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Effect Algebras and Extensions of Measures.

ANNA AVALLONE - ANNA DE SIMONE - PAOLO VITOLO

Sunto. – *Si affronta il problema dell'esistenza di un'estensione tipo Carathéodory per misure modulari definite su lattice-ordered effect algebras.*

Summary. – *We investigate the existence of a Carathéodory type extension for modular measures defined on lattice-ordered effect algebras.*

1. – Introduction.

The classic Carathéodory extension theorem says that every group-valued σ-additive exhaustive measure defined on a Boolean algebra $\mathcal{A}$ can be uniquely extended to a σ-additive measure on the σ-algebra generated by $\mathcal{A}$. A similar extension theorem has been proved for a class of measures (modular functions) on orthomodular lattices in the more general setting of non-commutative measure theory (see [24], [4]) and for measures on MV-algebras in fuzzy measure theory (see [9]) (see also [13], [20], [25]).

The aim of this paper is to study the extension problem for modular measures defined on lattice-ordered effect algebras (D-lattices), structures that represent a common generalization of orthomodular lattices and MV-algebras. Effect algebras were introduced by M.K. Bennett and D. Foulis in 1994 (see [11]) with the aim of modelling unsharp measurements in a quantum mechanical system, and they belong to a large class of non-Boolean structures frequently used because of their wide range of applications: in quantum physics (see [10]), in Mathematical Economics (see [17], [19], [30]), in fuzzy theory (see [12], [23]). For an extensive list of references see [16].

In the case of classic measure theory (i.e. additive functions defined on Boolean algebras), the fundamental tool used to obtain extension theorems are some topological methods based on the theory of Fréchet-Nikodym (FN) topologies (see e.g. [15], [21], [26]). Our approach in the present paper is based on similar topological methods, which have been adapted to the specific case. In this context the role of FN-topologies is played by D-uniformities (i.e. uniformities which make the lattice operations $\vee$, $\wedge$ and the $\ominus$ operation uniformly continuous) introduced in [2]. Differently from the Boolean case, in which the outer measure associated to a given measure naturally generates an FN-topology on the Boolean algebra, for the case of D-lattices it is not clear if the outer measure generates a D-uniformity. If this happens, we are able to establish an extension theorem (Theor. 6.2) which generalizes the one proved in [4] (Theorem 2.2.1) for modular functions defined on orthomodular lattices. The use of a Bartle-Dunford-Schwartz-type theorem in D-lattices proved in [5], allows to obtain an extension result for modular measures with values in a complete locally convex Hausdorff linear space (Cor. 6.8).

The paper is organized as follows: After stating the basic definition in the following section, we prove some technical results in Section 3, which allow to characterize in terms of additive classes the σ-sublattice generated by a given lattice. In Section 3 we study monotone functions which are useful in the next section, where we introduce an outer measure associated to a given measure, and we prove that, under suitable hypothesis, it has the properties we need (it is, in fact, a σ-submeasure). The results of Sections 3 and 4 are essential tools for obtaining the extension theorems proved in the last section.

2. – Preliminaries.

In this section we briefly describe the kind of measures we use and the structures on which they are defined. Let us start by the definition of effect algebras.

DEFINITION 2.1. – *An* effect algebra $(L, \oplus, 0, 1)$ *is a structure consisting of a set* L, *two special elements* 0 *and* 1, *and a partially defined binary operation* $\oplus$ *on* $L \times L$ *satisfying the following conditions for every* $a, b, c \in L$:

(1) *If* $a \oplus b$ *is defined, then* $b \oplus a$ *is defined and* $a \oplus b = b \oplus a$.

(2) *If* $b \oplus c$ *and* $a \oplus (b \oplus c)$ *are defined, then* $a \oplus b$ *and* $(a \oplus b) \oplus c$ *are defined and* $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3) *For every* $a \in L$, *there exists a unique* $a^\perp \in L$ *such that* $a \oplus a^\perp$ *is defined and* $a \oplus a^\perp = 1$.

(4) *If* $a \oplus 1$ *is defined, then* $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \leq b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and $\leq$ is a partial ordering in L, with 0 as the smallest element.

The sum allows to define orthogonality between two elements.

DEFINITION 2.2. – *We say that two elements* $a, b \in L$ *are* orthogonal, *and we write* $a \perp b$, *if* $a \oplus b$ *exists. Then* $a \perp b$ *if and only if* $a \leq b^\perp$.

For every $a, b \in L$, we have $a^\perp = 1 \ominus a$, $a \oplus b = (b^\perp \ominus a)^\perp$, and $a \leq b$ if and only if $a^\perp \geq b^\perp$.

In a natural way the sum of more than two elements is obtained: If $a_1, ..., a_n \in L$, we inductively define $a_1 \oplus ... \oplus a_n = (a_1 \oplus ... \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements and allows us to define orthogonality among more than two elements.

DEFINITION 2.3. – *A finite subset* $\{a_1, ..., a_n\}$ *of* L *is* orthogonal *if* $a_1 \oplus ... \oplus a_n$ *exists. For a sequence* $\{a_n\}$*, we say that it is orthogonal if, for every* n, $\bigoplus_{i \leq n} a_i$ *exists. If, moreover,* $\sup_n \bigoplus_{i \leq n} a_i$ *exists, we set* $\bigoplus_n a_n = \sup_n \bigoplus_{i \leq n} a_i$.

The measures we consider are all defined on a special type of effect algebras. The definition follows.

DEFINITION 2.4. – *If* $(L, \leq)$ *is a lattice, we say that the effect algebra is a* lattice ordered effect algebra *or a* D-lattice.

The notion of σ-continuity of a D-lattice is, as usual, expressed in terms of monotone sequences: We write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is decreasing and $a = \inf_n a_n$).

DEFINITION 2.5. – *The lattice* L *is said to be* σ*-continuous if* $a_n \uparrow a$ *implies* $a_n \wedge b \uparrow a \wedge b$ *(or, equivalently,* $a_n \downarrow a$ *implies* $a_n \vee b \downarrow a \vee b$*) for every* $b \in L$. L *is said to be* σ*-complete if every countable set has a supremum and an infimum.*

Let us set $\varDelta = \{(a, b) \in L \times L : a = b\}$. If $a, b \in L$, we put $a \triangle b = (a \vee b) \ominus (a \wedge b)$. Moreover, if $a \leq b$, we denote by $[a, b]$ the interval with these as extreme points:

$$[a, b] = \{c \in L : a \leq c \leq b\}$$

Our problem is to extend a function defined on a certain sublattice of a given D-lattice, the exact definition follows.

DEFINITION 2.6. – *If* $M \subseteq L$*, we say that* M *is a D-sublattice of* L *if it is a sublattice of* L, $1 \in M$ *and, for every* $a, b \in M$ *with* $a \leq b$*, the difference* $b \ominus a$ *belongs to* M.

A sublattice M of L is a D-sublattice if and only if, $0 \in M$, the complement $a^\perp$ of every element a of M belongs to M and, for every $a, b \in M$ with $a \perp b$, the sum $a \oplus b$ belongs to M.

In the sequel, we denote by L a *σ-continuous D-lattice*, and by G a *complete Hausdorff topological Abelian group*.

Measures on D-lattices are defined the usual way:

DEFINITION 2.7. – *A function* $\mu : L \to G$ *is said to be a* measure *if* $a \perp b$ *implies* $\mu(a \oplus b) = \mu(a) + \mu(b)$. *We say that* μ *is* σ-additive *if, for every orthogonal sequence* $\{a_n\}$ *in* L *such that* $\bigoplus_n a_n$ *exists,* $\mu\big(\bigoplus_n a_n\big) = \sum_{n=1}^{\infty} \mu(a_n)$.

It is easy to see that μ is a measure if and only if $a \leq b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$. We are interested, in particular, in modular measures:

DEFINITION 2.8. – *A function* $\mu : L \to G$ *is said to be* modular *if, for every* $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

If L is an orthomodular lattice, every modular function μ with $\mu(0) = 0$ is a measure. Conversely, if L is an MV-algebra, every measure is a modular function. In a D-lattice the two concepts are independent. The following result gives a characterization of modular measures by just an equation.

PROPOSITION 2.9. – *Let* $\mu : L \to G$ *be a function. Then* μ *is a modular measure if and only if, for every* $a, b \in L$, *the following equality holds*

$$(\,*\,) \qquad \mu(a) = \mu((a \vee b) \ominus b) + \mu(a \wedge b).$$

PROOF. – If μ is a modular measure, then for every $a, b \in L$, we have $\mu((a \vee b) \ominus b) = \mu(a \vee b) - \mu(b) = \mu(a) - \mu(a \wedge b)$, whence the assertion.

