p}(\mathbb{R}^N). \end{cases}$$

Then (52) *has at least a positive solution.*

PROOF. – Since the proof is similar to the proof of Theorem 4.4 we omit it. ∎

5. – Multiplicity of positive solutions.

To find multiplicity results for problem (52) we need the following extra hypotheses on k

($K1$) the set $\mathcal{C}(k) = \left\{ a \in \mathbb{R}^N \mid k(a) = \max_{x \in \mathbb{R}^N} k(x) \right\}$ is finite, say $\mathcal{C}(k) = \{ a_j \mid 1 \leq j \leq \mathrm{Card}\,(\mathcal{C}(k)) \}$;

($K2$) there exists $\theta \in \left(p, \dfrac{N}{p-1} \right)$ such that if $a_j \in \mathcal{C}(k)$ then $k(a_j) - k(x) = o(|x - a_j|)^\theta$ as $x \to a_j$.

Consider $0 < r_0 \ll 1$ such that $B_{r_0}(a_j) \cap B_{r_0}(a_i) = \emptyset$ for $i \neq j$, $1 \leq i$, $j \leq \mathrm{Card}\,(\mathcal{C}(k))$. Let $\delta = \dfrac{r_0}{3}$ and for any $1 \leq j \leq \mathrm{Card}\,(\mathcal{C}(k))$ define the following function

$$T_j(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \psi_j(x) \, |\nabla u|^p \, dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p \, dx} \quad \text{where } \psi_j(x) = \min\{1, |x - a_j|\}. \tag{55}$$

For the proof of the following separation lemma we refer to [1].

LEMMA 5.1. – *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$, such that $T_i(u) \leq \delta$ and $T_j(u) \leq \delta$, then $i = j$.*

Consider now the Nehari manifold,

$$M(\lambda) = \{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : u \not\equiv 0 \text{ and } \langle J'_\lambda(u), u \rangle = 0 \}, \tag{56}$$

namely $u \in M(\lambda)$ if and only if $u \not\equiv 0$ and

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \, dx = \int_{\mathbb{R}^N} k(x) \, |u|^{p^*} \, dx .$$

Notice that for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $u \not\equiv 0$, there exists $t > 0$ with $tu \in M(\lambda)$ and for all $u \in M(\lambda)$ we have

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \, dx < \frac{p^* - 1}{p - 1} \int_{\mathbb{R}^N} k(x) \, |u|^{p^*} \, dx . \tag{57}$$

Therefore we can prove easily the existence of $c_1 > 0$ such that

$$\forall u \in M(\lambda), \qquad \|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \geq c_1 .$$

DEFINITION 5.2. – *For any* $0 < \lambda < \Lambda_N$ *and* $1 \leq j \leq \mathrm{Card}(\mathcal{C}(k))$, *let us consider*

$M_j(\lambda) =$

$$\{u \in M(\lambda) : T_j(u) < \delta\} \text{ and its boundary } \Gamma_j(\lambda) = \{u \in M(\lambda) : T_j(u) = \delta\}.$$

We define

$$m_j(\lambda) = \inf\{J_\lambda(u) : u \in M_j(\lambda)\} \text{ and } \eta_j(\lambda) = \inf\{J_\lambda(u) : u \in \Gamma_j(\lambda)\}.$$

The following two lemmas give the behaviour of the functional with respect to the critical level $\tilde{c}$. The proofs can be obtained with a small modification of the arguments used in [1].

LEMMA 5.3. – *Suppose that* $(K0)$, $(K1)$, *and* $(K2)$ *hold, then* $M_j(\lambda) \neq \emptyset$ *and there exists* $\varepsilon_1 > 0$ *such that*

$$m_j(\lambda) < \tilde{c} \quad \text{for all } 0 < \lambda \leq \varepsilon_1 \text{ and } 1 \leq j \leq \mathrm{Card}(\mathcal{C}(k)). \tag{58}$$

LEMMA 5.4. – *Suppose that* $(K0)$, $(K1)$, *and* $(K2)$ *are satisfied, then there exists* ε_2 *such that for all* $0 < \lambda < \varepsilon_2$ *there holds*

$$\tilde{c} < \eta_j(\lambda).$$

We need now the following lemma that is suggested by the work of Tarantello [23]. See also [9].

