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On the Lower Semicontinuity of Supremal Functionals Defined on Measures.

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Sunto. – *In questo lavoro si considerano due particolari classi di funzionali supremali definiti sulle misure di Radon e si determinano alcune condizioni necessarie e sufficienti alla loro semicontinuità rispetto alla convergenza debole*. Vengono successivamente presentate alcune applicazioni di questi risultati alla minimizzazione di opportuni funzionali definiti su BV.*

Summary. – *In this paper we consider two particular classes of supremal functionals defined on Radon measures and we find necessary and sufficient conditions for their lower semicontinuity with respect to the weak* convergence. Some applications to the minimization of functionals defined on BV are presented.*

1. – Introduction.

The study of integral functionals defined on Radon measures was first approached by Goffman and Serrin in [21] and subsequently many authors developed this argument (see [4, 11, 15, 16, 18]). The simplest type of such functionals takes the form

$$\int_{\Omega} f(\lambda^a(x))dx + \int_{\Omega} f^{\infty}\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)d|\lambda^s|(x), \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$, f^{∞} is the recession function of f and $\lambda = \lambda^a \cdot \mathcal{L}^n + \lambda^s$ is the Radon-Nikodym decomposition of λ with respect to the Lebesgue measure $\mathcal{L}^n$.

As concerns the functionals of the form (1), it is well known that the convexity and the lower semicontinuity of the function f are necessary and sufficient conditions in order to have the lower semicontinuity with respect to the w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ convergence; moreover, in these hypotheses, (1) is also convex on $\mathcal{M}(\Omega, \mathbb{R}^m)$.

When λ is the gradient Du of $u \in BV(\Omega)$, the functional (1) can be written as

$$\int_{\Omega} f(\nabla u(x))dx + \int_{\Omega} f^{\infty}\left(\frac{dD^s u}{d|D^s u|}(x)\right)d|D^s u|(x), \tag{2}$$

where ∇u and $D^s u$ respectively are the absolutely continuous and the singular part of the decomposition of Du with respect to $\mathcal{L}^n$. The lower semicontinuity results proved for (1) allow to apply the direct methods in order to prove the existence of solutions of several minimization problems involving (2).

However, a great class of interesting functionals presents relevant effects of non convex terms so that they cannot be treated with functionals of the type (1) and (2). The typical example is given by the the functional

$$\int_\Omega |\nabla u(x)|^2 dx + \int_{S_u} d\mathcal{H}^{n-1}(x) + \int_\Omega |u(x) - g(x)|^2 dx, \tag{3}$$

that occurs in the theory of mechanics of fractures and in problems of computer vision (for further details see [3, 17]). Here $u \in BV(\Omega)$, $g \in L^2(\Omega)$, S_u is the set of the *jump points* of u and $\mathcal{H}^{n-1}$ is the $(n-1)$ dimensional Hausdorff measure.

Moving from these considerations Bouchitté and Buttazzo proposed a generalization of (1) given by

$$\int_\Omega f(\lambda^a(x))dx + \int_\Omega f^\infty\left(\frac{d\lambda^c}{d|\lambda^c|}(x)\right) d|\lambda^c|(x) + \sum_{x \in A_\lambda} \phi(\lambda^\#(x)), \tag{4}$$

where $\lambda = \lambda^a \cdot \mathcal{L}^n + \lambda^c + \lambda^\#$, $\lambda^c \perp \lambda^\#$, $\lambda^\#$ is purely atomic and A_λ is the set of the atoms of λ, and they were able to find necessary and sufficient conditions for its lower semicontinuity with respect to the w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ convergence (see [12, 13, 14]). Clearly, when the one dimensional case is considered, the results proved for (4) gives information on (3) too.

In the last decade, other types of functionals that show a non convex behavior have been studied by considering the sup norm: sometimes they are called *supremal* functionals (see [1, 2, 9, 10, 23]). The main example of such functionals is given by

$$\operatorname*{ess\,sup}_{x \in \Omega} f(\nabla u(x)), \tag{5}$$

where $u \in W^{1,\infty}(\Omega)$ and the essential supremum is considered with respect to $\mathcal{L}^n$.

As shown in [9], the lower semicontinuity of (5) with respect to the w^*-$W^{1,\infty}(\Omega)$ convergence is equivalent to ask the lower semicontinuity and the *level convexity* of f; we remember that f is level convex if every sub-level set of f is a convex set.