Conversely, if c and d are orthogonal elements in L, applying $(\,*\,)$ with $a = c \oplus d$ and $b = d$, we obtain $\mu(c \oplus d) = \mu(c) + \mu((c \oplus d) \wedge d) = \mu(c) + \mu(d)$. Hence μ is a measure and therefore, by $(\,*\,)$, we also obtain that μ is modular. $\square$

By [18], every modular function on a lattice generates a *lattice uniformity* $\mathcal{U}(\mu)$, i.e. a uniformity which makes the lattice operations $\vee$ and $\wedge$ uniformly continuous and $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous (see also 3.1 of [28]). Moreover, by 4.2 of [3], if μ is a modular measure on L, then $\mathcal{U}(\mu)$ is a *D-uniformity*, i.e. $\mathcal{U}(\mu)$ makes $\ominus$ (and therefore $\oplus$) uniformly continuous, too, and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $\{(a, b) \in L \times L : \mu(c) \in W$ for every $c \leq a \triangle b\}$, where W is a neighbourhood of 0 in G.

DEFINITION 2.10. – *A function* $\mu : L \to G$ *is said to be* exhaustive *if, for every monotone sequence* $\{a_n\}$ *in* L, $\{\mu(a_n)\}$ *is a Cauchy sequence in* G. *It is said* σ-order continuous (σ*-o.c.) if* $a_n \uparrow a$ *or* $a_n \downarrow a$ *in* L *implies* $\lim_n \mu(a_n) = \mu(a)$.

Some more terminology will be useful in the sequel: By 2.2 of [3], a measure μ is σ-additive if and only if μ is σ-o.c.

A D-uniformity $\mathcal{U}$ is said to be *exhaustive* if every monotone sequence in L is a Cauchy sequence in $\mathcal{U}$, *σ-order-continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L implies that $\{a_n\}$ converges to a in $\mathcal{U}$, and $\mathcal{U}$ *has the property* (σ) if, for every Cauchy sequence $\{a_n\}$ in $(L,\mathcal{U})$ such that $a_n \uparrow a$ or $a_n \downarrow a$, $\{a_n\}$ converges to a in $(L,\mathcal{U})$.

By 3.5 and 3.6 of [28], a modular function $\mu : L \to G$ is exhaustive (respectively, σ-o.c.) if and only if $\mathcal{U}(\mu)$ is exhaustive (respectively, σ-o.c.).

3. – σ-D-lattice generated by a D-sublattice.

The aim of this section is to characterize the σ-sublattice generated by a given lattice. To this aim, monotone subsets of D-lattices are needed.

DEFINITION 3.1. – *A subset M of L is said to be* monotone *if it is closed with respect to the limits (in L) of its monotone sequences.*

If $M \subseteq L$, we denote by $\sigma(M)$ the least monotone D-sublattice of L which contains M. The set $\sigma(M)$ is called *σ-D-sublattice generated by M*. Moreover we set

$$M_\sigma = \{a \in L : \exists \{a_n\} \subseteq M : a_n \uparrow a\},$$
$$M_\delta = \{a \in L : \exists \{a_n\} \subseteq M : a_n \downarrow a\}$$

and

$$M_{\sigma\delta} = (M_\sigma)_\delta.$$

We will be able to prove that for a given D-sublattice M of L, the set $\sigma(M)$ coincides with the least monotone subset $I(M)$ of L which contains M. This equality is an essential tool in the next sections.

We list below some relations regarding monotone sequences in D-lattices. They can be viewed as an immediate consequence of 1.8.7 and 1.8.8 of [16].

LEMMA 3.2. – (1) *If $b_n \uparrow b$ and $a \geq b_n$ for each n, then $a \ominus b_n \downarrow a \ominus b$.*
(2) *If $b_n \downarrow b$ and $a \geq b_n$ for each n, then $a \ominus b_n \uparrow a \ominus b$.*
(3) *If $b_n \uparrow b$ and $a \leq b_n$ for each n, then $b_n \ominus a \uparrow b \ominus a$.*
(4) *If $b_n \downarrow b$ and $a \leq b_n$ for each n, then $b_n \ominus a \downarrow b \ominus a$.*
(5) *If $a_n \uparrow a$, $b_n \uparrow b$ and $a \perp b$, then $a_n \perp b_n$ for each n and $a_n \oplus b_n \uparrow a \oplus b$.*

A description of the set $I(M)$ is contained in the following result.

PROPOSITION 3.3. – *Let $M \subseteq L$. Denote by ω_1 the set of all countable ordinal numbers, and let $\{M_\gamma : \gamma \in \omega_1\}$ be the transfinite sequence defined in the* *following way:*

(*a*) $M_0 = M$.
(*b*) $M_{a+1} = (M_a)_{\sigma\delta}$ *for* $a \in \omega_1$.
(*c*) $M_\gamma = \bigcup_{\beta<\gamma} M_\beta$ *if* $\gamma \in \omega_1$ *is a limit ordinal number and* $\gamma > 0$.

Then:

(1) $I(M) = \bigcup_{\gamma\in\omega_1} M_\gamma$.

(2) *If* M *is a sublattice of* L, *then* $I(M)$ *and every* $M_\gamma(\gamma \in \omega_1)$ *are sublattices of* L.

Proof. – Let us prove (1) in three steps:

(i) We first prove that $\{M_\gamma : \gamma \in \omega_1\}$ is increasing.

Let β, γ be in ω_1, with $\beta < \gamma$. By induction with respect to γ, suppose that, if $a' \leq a'' < \gamma$, then $M_{a'} \subseteq M_{a''}$. If γ is a limit ordinal number, then we have $M_\beta \subseteq \bigcup_{\lambda<\gamma} M_\lambda = M_\gamma$. On the other hand, if $\gamma = a + 1$, for some $a \in \omega_1$, then $\beta \leq a < \gamma$. By the inductive assumption, we have $M_\beta \subseteq M_a$. Since $M_\gamma = M_{a+1} = (M_a)_{\sigma\delta} \supseteq M_a$, then $M_\beta \subseteq M_\gamma$.

(ii) We now prove that $M_\gamma \subseteq I(M)$ for every $\gamma \in \omega_1$.

Trivially $M_0 = M \subseteq I(M)$. Let $\gamma > 0$ and suppose by induction that $M_\beta \subseteq I(M)$ for every $\beta < \gamma$. If γ is a limit ordinal number, then $M_\gamma = \bigcup_{\beta<\gamma} M_\beta \subseteq I(M)$. If $\gamma = a + 1$ for some $a \in \omega_1$, then $M_a \subseteq I(M)$ by the inductive assumption and therefore $M_\gamma = M_{a+1} = (M_a)_{\sigma\delta} \subseteq (I(M))_{\sigma\delta} = I(M)$.

(iii) We finally prove that $I(M) \subseteq \bigcup_{\gamma\in\omega_1} M_\gamma$.

Since $\bigcup_{\gamma\in\omega_1} M_\gamma \supseteq M$, it is sufficient to prove that $\bigcup_{\gamma\in\omega_1} M_\gamma$ is monotone. Let $\{a_n\} \subseteq \bigcup_{\gamma\in\omega_1} M_\gamma$ be a monotone sequence and let a be its order limit. For each n, let $\gamma_n \in \omega_1$ be such that $a_n \in M_{\gamma_n}$. By the properties of ω_1, there exists $\beta = \sup_n \gamma_n$ in ω_1. Since $\{M_a : a \in \omega_1\}$ is increasing and, for each n, $a_n \in M_{\gamma_n}$, we have $a_n \in M_\beta$ for each n. At this point $a \in (M_\beta)_\sigma$ if $\{a_n\}$ is increasing and $a \in (M_\beta)_\delta$ if $\{a_n\}$ is decreasing. In any case, $a \in (M_\beta)_{\sigma\delta} = M_{\beta+1} \subseteq \bigcup_{\gamma\in\omega_1} M_\gamma$.

Let us now pass to the proof of Statement (2). Suppose that M is a sublattice of L. We prove that, for every $\gamma \in \omega_1$, M_γ is a sublattice of L.

This is obvious if $\gamma = 0$, since in this case $M_0 = M$. Take then $\gamma > 0$ and assume, by induction, that for every $\beta < \gamma$, M_β is a sublattice of L. If $\gamma = a + 1$, for some $a \in \omega_1$, then M_a is a sublattice by induction and therefore $M_\gamma = (M_a)_{\sigma\delta}$ is a sublattice, too, since L is σ-continuous. If γ is a limit ordinal number, then $M_\gamma =$

$\bigcup_{\beta<\gamma} M_\beta$. By induction, for every $\beta < \gamma$, M_β is a sublattice. Then M_γ is a sublattice, too, since $\{M_\beta : \beta < \gamma\}$ is increasing by (i).

In a similar way we can prove that $I(M)$ is a sublattice of L. □

We are now ready to describe the desired characterization of $\sigma(M)$.