LEMMA 5.5. – *Assume that* $\lambda < \min\{\varepsilon_1, \varepsilon_2\}$ *where* ε_1, ε_2 *are given by Lemmas* 5.3 *and* 5.4. *Then for all* $u \in M_j(\lambda)$ *there exists* $\varrho_u > 0$ *and a differentiable function*

$$f : B(0, \varrho_u) \subset \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}$$

such that $f(0) = 1$ *and for all* $w \in B(0, \varrho_u)$ *there holds* $f(w)(u - w) \in M_j(\lambda)$. *Moreover for all* $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *we have*

$$\langle f'(0), v\rangle = - \frac{p \displaystyle\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx - p\lambda \int_{\mathbb{R}^N} \frac{|u|^{p-2} uv}{|x|^p} dx - p^* \int_{\mathbb{R}^N} k(x) |u|^{2^*-2} uv dx}{(p-1)\left[\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx\right] - (p^*-1) \int_{\mathbb{R}^N} k(x) |u|^{p^*} dx}. \tag{59}$$

PROOF. – Let $u \in M_j(\lambda)$ and let $G: \mathbb{R} \times \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be the function defined by

$$G(t, w) = t^{p-1}\left(\int_{\mathbb{R}^N} |\nabla(u-w)|^p dx - \lambda \int_{\mathbb{R}^N} \frac{|u-w|^p}{|x|^p} dx \right) - t^{p^*-1} \int_{\mathbb{R}^N} k(x)|u-w|^{p^*} dx .$$

Then $G(1, 0) = 0$ and

$$G_t(1, 0) = (p-1)\left[\int_{\mathbb{R}^N} |\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \right] - (p^*-1) \int_{\mathbb{R}^N} k(x)|u|^{p^*} dx \neq 0$$

in view of (57). Then by using the Implicit Function Theorem we get the existence of $\varrho_u > 0$ small enough and of a differentiable function $f: B(0, \varrho_u) \subset \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ such that $f(0) = 1$ and $G(f(w), w) = 0$ for all $w \in B(0, \varrho_u)$, which implies that $f(w)(u-w) \in M_j(\lambda)$. Moreover, we have

$$\langle f'(0), v \rangle = -\frac{\langle G_w(1,0), v\rangle}{G_t(1,0)}$$

$$= -\frac{p\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v dx - p\lambda \int_{\mathbb{R}^N} \frac{|u|^{p-2}uv}{|x|^p} dx - p^* \int_{\mathbb{R}^N} k(x)|u|^{p^*-2}uv dx}{(p-1)\left[\int_{\mathbb{R}^N} |\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \right] - (p^*-1) \int_{\mathbb{R}^N} k(x)|u|^{p^*} dx} .$$

The proof is thereby complete. ■

We are now in position to prove the main result of this section.

THEOREM 5.6. – *Assume that* $(K0)$, $(K1)$, *and* $(K2)$ *hold, then there exists* ε_3 *small such that for all* $0 < \lambda < \varepsilon_3$ *equation* (52) *has* $\mathrm{Card}(\mathcal{C}(k))$ *positive solutions* $u_{j,\lambda}$ *such that*

$$(60) \quad |\nabla u_{j,\lambda}|^p \to S^{N/p} \|k\|_\infty^{-(N-p)/p} \delta_{a_j} \ \textit{and} \ |u_{j,\lambda}|^{p^*} \to S^{N/p} \|k\|_\infty^{-N/p} \delta\, a_j \ \text{as } \lambda \to 0 .$$

PROOF. – Assume that $0 < \lambda < \varepsilon_3 = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, where ε_0, ε_1 and ε_2 are given by the Lemmas 4.8, 5.3 and 5.4. Let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for J_λ in $M_j(\lambda)$, i.e. $u_n \in M_j(\lambda)$ and $J_\lambda(u_n) \to m_j(\lambda)$ as $n \to \infty$. Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we can choose $u_n \geq 0$. It is not difficult to prove the existence of c_1, c_2 such that $c_1 \leq \|u_n\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \leq c_2$. By the Ekeland variational principle we get the existence of a subsequence denoted also by $\{u_n\}$ such that

$$J_\lambda(u_n) \leqslant m_j(\lambda) + \frac{1}{n} \text{ and } J_\lambda(w) \geqslant J_\lambda(u_n) - \frac{1}{n}\|w - u_n\| \text{ for all } w \in M_j(\lambda).$$

Let $0 < \varrho < \varrho_n \equiv \varrho_{u_n}$ and $f_n \equiv f_{u_n}$, where ϱ_{u_n} and f_{u_n} are given by Lemma 5.5. We set $v_\varrho = \varrho v$ where $\|v\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = 1$, then $v_\varrho \in B(0, \varrho_n)$ and we can apply Lemma 5.5 to obtain that $w_\varrho = f_n(v_\varrho)(u_n - v_\varrho) \in M_j(\lambda)$. Therefore we get

$$\frac{1}{n}\|w_\varrho - u_n\| \geqslant J_\lambda(u_n) - J_\lambda(w_\varrho) = \langle J_\lambda'(u_n), u_n - w_\varrho\rangle + o(\|u_n - w_\varrho\|)$$
$$\geqslant \varrho f_n(\varrho v)\langle J_\lambda'(u_n), v\rangle + o(\|u_n - w_\varrho\|).$$