It is worth noting that, supposing the lower semicontinuity of (5), the Dirichlet problem

$$\min\left\{ \operatorname*{ess\,sup}_{x \in \Omega} f(\nabla u(x)) : u \in W^{1,\infty}(\Omega), u = \varphi \text{ on } \partial\Omega \right\} \tag{6}$$

can be solved provided f is coercive, that is, there exists $\theta : [0, \infty) \to [0, \infty)$ such that

$$f(\xi) \geq \theta(|\xi|) \quad \text{and} \quad \lim_{t \to \infty} \theta(t) = \infty.$$

The celebrated problem to find of the best Lipschitz extension of a function defined on $\partial\Omega$ has clearly the form (6) with $f(\xi) = |\xi|$ (see for instance [8, 25]).

Having in mind the results proved for the integral case, in this paper we approach the problem to find suitable supremal functionals defined on $\mathcal{M}(\Omega, \mathbb{R}^m)$ that satisfy the property to be lower semicontinuous (briefly l.s.c.) with respect to the w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ convergence.

After several preliminary results presented in Section 2 and extensively used along all the paper, in Section 3 we consider the functional

$$F(\lambda, \Omega) = \left[\operatorname*{ess\,sup}_{x\in\Omega}\ f(\lambda^a(x))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega}\ f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)\right], \tag{7}$$

where the function $f^\natural$ is defined as

$$f^\natural(\xi) = \inf\left\{\liminf_{h\to\infty} f(t_h\xi_h) : \xi_h \to \xi, t_h \uparrow \infty\right\}.$$

Using techniques similar to the ones used in [21], we prove that F is l.s.c. with respect to the w^*-$\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ convergence if and only if f is l.s.c. and level convex (see Theorem 23), and that, under these hypotheses, F is also level convex on $\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, finding in this way a complete analogy between the integral and the supremal settings.

In Section 4 we carry on the study of another and more general type of functional, given, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, by

$$\mathbb{F}(\lambda, \Omega) = \left[\operatorname*{ess\,sup}_{x\in\Omega}\ f(\lambda^a(x))\right]$$

$$\vee \left[|\lambda^c|\text{-}\operatorname*{ess\,sup}_{x\in\Omega}\ f^\natural\left(\frac{d\lambda^c}{d|\lambda^c|}(x)\right)\right] \vee \left[\bigvee_{x\in A_\lambda} \phi(\lambda^\#(x))\right]. \tag{8}$$

Note that, even if f is l.s.c. and level convex, in general $\mathbb{F}$ fails to be level convex on $\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, because of the dependence on ϕ.

After having proved that in fact $\mathbb{F}$ generalizes F (that is, $\mathbb{F} \equiv F$ whenever $\phi \equiv f^\natural$), we approach the problem to find necessary and sufficient conditions for the lower semicontinuity of $\mathbb{F}$ with respect to the w^*-$\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ convergence (see Theorems 6, 8 and 9).

Finally, Section 5 is devoted to some simple applications of the results obtained in Sections 3 and 4 to the BV setting. In particular, by means of the results about F, we propose a generalized version of the Dirichlet problem (6) defined on $BV(\Omega)$ and, even in the case in which f is not coercive, we find conditions for the existence of a minimum point.

Subsequently, thanks to the results proved for $\mathbb{F}$, we prove the existence of a minimum point also for a particular class of one dimensional functionals defined on BV and similar to (3). We point out that this topic has been already approached by Alicandro, Braides and Cicalese [2].

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2. – Notations and preliminary results.

2.1. *Some tools from measure theory.*

Let us consider the measurable space $(\Omega, \mathcal{B}(\Omega))$ where $\Omega \subseteq \mathbb{R}^n$ is an open set and $\mathcal{B}(\Omega)$ is the σ-algebra of the Borel subsets of Ω.

We say that K is *well contained* in Ω (briefly $K \subset\subset \Omega$) if $\mathrm{cl}(K) \subseteq \Omega$, that is, the *closure* of K is a subset of Ω. Moreover, for every $x \in \mathbb{R}^n$ and $\rho > 0$, we set

$$B(x, \rho) = \{y \in \mathbb{R}^n : |x - y| < \rho\} \quad \text{and} \quad S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

If $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ is a positive measure such that, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$, it is $\mu(K) < \infty$, then μ is called a *positive Radon measure* on Ω: the set of the positive Radon measures on Ω is denoted by $\mathcal{M}^+(\Omega)$. Obviously the Lebesgue measure on $\mathbb{R}^n$, denoted with $\mathcal{L}^n$, belongs to $\mathcal{M}^+(\mathbb{R}^n)$ so as the standard Dirac measure centered on $x \in \mathbb{R}^n$ denoted with δ_x.