THEOREM 3.4. – *Let M be a D-sublattice of L and $\sigma(M)$ the σ-D-sublattice generated by M. Then $\sigma(M) = I(M)$.*

PROOF. – Since $\sigma(M) \supseteq I(M)$, we only have to prove that $I(M)$ is a D-sublattice of L.

By Prop 3.3, $I(M)$ is a sublattice of L. Moreover $0, 1 \in I(M)$ since $M \subseteq I(M)$.

Let $\{M_\gamma : \gamma \in \omega_1\}$ be as in Prop 3.3. We will prove that, for every $\alpha \in \omega_1$ there exists a natural number k such that, if a and b are elements of M_α with $a \geq b$, then $a \ominus b \in M_{\alpha+k}$, This will imply the thesis.

Let $\alpha \in \omega_1$ and $a, b \in M_\alpha$ with $a \geq b$. We proceed by induction. We may assume that α is the minimum ordinal number such that $a, b \in M_\alpha$. If $\alpha = 0$, the assertion is trivial since in this case $M_0 = M$. We can then assume $\alpha \neq 0$. If α is a limit ordinal number, by the equality $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, we can find β_1, β_2 in ω_1 such that $\beta_1 < \alpha, \beta_2 < \alpha$, $a \in M_{\beta_1}$ and $b \in M_{\beta_2}$.

If we set $\beta = \max\{\beta_1, \beta_2\}$, we obtain $\beta < \alpha$ and $a, b \in M_\beta$, a contradiction with the definition of α. Then there exists $\gamma \in \omega_1$ such that $\alpha = \gamma + 1$, which gives $M_\alpha = (M_\gamma)_{\sigma\delta}$. Take two double sequences $\{a_{m,n}\}, \{b_{m,n}\}$ in M_γ and two sequences $\{a_n\}, \{b_n\}$ in L such that, for each n, $a_{m,n} \uparrow_m a_n$, $b_{m,n} \uparrow_m b_n$ and, moreover, $a_n \downarrow a$, $b_n \downarrow b$. By Prop 3.3 M_γ is a sublattice of L, this allows us to assume that, for each m, the sequence $\{a_{m,n}\}$ is decreasing with respect to n (if not, we can replace $\{a_{m,n}\}$ with the sequence $\{c_{m,n}\}$ defined as $c_{m,n} = \inf_{i \leq n} a_{m,i}$).

Choose now a natural number k corresponding to γ according to the inductive hypothesis and fix $m, n, p, q, r \geq p$ and $s \geq q$. Then, by the inductive assumption,

$$a_{r,q} \ominus (b_{m,n} \wedge a_{p,s}) \in M_{\gamma+k}.$$

Since Statement (3) of Lemma 3.2 gives

$$a_{r,q} \ominus (b_{m,n} \wedge a_{p,s}) \uparrow_r a_q \ominus (b_{m,n} \wedge a_{p,s}),$$

we obtain $a_q \ominus (b_{m,n} \wedge a_{p,s}) \in (M_{\gamma+k})_\sigma$. Moreover, since $b_{m,n} \uparrow_m b_n$ and L is σ-continuous, (1) of Lemma 3.2 implies $a_q \ominus (b_{m,n} \wedge a_{p,s}) \downarrow_m a_q \ominus (b_n \wedge a_{p,s})$. At this point $a_q \ominus (b_n \wedge a_{p,s}) \in (M_{\gamma+k})_{\sigma\delta} = M_{\gamma+k+1} = M_{\alpha+k}$.

Similarly, since $a_{p,s} \uparrow_p a_s$ and L is σ-continuous, using again (1) of Lemma 3.2 we obtain that $a_q \ominus (b_n \wedge a_{p,s}) \downarrow_p a_q \ominus (b_n \wedge a_s)$. Then $a_q \ominus (b_n \wedge a_s) \in (M_{\gamma+k})_{\sigma\delta\delta} = (M_{\gamma+k})_{\sigma\delta} = M_{\alpha+k}$. From $b_n \downarrow b$, by (2) of the same lemma it follows that $a_q \ominus (b_n \wedge a_s) \uparrow_n a_q \ominus (b \wedge a_s)$. Then $a_q \ominus (b \wedge a_s) \in (M_{\alpha+k})_\sigma$. Since $a_s \downarrow a$, using again (2) of

Lemma 3.2 we obtain $a_q \ominus (b \wedge a_s) \uparrow a_q \ominus (b \wedge a) = a_q \ominus b$. Hence $a_q \ominus b \in (M_{a+k})_{\sigma\sigma} = (M_{a+k})_\sigma$. Finally, since $a_q \downarrow a$, Statement (4) of the same lemma gives $a_q \ominus b \downarrow a \ominus b$, from which $a \ominus b \in (M_{a+k})_{\sigma\delta} = M_{a+k+1}$ follows. $\square$

4. – Monotone functions.

In this section we study some properties of $[0, +\infty]$-valued functions on L which will be useful in the next section.

For a function $\eta : L \to [0, +\infty]$, we say that η is:

- *upper-continuous* if $a_n \uparrow a$ implies $\lim_n \eta(a_n) = \eta(a)$,
- *lower-continuous* if $a_n \downarrow a$ implies $\lim_n \eta(a_n) = \eta(a)$,
- *upper-D-continuous* if, for every $a, b \in L$ and every sequence $\{b_n\}$ in L, $b_n \uparrow b$ or $b_n \downarrow b$ imply $\eta((a \vee b) \ominus b) \leq \limsup_n \eta((a \vee b_n) \ominus b_n)$,
- *exhaustive* if, for every monotone sequence $\{a_n\}$ in L, $\lim_n \eta(a_n \triangle a_{n+p}) = 0$, $\forall p \in \mathbb{N}$,
- σ-order continuous if $a_n \uparrow a$ or $a_n \downarrow a$ imply $\lim_n \eta(a_n \triangle a) = 0$,
- *has the property* (σ) if $a_n \uparrow a$ or $a_n \downarrow a$ and $\lim_{n,m} \eta(a_n \triangle a_m) = 0$ imply $\lim_n \eta(a_n \triangle a) = 0$,
- *subadditive* if, for every $a, b \in L$ with $a \perp b$, $\eta(a \oplus b) \leq \eta(a) + \eta(b)$,
- σ-*subadditive* if, for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $\eta\big(\bigoplus_n a_n\big) \leq \sum_{n=1}^{\infty} \eta(a_n)$.

It is easy to see that η is upper D-continuous if and only if, for every $a, b \in L$ and every sequence $\{b_n\}$ in L, any of the relations $b_n \uparrow b$ or $b_n \downarrow b$ implies $\eta((a \vee b) \ominus b) \leq \sup_n \eta((a \vee b_n) \ominus b_n)$. A study of the relation among the previous properties follows. Let us start by verifying that σ-subadditivity is a stronger condition than property (σ).

PROPOSITION 4.1. – *Let* $\eta : L \to [0, +\infty]$ *be* σ*-subadditive and monotone. Then* η *has the property* (σ).

PROOF. – Take a sequence $a_n \uparrow a$ and assume that $\lim_{n,m} \eta(a_n \triangle a_m) = 0$. For $\varepsilon > 0$ and, for every k, let n_k be such that, for every $i \geq j \geq n_k$, $\eta(a_i \ominus a_j) < \varepsilon/2^k$. We may assume that the sequence $\{n_k\}$ is strictly increasing. If we set $b_k = a_{n_k}$, we have $b_k \uparrow a$ and $\eta(b_i \ominus b_j) < \varepsilon/2^k$ for every $i \geq j \geq k$ and every k. Set $c_n = b_n \ominus b_{n-1}$ for each $n \geq 2$. From 2.1 of [3] we have $\bigoplus_{n=2}^{k} c_n = b_k \ominus b_1$ for every k.

Then, by (3) of Lemma 3.2, we have

$$\bigoplus_{n=2}^{\infty} c_n = \sup_k \bigoplus_{n=2}^{k} c_k = \sup_k (b_k \ominus b_1) = a \ominus b_1.$$

Since η is σ-subadditive, we have $\eta(a \ominus b_1) \leq \sum_{n=2}^{\infty} \eta(c_n) < \sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}} = \varepsilon$. Since η is monotone, we have $\eta(a \ominus a_n) \leq \eta(a \ominus b_1) < \varepsilon$ for each $n \geq n_1$. Then $\lim_{\eta} (a_n \triangle a) = \lim_n \eta(a \ominus a_n) = 0$.

If $a_n \downarrow a$ and $\lim_{n,m} \eta(a_n \triangle a_m) = 0$, then $a_n^{\perp} \uparrow a^{\perp}$. From the previous result we obtain $\lim_n \eta(a_n \triangle a) = \lim_n \eta(a_n \ominus a) = \lim_n \eta(a^{\perp} \ominus a_n^{\perp}) = \lim_n \eta(a^{\perp} \triangle a_n^{\perp}) = 0$. This completes the proof. $\square$

The following two results compare the notions of upper continuity and upper D-continuity for monotone functions.