Hence we conclude that

$$\langle J_\lambda'(u_n), v\rangle \leqslant \frac{1}{n}\frac{\|w_\varrho - u_n\|}{\varrho f_n(\varrho v)}(1 + o(1)).$$

Since $|f_n(\varrho v)| \to |f_n(0)| \geqslant c$ as $\varrho \to 0$ and

$$\frac{\|w_\varrho - u_n\|}{\varrho} = \frac{\|f_n(0)u_n - f_n(\varrho v)(u_n - \varrho v)\|}{\varrho}$$
$$\leqslant \frac{\|u_n\| |f_n(0) - f_n(\varrho v)| + |\varrho| |f_n(\varrho v)|}{\varrho} \leqslant C|f_n'(0)| \|v\| + c_3 \leqslant c.$$

Therefore we conclude that $J_\lambda'(u_n) \to 0$ as $n \to \infty$. Hence $\{u_n\}$ is a Palais-Smale sequence for J_λ. Since $m_j(\lambda) < \tilde{c}$ and $\tilde{c} = \tilde{c}(\lambda)$ for $\lambda \leqslant \varepsilon_0$, then from Lemma 4.6 we get the existence result.

Let us now prove (60). Assume $\lambda_n \to 0$ as $n \to \infty$ and let $u_n \equiv u_{j_0, \lambda_n} \in M_{j_0}(\lambda_n)$ be a solution to problem (52) with $\lambda = \lambda_n$. Then up to a subsequence we get the existence of $\ell > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|u_n|^{p^*}\,dx = \ell.$$

From Sobolev inequality, it follows that $\ell \geqslant S^{N/p}\|k\|_\infty^{-\frac{N-p}{p}}$. On the other hand since $u_n \in M(\lambda_n)$ we have

$$\frac{\ell}{N} + o(1) = J_{\lambda_n}(u_n) \leqslant \frac{1}{N}S^{N/p}\|k\|_\infty^{-\frac{N-p}{p}} + o(1)$$

which yields $\ell \leqslant S^{N/p}\|k\|_\infty^{-\frac{N-p}{p}}$. Therefore $\ell = S^{N/p}\|k\|_\infty^{-\frac{N-p}{p}}$ and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\|k\|_\infty - k(x))|u|_n^{p^*}\,dx = 0.$$

We set $w_n = \frac{u_n}{\|u_n\|_{p^*}}$, then $\|w_n\|_{p^*} = 1$ and $\lim_{n\to\infty}\|w_n\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = S$. Hence we get the existence of $w_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that one of the following alternatives holds

1. $w_0 \not\equiv 0$ and $w_n \to w_0$ strongly in the $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

2 $w_0 \equiv 0$ and either

i) $|\nabla w_n|^p \rightharpoonup d\mu = S\delta_{x_0}$ and $|w_n|^{p^*} \to d\nu = \delta_{x_0}$

or

ii) $|\nabla w_n|^p \rightharpoonup d\mu_\infty = S\delta_\infty$ and $|w_n|^{p^*} \rightharpoonup d\nu_\infty = \delta_\infty$.

Arguing as in [1, Lemma 3.11] it is possible to show that the alternative 1 and the alternative 2 ii) can not hold. Then we conclude that the unique possible behaviour is the alternative 2. i), namely we get the existence of $x_0 \in \mathbb{R}^N$ such that

$$|\nabla w_n|^p \rightharpoonup d\mu = S\delta_{x_0} \text{ and } |w_n|^{p^*} \rightharpoonup d\nu = \delta_{x_0}.$$

Since

$$\int_{\mathbb{R}^N} |\nabla w_n|^p\, dx = S + o(1) = S \int_{\mathbb{R}^N} |w_n|^{p^*}\, dx + o(1) = \frac{S}{\|k\|_\infty} \int_{\mathbb{R}^N} k(x)\, |w_n|^{p^*}\, dx + o(1)$$

$$= \frac{S}{\|k\|_\infty} k(x_0) + o(1),$$

then we obtain that $x_0 \in \mathcal{C}(k)$. Using Lemma 5.1, we conclude that $x_0 = a_{j_0}$ and the result follows. ∎

6. – Further results.