Let us fix $m \in \mathbb{N}$: if $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is a vector measure then it is called a *finite Radon measure* on Ω while if $\lambda : \{K \in \mathcal{B}(\Omega) : K \subset\subset \Omega\} \to \mathbb{R}^m$ and, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$, λ is a vector measure on $\mathcal{B}(K)$ then it is called a *Radon measure* on Ω. We denote the set of Radon measures (resp. finite Radon measures) with $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}(\Omega, \mathbb{R}^m)$).

Let us consider now $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$. For every $B \in \mathcal{B}(\Omega)$ the *total variation* of λ on B is defined by (1).

$$|\lambda|(B) = \sup\left\{ \sum_{j=1}^{r} |\lambda(B_j)| : B_j \in \mathcal{B}(\Omega), B_j \subset\subset \Omega,\ \bigcup_{j=1}^{r} B_j \subseteq B, B_i \cap B_j = \emptyset,\ \forall i \neq j \right\}. \tag{9}$$

It is well known that $|\lambda| \in \mathcal{M}^+(\Omega)$, that $\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$ implies $|\lambda|(\Omega) < \infty$ and that if $\lambda \in \mathcal{M}^+(\Omega)$ then $|\lambda| = \lambda$.

Given now $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ we say that λ is *concentrated* on a set $B \in \mathcal{B}(\Omega)$ if $|\lambda|(\Omega \setminus B) = 0$. If $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ we say that λ_1 is *singular* with respect to λ_2 (briefly $\lambda_1 \perp \lambda_2$) if there exist $B_1, B_2 \in \mathcal{B}(\Omega)$ such that λ_1 is concentrated in B_1, λ_2 is concentrated in B_2 and $B_1 \cap B_2 = \emptyset$.

(1) When $x \in \mathbb{R}^n$, $|x|$ means $\|x\|_{\mathbb{R}^n}$ the standard Euclidean norm on $\mathbb{R}^n$.

If now we consider $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(\Omega)$ we say that λ is *absolutely continuous* with respect to μ (briefly $\lambda << \mu$) if $\mu(B) = 0$ implies $|\lambda|(B) = 0$. Moreover, fixed $E \in \mathcal{B}(\Omega)$, we denote with $\lambda \llcorner E$ the *restriction* of λ to E, that is, the element of $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ defined, for every $B \in \mathcal{B}(\Omega)$ (when $\lambda \notin \mathcal{M}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$, we need $B \subset\subset \Omega$ too), as $\big(\lambda \llcorner E\big)(B) = \lambda(E \cap B)$. For every $E \in \mathcal{B}(\mathbb{R}^n)$ we will always denote with $\mathcal{L}^n$ the measure $\mathcal{L}^n \llcorner E$.

We write for short $\lambda(x)$ instead of $\lambda(\{x\})$ and we denote

$$A_\lambda = \{x \in \Omega : \lambda(x) \neq 0\},$$

the *set of the atoms* of λ. Finally, if $\mu \in \mathcal{M}^+(\Omega)$, we define the *support* of μ as the set

$$\mathrm{spt}(\mu) = \mathrm{cl}\Big(\Big\{x \in \Omega : \forall \rho > 0, \mu(B(x, \rho)) > 0\Big\}\Big),$$

while, if $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, we set $\mathrm{spt}(\lambda) = \mathrm{spt}(|\lambda|)$: note that λ is concentrated on $\mathrm{spt}(\lambda)$.

If we consider now $\mu \in \mathcal{M}^+(\Omega)$ and a function $u \in L^1_\mu(\Omega, \mathbb{R}^m)$ (resp. $L^1_{\mathrm{loc},\mu}(\Omega, \mathbb{R}^m)$) we denote with $u \cdot \mu$ the element of $\mathcal{M}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$) defined, for every $B \in \mathcal{B}(\Omega)$ (resp. $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$), as

$$(u \cdot \mu)(B) = \int_B u(x) d\mu(x);$$

we have $|u \cdot \mu| = |u| \cdot \mu$ and $u \cdot \mu << \mu$. A standard result of measure theory (see [6] Theorem 1.28) says that, given $\mu \in \mathcal{M}^+(\Omega)$, we can decompose in a unique way a measure $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ as