PROPOSITION 4.2. – *Let $\eta : L \to [0, +\infty]$ be a monotone function. If η is upper D-continuous, then η is upper-continuous.*

PROOF. – Let $\{a_n\}$ be a sequence in L such that $a_n \uparrow a$. Since $a_n^{\perp} \downarrow a^{\perp}$, we have $\eta(a) = \eta((1 \vee a^{\perp}) \ominus a^{\perp}) \leq \sup_n \eta((1 \vee a_n^{\perp}) \ominus a_n^{\perp}) = \sup_n \eta(a_n) \leq \eta(a)$. $\square$

PROPOSITION 4.3. – *Let $\eta : L \to [0, +\infty]$ be a monotone subadditive function. If η is upper and lower-continuous, then η is upper D-continuous.*

PROOF. – Take two elements a and b in the lattice $\sigma(M)$ and a sequence $\{b_n\}$ in the same set increasing to b. By (1) of Lemma 3.2, we have that the sequence $(a \vee b) \ominus (a \vee b_n)$ decreases to zero. Then for each positive ε we can choose n such that $\eta((a \vee b) \ominus (a \vee b_n)) < \varepsilon$. By 1.1.6-iv of [16], we have

$$(a \vee b) \ominus b_n = ((a \vee b_n) \ominus b_n) \oplus ((a \vee b) \ominus (a \vee b_n)).$$

We then obtain

$$\begin{aligned}\eta((a \vee b) \ominus b) &\leq \eta((a \vee b) \ominus b_n) \leq \eta((a \vee b_n) \ominus b_n) + \eta((a \vee b) \ominus (a \vee b_n)) \\ &< \varepsilon + \eta((a \vee b_n) \ominus b_n) \leq \varepsilon + \sup_n \eta((a \vee b_n) \ominus b_n).\end{aligned}$$

Let us now take $b_n \downarrow b$. By (4) of Lemma 3.2, we have $b_n \ominus b \downarrow 0$. Hence, if $\varepsilon > 0$, we can choose n such that $\eta(b_n \ominus b) < \varepsilon$. Again, the use of 1.1.6-iv of [16], gives $\eta((a \vee b) \ominus b) \leq \eta((a \vee b_n) \ominus b) \leq \eta((a \vee b_n) \ominus b_n) + \eta(b_n \ominus b) < \varepsilon + \sup_n \eta((a \vee b_n) \ominus b_n)$, which implies the assertion. $\square$

For some functions defined on an orthomodular lattice, upper continuity is equivalent to upper D-continuity.

Proposition 4.4. – *Let $\eta : L \to [0, +\infty]$ be a monotone subadditive function. If L is an orthomodular lattice, then η is upper D-continuous if and only if η is upper-continuous.*

Proof. – By Prop 4.2, we only have to prove that, if η is upper-continuous, then η is upper-D-continuous.

Take $a, b \in L$ and $\{b_n\} \subseteq L$ such that $b_n \uparrow b$. Since L is σ-continuous, we have $(a \vee b_n) \wedge b^{\perp} \uparrow (a \vee b) \wedge b^{\perp}$. Then $\eta((a \vee b) \ominus b) = \eta((a \vee b) \wedge b^{\perp}) = \sup_n \eta((a \vee b_n) \wedge b^{\perp}) \leq \sup_n \eta((a \vee b_n) \wedge b_n^{\perp}) = \sup_n \eta((a \vee b_n) \ominus b_n)$.

In case $b_n \downarrow b$, σ-continuity of L implies $(a \vee b) \wedge b_n^{\perp} \uparrow (a \vee b) \wedge b^{\perp}$. Therefore we obtain

$$\begin{aligned} \eta((a \vee b) \ominus b) &= \eta((a \vee b) \wedge b^{\perp}) = \sup_n \eta((a \vee b) \wedge b_n^{\perp}) \\ &\leq \sup_n \eta((a \vee b_n) \wedge b_n^{\perp}) = \sup_n \eta((a \vee b_n) \ominus b_n). \end{aligned}$$

□

We now introduce the concept of submeasure, which plays an important role in the next sections.

Definition 4.5. – *A function $\eta : L \to [0, +\infty]$ is said to be a* submeasure *if η is monotone and subadditive, $\eta(0) = 0$ and $\eta((a \vee b) \ominus b) \leq \eta(a)$ for every $a, b \in L$.*

*A σ-subadditive submeasure is called σ-*submeasure.

Observe that every positive real-valued modular measure λ is a submeasure since $\lambda((a \vee b) \ominus b) = \lambda(a \vee b) - \lambda(b) = \lambda(a) - \lambda(a \wedge b) \leq \lambda(a)$.

By 3.2 of [6], every submeasure η on L generates a D-uniformity $\mathcal{U}(\eta)$ on L, which is the weakest D-uniformity which makes η uniformly continuous, and a base of $\mathcal{U}(\eta)$ is the family consisting of the sets $\{(a, b) \in L \times L : \eta(a \triangle b) < \varepsilon\}$, where $\varepsilon > 0$.

It is easy to see that a submeasure $\eta : L \to [0, +\infty]$ is *exhaustive* or *σ-o.c.* or *has the property* (σ) if and only if $\mathcal{U}(\eta)$ is exhaustive or σ-o.c. or has the property (σ).

We need the following results to give in Prop 4.8 below a characterization of the submeasures by just an inequality in similar way as that obtained in Prop 2.9.

Lemma 4.6. – *Take two elements a and b in L. Define $c = a \ominus (a \wedge b)$ and $d = b \ominus (a \wedge b)$. Then $(c \vee d) \ominus d = (a \vee b) \ominus b$.*

PROOF. – By 1.1.6-iv and 1.8.4 of [16], we have

$$a \triangle b = ((a \vee b) \ominus b) \oplus (b \ominus (a \wedge b)) = (a \ominus (a \wedge b)) \vee (b \ominus (a \wedge b)).$$

We then obtain

$$(a \vee b) \ominus b = (a \triangle b) \ominus (b \ominus (a \wedge b))$$
$$= ((a \ominus (a \wedge b)) \vee (b \ominus (a \wedge b))) \ominus (b \ominus (a \wedge b)) = (c \vee d) \ominus d.$$

□

LEMMA 4.7. – *Let $\eta : L \to [0, +\infty]$ be a monotone function. Then the following conditions are equivalent:*

(1) *For every $a, b \in L$, $\eta((a \vee b) \ominus b) = \eta(a \ominus (a \wedge b))$.*
(2) *For every $a, b \in L$, $\eta((a \vee b) \ominus b) \leq \eta(a)$.*
(3) *$a \wedge b = 0 \Rightarrow \eta((a \vee b) \ominus b) \geq \eta(a)$.*

PROOF. – Statement (1) is obviously stronger than (2) and (3). To prove that (2) $\Rightarrow$ (1), let us take $a, b \in L$ and set $c = a \ominus (a \wedge b)$, $d = b \ominus (a \wedge b)$. The previous lemma ensures that $\eta((a \vee b) \ominus b) = \eta((c \vee d) \ominus d) \leq \eta(c) = \eta(a \ominus (a \wedge b))$. Moreover, since $a \ominus (a \wedge b) = (a \wedge b)^{\perp} \ominus a^{\perp} = (a^{\perp} \vee b^{\perp}) \ominus a^{\perp}$, we obtain $\eta(a \ominus (a \wedge b)) \leq \eta(b^{\perp} \ominus (a^{\perp} \wedge b^{\perp})) = \eta((a \vee b) \ominus b)$.

Let us now show that (3) $\Rightarrow$ (1). Take elements a, b in L and set $c = a \ominus (a \wedge b)$, $d = b \ominus (a \wedge b)$. By 1.8.5 of [16], we have $c \wedge d = 0$. By the previous lemma we obtain $\eta((a \vee b) \ominus b) = \eta((c \vee d) \ominus d) \geq \eta(c) = \eta(a \ominus (a \wedge b))$. Then $\eta(a \ominus (a \wedge b)) = \eta((a^{\perp} \vee b^{\perp}) \ominus a^{\perp}) \geq \eta(b^{\perp} \ominus (a^{\perp} \wedge b^{\perp})) = \eta((a \vee b) \ominus b)$. □

We can conclude this section by the desired result:

PROPOSITION 4.8. – *Let $\eta : L \to [0, +\infty]$ be a monotone function, with $\eta(0) = 0$. Then η is a submeasure if and only if, for every $a, b \in L$, the following inequality holds*

$$(\ast) \qquad \eta(a) \leq \eta((a \vee b) \ominus b) + \eta(a \wedge b).$$

PROOF. – Let η be a submeasure. We can suppose $\eta(a \wedge b) < +\infty$. Then we have

$$\eta(a) \leq \eta(a \ominus (a \wedge b)) + \eta(a \wedge b).$$

By Lemma 4.7 above, it follows that $\eta(a \ominus (a \wedge b)) = \eta((a \vee b) \ominus b)$, from which the assertion follows.