In this section we use the Lusternik-Schnirelman category theory to get multiplicity results for problem (52), we refer to [4] for a complete discussion. We follow the argument by Musina see [17]. We assume that k is a nonnegative function and that $0 < \lambda < \bar{\varepsilon}_0$ where $\bar{\varepsilon}_0$ is chosen in such a way that $\left(\frac{S_{\bar{\varepsilon}_0}}{S}\right)^{N/p} > \frac{1}{2}$ and $\bar{\varepsilon}_0 \leq \varepsilon_0$, being ε_0 given in Lemma 4.8. We set for $\delta > 0$

$$\mathcal{C}(k) = \{a \in \mathbb{R}^N \,|\, k(a) = \|k(x)\|_\infty\} \text{ and } \mathcal{C}_\delta(k) = \{x \in \mathbb{R}^N : dist(x, \mathcal{C}(k)) \leq \delta\}.$$

We suppose that $(K2)$ and the following assumption

$(K3)$ there exist R_0, $d_0 > 0$ such that $\sup_{|x| > R_0} |k(x)| \leq \|k\|_\infty - d_0$

hold. Let $M(\lambda)$ be defined by (56). Consider

$$\widetilde{M}(\lambda) \equiv \{u \in M(\lambda) : J_\lambda(u) < \tilde{c}\}.$$

Then we have the following results.

LEMMA 6.1. – *Let $\{v_n\}_{n\in\mathbb{N}} \subset M(\lambda)$ be such that $J_\lambda(v_n) \to c < \tilde{c}$ and $J'_{\lambda|M(\lambda)}(v_n) \to 0$, then $\{v_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. Moreover there exists $\bar{\varepsilon}_1 > 0$ such that if $0 < \lambda < \lambda_0 := \min\{\bar{\varepsilon}_0, \bar{\varepsilon}_1\}$, then $\widetilde{M}(\lambda) \neq \emptyset$ and* *for any* $\{\lambda_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ *such that* $\lambda_n\to 0$ *as* $n\to\infty$ *and* $\{v_n\}_{n\in\mathbb{N}}\subset\widetilde{M}(\lambda_n)$, *there exist* $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ *and* $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ *such that* $x_n\to x_0\in\mathcal{C}(k)$, $r_n\to 0$ *as* $n\to\infty$ *and*

$$v_n-\left(\frac{S}{\|k\|_\infty}\right)^{\frac{N-p}{p^2}}u_{r_n}(\cdot-x_n)\to 0 \;\; in \;\; \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{61}$$

where

$$u_r(x)=\frac{C_r}{\left(r^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}} \tag{62}$$

and C_r *is the normalizing constant to be* $\|u_r\|_{p^*}=1$.

PROOF. – The proof is a direct modification of the arguments used in [1] and it will be omitted. ∎

REMARK 6.2. – *Notice that as a consequence of the above lemma we obtain the existence of at least cat*$(\widetilde{M}(\lambda))$ *solutions that eventually can change sign.*

The main result of this section is the following Theorem, for the proof of which we refer to [1].

THEOREM 6.3. – *Assume that hypotheses* $(K0)$, $(K2)$ *and* $(K3)$ *hold and let* $\delta>0$. *Then there exists* $\lambda_0>0$ *such that for all* $0<\lambda<\lambda_0$, *equation* (52) *has at least* $cat_{\mathcal{C}_\delta(k)}\mathcal{C}(k)$ *positive solutions.*

REMARK 6.4.

i) *If* $\mathcal{C}(k)$ *is finite, then for* λ *small, equation* (52) *has at least* $\mathrm{Card}(\mathcal{C}(k))$ *solutions.*

ii) *We give now a typical example where equation* (52) *has infinitely many solutions. Let* $\eta:\mathbb{R}\to\mathbb{R}_+$ *be such that* η *is regular,* $\eta(0)=0$ *and* $\eta(r)=1$ *for* $r\geq\frac{1}{2}$. *We define* k_1 *on* $[0,1]\subset\mathbb{R}$ *by*

$$k_1(r)=\begin{cases}1 & \text{if } r=0,\\ 1-\eta(r)\left|\sin\frac{1}{r}\right|^{\theta} & \text{if } 0<r\leq 1,\end{cases}$$

where $p<\theta<N$. *Notice that* k_1 *has infinitely many global maxima achieved on the set*

$$\mathcal{C}(k_1)=\left\{r_n=\frac{1}{n\pi}\text{for } n\geq 1\right\}.$$

Now we define k to be any continuous bounded function such that $k(x) = k_1(|x|)$ if $|x| \leq 1$, $\|k\|_\infty \leq 1$ and $\lim_{|x| \to \infty} k(x) = 0$. Since for all $m \in \mathbb{N}$ there exists $\delta(m)$ such that $cat(\mathcal{C})_{\mathcal{C}_\delta} = m$, then we conclude that equation (52) has at least m solutions for $\lambda < \lambda(\delta)$.

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