Conversely, by $(\ast)$, we have that, if $a \wedge b = 0$, then $\eta((a \vee b) \ominus b) \geq \eta(a)$. Again, the use of Lemma 4.7 gives

$$\eta((a \vee b) \ominus b) \leq \eta(a) \qquad \forall a, b \in L.$$

Moreover, if c and d are orthogonal elements in L, then, application of $(*)$ with $a = c \oplus d$ and $b = d$, gives $\eta(c \oplus d) \leq \eta(c) + \eta(d)$. This is sufficient to say that η is a submeasure, as wanted. $\square$

5. – The outer measure.

In this section $M \subseteq L$ is a D-sublattice of L, $\|\cdot\|$ is a fixed seminorm on G and $\mu : M \to G$ is a σ-additive exhaustive modular measure.

Following the classical way, we define an outer measure μ^* associated to μ. We prove that, under suitable hypothesis, μ^* is a σ-submeasure on $\sigma(M)$. This fact yields, by 3.2 of [6], the existence of a D-uniformity on $\sigma(M)$ which makes μ^* uniformly continuous, which is the starting point for establishing the desired extension in the next section.

In the following, we set, for $a \in L$,

$$\tilde{\mu}(a) = \sup\{\|\mu(b)\| : b \in M, b \leq a\}$$

and

$$\mu^*(a) = \inf\{\tilde{\mu}(b) : b \in M_\sigma, b \geq a\}.$$

We say that μ is *upper-D-continuous* if μ^* is upper D-continuous on $\sigma(M)$.

Let us start by stating some properties of $\tilde{\mu}$.

PROPOSITION 5.1. – *The function $\tilde{\mu}$ is a submeasure on M.*

PROOF. – It is clear that $\tilde{\mu}$ is monotone and that $\tilde{\mu}(0) = 0$. Take two orthogonal elements a and b in M. We first prove that $\tilde{\mu}(a \oplus b) \leq \tilde{\mu}(a) + \tilde{\mu}(b)$. Let c be an element of M with $c \leq a \oplus b$. Set $d = c \wedge a$ and $e = (c \vee a) \ominus a$. By the relation

$$e \leq a^\perp \leq a^\perp \vee c^\perp = (a \wedge c)^\perp$$

we obtain $d \perp e$. Moreover we have

$$\begin{aligned}\mu(d \oplus e) &= \mu(c \wedge a) + \mu((c \vee a) \ominus a)\\ &= \mu(c \wedge a) + \mu(c \vee a) - \mu(a) = \mu(c).\end{aligned}$$

Since $d \leq a$ and $e \leq ((a \oplus b) \vee a) \ominus a = (a \oplus b) \ominus a = b$, then $\|\mu(c)\| = \|\mu(d \oplus e)\| \leq \|\mu(d)\| + \|\mu(e)\| \leq \tilde{\mu}(a) + \tilde{\mu}(b)$, the assertion follows.

Let us now check that $\tilde{\mu}((a \vee b) \ominus b) \leq \tilde{\mu}(a)$. Let $c \in M$ be such that $c \leq (a \vee b) \ominus b$. Set $s = b \oplus (((a \vee b) \ominus b) \ominus c)$. If we prove that $c = (a \vee s) \ominus s$, it will follow that $\mu(c) = \mu(a \vee s) - \mu(s) = \mu(a) - \mu(a \wedge s) = \mu(a \ominus (a \wedge s))$, whence $\|\mu(c)\| \leq \tilde{\mu}(a)$ and therefore the thesis.

Obviously $b \leq s$; from 1.1.6-iv of [16], it follows that $s = b \oplus (((a \vee b) \ominus c) \ominus b) = (a \vee b) \ominus c \leq a \vee b$. This means that $a \vee s = a \vee b$, whence $(a \vee s) \ominus c = (a \vee b) \ominus c = s$. At this point $(a \vee s) \ominus s = (a \vee s) \ominus ((a \vee s) \ominus c) = c$. $\square$

REMARK 5.2. – *Note that $\mathcal{U}(\mu)$ and $\mathcal{U}(\tilde{\mu}_{|M})$ have the same base of neighbourhoods of* 0*. Hence, by* 2.4 *of* [6]*, we have $\mathcal{U}(\mu) = \mathcal{U}(\tilde{\mu}_{|M})$.*

PROPOSITION 5.3. – (1) *The function $\tilde{\mu}$ is upper-continuous on M_σ.*

(2) *If a and b are orthogonal elements in M_σ, then $a \oplus b \in M_\sigma$ and $\tilde{\mu}(a \oplus b) \leq \tilde{\mu}(a) + \tilde{\mu}(b)$.*

PROOF. – (1) Take $a_n \uparrow a$ in M_σ. Suppose first that $a_n \in M$ for each n. Let b be an element of M such that $b \leq a$. Since L is σ-continuous, we have that $b \wedge a_n \uparrow b$. σ-order continuity of μ ensures that $\mu(b) = \lim_n \mu(b \wedge a_n)$. Let $\varepsilon > 0$ and n be such that $\|\mu(b) - \mu(b \wedge a_n)\| < \varepsilon$. Then

$$\|\mu(b)\| < \varepsilon + \|\mu(b \wedge a_n)\| \leq \varepsilon + \tilde{\mu}(a_n) \leq \varepsilon + \sup_n \tilde{\mu}(a_n).$$

Hence $\tilde{\mu}(a) \leq \sup_n \tilde{\mu}(a_n)$. On the other hand, we have $\tilde{\mu}(a) \geq \tilde{\mu}(a_n)$ for each n since $\tilde{\mu}$ is monotone. Then $\tilde{\mu}(a) = \sup_n \tilde{\mu}(a_n)$.

Take now a sequence $\{a_n\}$ in $\subseteq M_\sigma$. For each n, let $\{b_{n,m}\} \subseteq M$ be such that $b_{n,m} \uparrow_m a_n$. Set $c_n = \sup_{n \leq m} b_{n,m}$. We have that $\{c_n\} \subseteq M$, $c_m \leq a_m$ for each m and $c_m \uparrow a$. From the previous result, it follows that

$$\tilde{\mu}(a) = \sup_m \tilde{\mu}(c_m) \leq \sup_m \tilde{\mu}(a_m) \leq \tilde{\mu}(a),$$

which implies $\tilde{\mu}(a) = \sup_m \tilde{\mu}(a_m)$.

(2) If a and b are two orthogonal elements in M_σ, their sum $a \oplus b$ belongs to M_σ, thank to (5) of Lemma 3.2. Suppose first that $a \in M$ and choose a sequence $\{b_n\}$ in M such that $b_n \uparrow b$. Of course, $b_n \perp a$ for each n, $a \oplus b_n \uparrow a \oplus b$ and, by (1), $\tilde{\mu}(a \oplus b) = \lim_n \tilde{\mu}(a \oplus b_n) \leq \tilde{\mu}(a) + \lim_n \tilde{\mu}(b_n) = \tilde{\mu}(a) + \tilde{\mu}(b)$.

Suppose now $a \in M_\sigma$. Choose $\{a_n\}$ in M such that $a_n \uparrow a$. Then $a_n \oplus b \uparrow a \oplus b$ and $\tilde{\mu}(a \oplus b) = \lim_n \tilde{\mu}(a_n \oplus b) \leq \lim_n \tilde{\mu}(a_n) + \tilde{\mu}(b) = \tilde{\mu}(a) + \tilde{\mu}(b)$. □

We continue by stating the properties of the function μ^*.

LEMMA 5.4. – *Let $b \in M$ and suppose that, for every $a \in M$, $\mu^*((a \vee b) \ominus b) \leq \mu^*(a)$. Then, for every $a \in L$, $\mu^*((a \vee b) \ominus b) \leq \mu^*(a)$.*

PROOF. – Take a in L and an element c in M_σ with $c \geq a$. We have to prove that $\mu^*((a \vee b) \ominus b) \leq \tilde{\mu}(c)$. Choose a sequence $\{b_n\}$ in M such that $b_n \uparrow c$. Then, by (3) of Lemma 3.2, we have $(b_n \vee b) \ominus b \uparrow (c \vee b) \ominus b$. Since the elements $(b_n \vee b) \ominus b$ belong to M, the relation $(c \vee b) \ominus b \in M_\sigma$ holds. Upper continuity of μ^* on M_σ, due to Prop. 5.3 above, gives $\mu^*((c \vee b) \ominus b) = \sup_n \mu^*((b_n \vee b) \ominus b)$. From

$\{b_n\} \subseteq M$, it follows that $\mu^*((b_n \vee b) \ominus b) \leq \mu^*(b_n) \leq \tilde{\mu}(c)$. At this point, the relation $a \leq c$, implies $\mu^*((a \vee b) \ominus b) \leq \mu^*((c \vee b) \ominus b) \leq \tilde{\mu}(c)$, as wanted. □

LEMMA 5.5. – *For every $a \in L$, there exists a decreasing sequence $\{a_n\}$ in M_σ such that $a_n \geq a$ for each n and $\mu^*(a) = \lim_n \mu^*(a_n)$.*

PROOF. – Let $a \in L$. By definition of $\mu^*(a)$, we can find for each n an element b_n in M_σ such that $b_n \geq a$ and $\mu^*(a) > \tilde{\mu}(b_n) - 1/n$.

Set $a_n = \inf_{k \leq n} b_k$. Then $\{a_n\}$ is a decreasing sequence in M_σ and, for each n, it is $a_n \geq a$ and $\tilde{\mu}(a_n) \leq \tilde{\mu}(b_n) < \mu^*(a) + 1/n$. Since $\mu^*(a) \leq \tilde{\mu}(a_n)$ for each n, we get $\mu^*(a) = \lim_n \mu^*(a_n)$. □

LEMMA 5.6. – *Let $A \subseteq L$ be a set with the following properties:*

(1) *$M \subseteq A$.*
(2) *If $S \subseteq A$ is a sublattice of L, then $S_\sigma \subseteq A$.*
(3) *If $S \subseteq A$ is a sublattice of L and $M_\sigma \subseteq S$, then $S_\delta \subseteq A$.*

Then $\sigma(M) \subseteq A$.

PROOF. – By assumption, we have that, if $S \subseteq A$ is a sublattice of L with $M \subseteq S$, then $S_{\sigma\delta} \subseteq A$.

Let $\{M_\gamma : \gamma \in \omega_1\}$ be as in Prop 3.3. By this proposition and Theor. 3.4, we have

$$\sigma(M) = \bigcup_{\gamma \in \omega_1} M_\gamma.$$

All we need to prove is that $M_\gamma \subseteq A$ for each $\gamma \in \omega_1$. By assumption, $M_0 = M \subseteq A$. Take now $\gamma > 0$ and suppose by induction that $M_\alpha \subseteq A$ for each $\alpha < \gamma$. If γ is a limit ordinal, we trivially obtain $M_\gamma \subseteq A$. If γ is of the type $\gamma = \alpha + 1$ for some $\alpha \in \omega_1$, then $M_\gamma \subseteq A$ since in this case, due to Prop 3.3 M_γ is a sublattice of L. □

We can now obtain σ-subadditivity of μ^* as a consequence of its upper continuity on $\sigma(M)$.

THEOREM 5.7. – *Suppose that μ^* is upper-continuous on $\sigma(M)$. Then μ^* is σ-subadditive on $\sigma(M)$.*

PROOF. – *Step 1*: We first prove that μ^* is subadditive on $\sigma(M)$.

Set

$$A = \{b \in L : \ \forall\ a \in \sigma(M) \ \text{ with } \ a \perp b, \ \mu^*(a \oplus b) \leq \mu^*(a) + \mu^*(b)\}.$$

We prove that A has the properties (1), (2) and (3) of Lemma 5.6 above, whence we will obtain that $\sigma(M) \subseteq A$ and then that μ^* is subadditive on $\sigma(M)$.

(1) Let us start by proving that $M \subseteq A$.

Take $a \in \sigma(M)$ and $b \in M$ such that $a \perp b$. We use Lemma 5.5 to find a decreasing sequence $\{a_n\}$ in M_σ such that $a_n \geq a$ and $\mu^*(a) = \lim_n \mu^*(a_n)$. Set $c_n = a_n \wedge b^\perp$. We have:

$$c_n \in M_\sigma \text{ and } c_n \geq a \qquad \forall n \in \mathbb{N}$$

At this point $\mu^*(a) \leq \tilde{\mu}(c_n) \leq \tilde{\mu}(a_n)$, whence $\lim_n \tilde{\mu}(c_n) = \mu^*(a)$. Observe that $c_n \perp b$ for each n, and set $b_n = c_n \oplus b$. By 1.8.7 of [16], we have that $b_n \in M_\sigma$ and, moreover, $b_n \geq a \oplus b$. Then we obtain $\tilde{\mu}(b_n) \geq \mu^*(a \oplus b)$. Moreover, by (2) of Prop 5.3, we have $\mu^*(a)+\mu^*(b) = \lim_n \tilde{\mu}(c_n)+\tilde{\mu}(b) \geq \lim_n \tilde{\mu}(b_n)$. Therefore $\mu^*(a \oplus b) \leq \mu^*(a) + \mu^*(b)$.

(2) We prove in fact that, if $\{b_n\}$ is a sequence in A increasing to b, then $b \in A$.

Take an element a in $\sigma(M)$ orthogonal to b. Since $b_n \leq b$ for each n, then a is orthogonal to each b_n and, by 1.8.7 of [16],

$$b_n \oplus a \uparrow b \oplus a.$$

The fact that the elements of the sequence $\{b_n\}$ are in A, gives

$$\mu^*(a \oplus b) = \lim_n \mu^*(a \oplus b_n) \leq \lim_n \mu^*(b_n) + \mu^*(a) = \mu^*(b) + \mu^*(a).$$

(3) Let $S \subseteq A$ be a sublattice of L such that $M_\sigma \subseteq S$. We want to prove that $S_\delta \subseteq A$.

Take $b \in S_\delta$.

(a) We first prove that there exists a sequence $\{c_n\} \subseteq S$ such that $c_n \downarrow b$ and $\lim_n \mu^*(c_n) = \mu^*(b)$.

Choose a sequence $\{b_n\}$ in S decreasing to b. By Lemma 5.5 we can find a decreasing sequence $\{a_n\}$ in M_σ such that $a_n \geq b$ for each n, and $\lim_n \mu^*(a_n) = \mu^*(b)$. Set $c_n = a_n \wedge b_n$. Since M_σ is a subset of S, we have, for each n, $c_n \in S$ and $b \leq c_n \leq a_n$. Then $\mu^*(b) \leq \mu^*(c_n) \leq \mu^*(a_n)$, whence $\lim_n \mu^*(c_n) = \mu^*(b)$. Moreover, since $b \leq c_n \leq b_n$, then c_n decreases to b.

(b) Take in $\sigma(M)$ an element a orthogonal to b, and choose a sequence $\{c_n\} \subseteq S$ as in (a). Since $c_n^\perp \uparrow b^\perp$ and L is σ-continuous, we have

$$a \wedge c_n^\perp \uparrow a \wedge b^\perp = a$$

and

$$a \wedge c_n^\perp \leq a \leq b^\perp.$$

Then, by 1.8.7 of [16]

$$(a \wedge c_n^\perp) \oplus b \uparrow a \oplus b.$$

Hence $\mu^*(a \oplus b) = \lim_n \mu^*((a \wedge c_n^\perp) \oplus b)$. On the other hand, since $\{c_n\} \subseteq A$ and $b \leq c_n$, we have $\mu^*((a \wedge c_n^\perp) \oplus b) \leq \mu^*((a \wedge c_n^\perp) \oplus c_n) \leq \mu^*(a \wedge c_n^\perp) + \mu^*(c_n) \leq \mu^*(a) + \mu^*(c_n)$. Therefore we obtain $\mu^*(a \oplus b) \leq \mu^*(a) + \mu^*(b)$, which implies that $b \in A$.

Step 2: We now prove that μ^* is σ-subadditive.

By Step 1 we obtain by induction that, if $\{a_1, \cdots, a_n\} \subseteq \sigma(M)$ is orthogonal, then $\mu^*(a_1 \oplus \cdots \oplus a_n) \leq \mu^*(a_1) + \cdots + \mu^*(a_n)$. Now let $\{a_n\} \subseteq \sigma(M)$ be such that $a = \bigoplus_n a_n$ exists. Set $b_n = \bigoplus_{i \leq n} a_i$ for each n. Then b_n increases to a and b_n belongs to $\sigma(M)$ for each n since $\sigma(M)$ is a D-sublattice of L. Since μ^* is upper-continuous on $\sigma(M)$, we obtain

$$\mu^*(a) = \sup_n \mu^*(b_n) \leq \sup_n \sum_{i \leq n} \mu^*(a_i) = \sum_{n=1}^{\infty} \mu^*(a_n).$$

□

The work done up to this point allows to give a sufficient condition for μ^* to be a σ-submeasure.

THEOREM 5.8. – *If μ is upper-D-continuous, then μ^* is a σ-submeasure on $\sigma(M)$.*

PROOF. – By Prop 4.2 and Theor. 5.7, the function μ^* is σ-subadditive on $\sigma(M)$. We prove that, for every $a \in L$ and every $b \in \sigma(M)$, the following inequality holds

$$\mu^*((a \vee b) \ominus b) \leq \mu^*(a).$$

Set $A = \{b \in \sigma(M) : \forall a \in L,\ \mu^*((a \vee b) \ominus b) \leq \mu^*(a)\}$. Prop 5.1 ensures that $\mu^*((a \vee b) \ominus b) \leq \mu^*(a)$ for every $a, b \in M$. Then, Lemma 5.4 gives $M \subseteq A$. Moreover upper-D-continuity of μ implies that A is monotone. Then the assertion follows by Theor. 3.4. □

We can say something more if the function we start with takes values in $[0, +\infty[$.

PROPOSITION 5.9. – *Let $\mu : L \to [0, +\infty[$. Then:*

(1) *μ^* is upper-continuous on L and lower-continuous on $\sigma(M)$ (and therefore, by* Prop. 4.3, *μ is upper-D-continuous).*

(2) *If $\{a_n\}$ and $\{b_n\}$ are sequences in M such that $a_n \uparrow a$, $b_n \downarrow b$ and $a \geq b$, then $\inf_n \mu(b_n) \leq \sup_n \mu(a_n)$.*

PROOF. – (1) By 3.1 of [1], μ^* is upper-continuous on L. We prove that μ^* is lower-continuous on $\sigma(M)$.

For every $a \in L$ define

$$\underset{\sim}{\mu}(a) = \inf\{\mu(b) : b \in M, b \geq a\}$$

and

$$\mu_*(a) = \sup\{\underset{\sim}{\mu}(b) : b \in M_\delta, b \leq a\}.$$

As in Prop 5.3, we can prove that $\underset{\sim}{\mu}$ is lower-continuous on M_δ. Hence, by (3.1) of [1], μ_* is lower-continuous on L. We prove that $\mu^* = \mu_*$ on $\sigma(M)$, whence the assertion.

The proof consists of four steps:

(i) We first prove that, for every $a \in L$, $\underset{\sim}{\mu}(a) = \mu(1) - \tilde{\mu}(a^\perp)$. Let $b \in M$ with $b \geq a$. Then $\tilde{\mu}(a^\perp) \geq \tilde{\mu}(b^\perp) = \mu(1) - \mu(b)$, whence $\mu(1) - \tilde{\mu}(a^\perp) \leq \mu(b)$. Hence $\mu(1) - \tilde{\mu}(a^\perp) \leq \underset{\sim}{\mu}(a)$.

Conversely, let $c \in M$ with $c \leq a^\perp$. Then $\underset{\sim}{\mu}(a) \leq \mu(c^\perp) = \mu(1) - \mu(c)$, whence $\tilde{\mu}(a^\perp) \leq \mu(1) - \underset{\sim}{\mu}(a)$.

(ii) We now prove that, for every $a \in L$, $\mu_*(a) = \mu(1) - \mu^*(a^\perp)$.

Let $b \in M_\delta$ with $b \leq a$. Then, by (i), we have $\mu^*(a^\perp) \leq \tilde{\mu}(b^\perp) = \mu(1) - \underset{\sim}{\mu}(b)$, whence $\mu(1) - \mu^*(a^\perp) \geq \mu_*(a)$. Conversely, let $c \in M_\sigma$ with $c \geq a^\perp$. Again by (i), we have $\mu_*(a) \geq \underset{\sim}{\mu}(c^\perp) = \mu(1) - \tilde{\mu}(c)$, whence $\mu^*(a^\perp) \geq \mu(1) - \mu_*(a)$.

(iii) We prove that $\mu_* \leq \mu^*$.

By (ii) and Theor. 5.7, we have, for every $a \in L$, $\mu_*(a) = \mu(1) - \mu^*(a^\perp) \leq \mu^*(a) + \mu^*(a^\perp) - \mu^*(a^\perp) = \mu^*(a)$.

(iv) We finally prove that $\mu^* = \mu_*$ on $\sigma(M)$.

Set $A = \{a \in L : \mu^*(a) = \mu_*(a)\}$. Since $\mu^* = \mu_*$ on M, by Theor. 3.4 it is sufficient to prove that A is monotone. Let $\{a_n\} \subseteq A$ be such that $a_n \uparrow a$. Then, since μ^* is upper-continuous on L, we have $\mu^*(a) = \lim_n \mu^*(a_n) = \lim_n \mu_*(a_n) \leq \mu_*(a)$. By (iii), we obtain that $a \in A$. On the other side, take a sequence $\{a_n\} \subseteq A$ such that $a_n \downarrow a$. Since μ_* is lower-continuous on L, we have $\mu_*(a) = \lim_n \mu_*(a_n) = \lim_n \mu^*(a_n) \geq \mu^*(a)$. Then, by (iii), a belongs to A.

Statement (2) trivially follows from (1). □

6. – The extension theorem.

We are now ready to establish the main result.

As in Section 5, M is a D-sublattice of L and $\mu : M \to G$ is a σ-additive exhaustive modular measure.

The next result (6.14 and 8.2.1 of [27]) is an essential tool for the proof of our main theorem.

THEOREM 6.1. – *Let L' be a lattice, $S \subseteq L'$ a sublattice and $\mathcal{U}$ a lattice uniformity on L'. Then:*

(1) *The closure of S in $(L',\mathcal{U})$ is a sublattice of L'.*
(2) *If $\mathcal{U}$ is exhaustive on S, then $\mathcal{U}$ is exhaustive on $\overline{S}$.*
(3) *If $\mathcal{U}$ has the property (σ) and the restriction of $\mathcal{U}$ to S is exhaustive, then $\overline{S}$ is monotone.*

THEOREM 6.2. – *Suppose that the group G is normed and that the measure μ is upper-D-continuous. Then μ can be uniquely extended to a σ-additive exhaustive modular measure $\overline{\mu} : \sigma(M) \to G$, and M is dense in $(\sigma(M),\mathcal{U}(\overline{\mu}))$.*

PROOF. – Theor. 5.8, ensures that μ^* is a σ-submeasure on $\sigma(M)$. By 3.2 of [6], it generates a D-uniformity $\mathcal{U}(\mu^*)$ on $\sigma(M)$ which makes μ^* uniformly continuous. Rem. 5.2 gives $\mathcal{U}(\mu) = \mathcal{U}(\tilde{\mu}_{|M}) = \mathcal{U}(\mu^*_{|M})$. This means that μ is uniformly continuous with respect to $\mathcal{U}(\mu^*)$, too. Therefore, denoting by $\overline{M}$ the closure of M in $(\sigma(M),\mathcal{U}(\mu^*))$, μ can be extended to a function $\overline{\mu} : \overline{M} \to G$, which is uniformly continuous with respect to $\mathcal{U}(\mu^*)$. Observe that $\mathcal{U}(\mu^*_{|M}) = \mathcal{U}(\mu)$ is exhaustive and that, by Prop 4.1 and Theor. 5.7, $\mathcal{U}(\mu^*)$ has the property (σ). At this point we use Theor. 6.1 to obtain that $\mathcal{U}(\mu^*)$ is exhaustive and $\overline{M}$ is closed with respect to the limits of monotone sequences. Hence $\overline{M} = \sigma(M)$. Moreover, since μ^* is exhaustive and has the property (σ), by 8.1.2 of [27] μ^* is σ-o.c. Hence $\overline{\mu}$ is exhaustive and σ-o.c., too, since $\overline{\mu}$ is uniformly continuous with respect to $\mathcal{U}(\mu^*)$.

Let us show now that $\overline{\mu}$ is a modular measure. Take two orthogonal elements a, b in $\sigma(M)$. Choose sequences $\{a_n\}$, $\{b_n\}$ in M which converge, respectively, to a and b in $(\sigma(M),\mathcal{U}(\mu^*))$. We may assume that, for each n, $a_n \perp b_n$ since we can replace $\{a_n\}$ by $\{a_n \wedge b_n^{\perp}\}$. Since $\mathcal{U}(\mu^*)$ is a D-uniformity, we have that $\{a_n \oplus b_n\}$ converges to $a \oplus b$ in $\mathcal{U}(\mu^*)$. Since $\overline{\mu}$ is continuous with respect to $\mathcal{U}(\mu^*)$, the sequence $\{\overline{\mu}(a_n \oplus b_n)\}$ converges to $\overline{\mu}(a \oplus b)$. On the other hand, since $\overline{\mu} = \mu$ on M, then

$$\overline{\mu}(a_n \oplus b_n) = \mu(a_n) + \mu(b_n) \to \overline{\mu}(a) + \overline{\mu}(b).$$

Then $\overline{\mu}(a \oplus b) = \overline{\mu}(a) + \overline{\mu}(b)$.

In the same way we see that $\overline{\mu}$ is modular.

From $\mathcal{U}(\overline{\mu}) \leq \mathcal{U}(\mu^*)$, it follows that M is dense in $(\sigma(M),\mathcal{U}(\overline{\mu}))$.

Now let $\mu_1, \mu_2 : \sigma(M) \to G$ be σ-additive modular measures such that $\mu_1 = \mu_2 = \mu$ on M. Set $\upsilon = \mu_1 - \mu_2$. Since υ is a σ-o.c. modular measure, υ has σ by 8.1.2 of [27], and $\upsilon_{|M} = 0$ is exhaustive. By Theor. 6.1, M is dense in $(\sigma(M),\mathcal{U}(\upsilon))$, Hence $\upsilon = 0$ on $\sigma(M)$. $\square$

REMARK 6.3. – *Note that, by Prop* 4.4, *Theor.* 6.2 *represents an extension of* 2.2.1 *of* [4].

A Carathéodory type extension theorem for positive real-valued modular measures immediately follows from the latter result.

COROLLARY 6.4. – *Let $\hat{\mu} : M \to [0, +\infty[$ be a σ-additive modular measure- Then $\hat{\mu}$ can be uniquely extended to a σ-additive modular measure $\overline{\mu} : \sigma(M) \to [0, +\infty[$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$.*

PROOF. – By 2.6 of [7], $\hat{\mu}$ is exhaustive. Moreover, by Prop 5.9, $\hat{\mu}$ is upper-D-continuous. The assertion follows by Theor. 6.2. □

REMARK 6.5. – *Note that* Cor. 6.4 *above can be also obtained as a consequence of Theor.* 5 *of* [1]. *It is sufficient to observe that Condition* 4.4 *of* [1], *i.e. Statement* (2) *of Prop* 5.9, *is satisfied and that, by the density of M in $(\sigma(M), \mathcal{U}(\overline{\mu}))$, the extension of* [1] *is also a measure.*

Again, as a consequence of the previous Theor. 6.2, we obtain a further Carathéodory type extension theorem.

In the next result a *control measure* for a modular measure $\mu : M \to G$ is a modular measure $\upsilon : M \to [0, +\infty[$ such that $\mathcal{U}(\upsilon) = \mathcal{U}(\mu)$.

COROLLARY 6.6. – *Suppose that $\mu : M \to G$ has a control measure. Then μ can be uniquely extended to a σ-additive exhaustive G-valued modular measure $\overline{\mu}$ on $\sigma(M)$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$.*

PROOF. – Let $\upsilon : M \to [0, +\infty[$ be a control measure for μ. The equality $\mathcal{U}(\mu) = \mathcal{U}(\upsilon)$ ensures that υ is exhaustive and σ-o.c., hence σ-additive by 2.2 of [3]. Cor. 6.4, ensures that υ can be extended to a σ-additive exhaustive modular measure $\overline{\upsilon} : \sigma(M) \to [0, +\infty[$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\upsilon}))$. Moreover, since μ is uniformly continuous with respect to $\mathcal{U}(\upsilon)$ and $\mathcal{U}(\upsilon) \leq \mathcal{U}(\overline{\upsilon})_{|M \times M}$, we have that μ is uniformly continuous with respect to $\mathcal{U}(\overline{\upsilon})$. This means that μ can be extended to a function $\overline{\mu} : \sigma(M) \to G$ which is uniformly continuous with respect to $\mathcal{U}(\overline{\upsilon})$.

As in Theor. 6.2, using the fact that M is dense in $(\sigma(M), \mathcal{U}(\overline{\upsilon}))$ and that $\mathcal{U}(\overline{\upsilon})$ is a D-uniformity, we can prove that $\overline{\mu}$ is a modular measure. The uniform continuity of $\overline{\mu}$ with respect to $\mathcal{U}(\overline{\upsilon})$, implies that $\overline{\mu}$ is exhaustive and σ-additive and that M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$.

The uniqueness of $\overline{\mu}$ can be proved as in Theor. 6.2. □

We recall that, if $\| \cdot \|$ is a seminorm on G, for a modular measure $\mu : L \to G$, the *total variation* is the function $|\mu| : L \to [0, +\infty]$ defined as

$$|\mu|(a) = \sup\left\{ \sum_{i=0}^{n-1} \|\mu(a_{i+1}) - \mu(a_i)\| : a_0 \leq a_1 \leq \cdots \leq a_n = a \right\},\ a \in L.$$

By 1.3.10 of [28], $|\mu|$ is a modular function. Moreover, with a proof similar to 3.11 of [8], it is possible to prove that $|\mu|$ is a measure.

COROLLARY 6.7. – *Suppose that G is normed and that μ has bounded variation. Then μ can be uniquely extended to a σ-additive exhaustive modular measure $\overline{\mu} : \sigma(M) \to G$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$.*

PROOF. – By 1.3.11 of [29], the total variation $|\mu|$ of μ generates the same topology and, by (2.4) of [6], the same uniformity. Hence $|\mu|$ is a control measure for μ. At this point, it is sufficient to apply Cor. 6.6 to obtain the thesis. □

Cor. 6.6 also allows to obtain the extension result for functions with values in Hausdorff linear spaces. The way of reasoning is the same as that of [4].

COROLLARY 6.8. – *Suppose that G is a complete Hausdorff locally convex linear space. Then μ can be uniquely extended to a σ-additive exhaustive modular measure $\overline{\mu} : \sigma(M) \to G$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$.*

PROOF. – If G is metrizable, we can apply Cor. 6.6 since, by 4.2 and 3.3 of [5], μ has a control measure.

Assume then that G is not metrizable. In this case, G can be embedded in a product $\prod_{a \in A} G_a$ of metric spaces and therefore μ can be seen as a function $(\mu_a)_{a \in A}$ with values in $\prod_{a \in A} G_a$.

For each $a \in A$, μ_a can be extended to a σ-additive exhaustive modular measure $\overline{\mu}_a : \sigma(M) \to G_a$, and M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}_a))$. Set $\overline{\mu} = (\overline{\mu}_a)_{a \in A}$. Then $\overline{\mu} : \sigma(M) \to \prod_{a \in A} G_a$ is a σ-additive exhaustive modular measure which extends μ. Using the fact that $\mathcal{U}(\overline{\mu}_{|M}) = \mathcal{U}(\mu)$ is exhaustive and that, given 8.1.2 of [27], $\mathcal{U}(\overline{\mu})$ has the property (σ) , we obtain by an application of Theor. 6.1 that M is dense in $(\sigma(M), \mathcal{U}(\overline{\mu}))$. At this point

$$\overline{\mu}(\sigma(M)) \subseteq \overline{\mu(M)} \subseteq G$$

and therefore $\overline{\mu}$ is a G-valued extension of μ to $\sigma(M)$.

The uniqueness of $\overline{\mu}$ can be proved as in Theor. 6.2. □

We can now prove the following generalization of [14] (Corollary 1.5.2).

THEOREM 6.9. – *Let L_1 be a D-lattice, X a complete metrizable locally convex linear space and $\lambda : L_1 \to X$ a modular measure. Then the following are equivalent:*

(1) *λ is exhaustive;*

(2) $\lambda(L_1)$ *is relatively weakly compact;*

(3) λ *has a control measure.*

Moreover, if L_2 *is a* σ*-continuous D-lattice which contains* L_1 *as a D-sublattice and* λ *is* σ*-additive, they are also equivalent to the following:*

(4) λ *has a (unique) extension to a* σ*-additive exhaustive modular measure on the* σ*-D-sublattice* $\sigma(L_1)$ *of* L_2 *generated by* L_1.

PROOF. – (1) $\Leftrightarrow$ (2). By 4.1 of [28], the equivalence between exhaustivity and relative weak compactness of the range holds for any modular function λ on any lattice L which satisfies the following condition: for every finite chain $a_0 \leq ... \leq a_n$ in L and for every $I \subseteq \{1, ..., n\}$,

$$(*) \qquad \sum_{i \in I} [\lambda(a_i) - \lambda(a_{i-1})] \in \lambda(L) - \lambda(L).$$

By the proof of 2.6 of [7], every modular measure on L_1 satisfies $(*)$, hence the equivalence holds.

(3) $\Rightarrow$ (1). Let υ be a control measure for λ. By 2.6 of [7], υ is exhaustive. Since $\mathcal{U}(\lambda) = \mathcal{U}(\upsilon)$, then λ is exhaustive, too.

(1) $\Rightarrow$ (3) follows by 4.2 of [5].

For the second part, we observe that (4) follows from (1) given Cor. 6.8, the reverse implication, (4) $\Rightarrow$ (1) is trivial. □

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