$$\lambda = \lambda^a \cdot \mu + \lambda^s, \tag{10}$$

where $\lambda^a \in L^1_{\mathrm{loc},\mu}(\Omega, \mathbb{R}^m)$ and $\lambda^s \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with $\lambda^s \perp \mu$: we call $\lambda^a \cdot \mu$ the *absolutely continuous part* of λ while λ^s its *singular part*. With the notation quoted above, we can also write

$$\lambda = \lambda^a \cdot \mu + \lambda^c + \lambda^\#, \tag{11}$$

where $\lambda^c = \lambda^s \llcorner (\Omega \setminus A_\lambda)$ is said the *Cantor part* of λ while $\lambda^\# = \lambda^s \llcorner A_\lambda$ is said its *atomic part*. These decompositions of λ obviously depend on μ even if in the notations λ^a, λ^s, λ^c and $\lambda^\#$ the measure μ is not expressly named: however, every time one of these decompositions is used, the measure μ will be clear by the context. Finally note that, if $\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$, then $\lambda^a \in L^1_\mu(\Omega, \mathbb{R}^m)$ and $\lambda^s, \lambda^c, \lambda^\# \in \mathcal{M}(\Omega, \mathbb{R}^m)$.

Let $\lambda_h, \lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$): we say $\lambda_h \to \lambda$ in w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ (resp. w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$) if

$$\lim_{h \to \infty} \int_\Omega \varphi(x) d\lambda_h(x) = \int_\Omega \varphi(x) d\lambda(x),$$

for every $\varphi \in C_0(\Omega)$, that is, the set of the continuous functions on Ω with the property that, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq \Omega$ such that, for every $x \in \Omega \setminus K_\varepsilon$, $|\varphi(x)| < \varepsilon$ (resp. for every $\varphi \in C_c(\Omega)$, that is, the set of the continuous functions on Ω with compact support).

The compactness theorems about this kind of convergence are considered known (see for instance [6] Theorem 1.59 and Corollary 1.60).

It's suitable at this point to introduce also the following notion of convergence. Let $\{\Omega_h\}_{h=1}^{\infty}$ be a family of open subsets of $\mathbb{R}^n$ and let $\lambda_h \in \mathcal{M}_{\mathrm{loc}}(\Omega_h, \mathbb{R}^m)$, $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$: we say $\lambda_h \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ if, for every $K \subset\subset \Omega$, we have $K \subseteq \Omega_h$ when h is large enough and, for every $\varphi \in C_c(\Omega)$,

$$\lim_{h\to\infty} \int_{\Omega_h} \varphi(x)d\lambda_h(x) = \int_{\Omega} \varphi(x)d\lambda(x).$$

Given a measure $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $\rho > 0$, we define the *convolution* of λ with step ρ as

$$\lambda_\rho(x) = \int_{B(x,\rho)} \rho^{-n} k\left(\frac{x-y}{\rho}\right) d\lambda(y) : \Omega_\rho \to \mathbb{R}^m. \tag{12}$$

In the previous definition $k : \mathbb{R}^n \to [0, \infty)$ is a *convolution kernel*, that is $k \in C_c^\infty(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$, $k(x) = k(-x)$, $\mathrm{spt}(k) \subseteq B(0,1)$ and $\int_{\mathbb{R}^n} k(x)dx = 1$ (we require also that $k(0) \neq 0$), and

$$\Omega_\rho = \{x \in \Omega : d(x, \partial\Omega) > \rho\},$$

where $d(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}$ and $\partial\Omega$ is the topological boundary of Ω.

It is well known that $\lambda_\rho \in C^\infty(\Omega_\rho, \mathbb{R}^m)$ and it can be simply proved using Fubini's Theorem that $\lambda_\rho \cdot \mathcal{L}^n \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ as $\rho \to 0$ (see [6] Theorem 2.2). In the following, referring to a convolution kernel k, we will set $k_\rho(x) = \rho^{-n} k(\rho^{-1}x)$.

Now we propose a theorem on measures that describes the point-wise behavior of the convolutions of a measure and that will be fundamental in proving some results of the following chapters. The principal tools used in its proof are the Besicovich's Derivation Theorem for Radon measures and the Lebesgue's points Theorem (see [6] Theorem 2.22 and Corollary 2.23). This theorem generalizes Theorem 3 in [22].

THEOREM 1. – *Let $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, referring to the decomposition* (10) *with respect to $\mathcal{L}^n$, we have*

(i) *for $\mathcal{L}^n$-a.e. $x \in \Omega$, $\lim_{h\to\infty} |\lambda_\rho(x) - \lambda^a(x)| = 0$;*

(ii) *for $|\lambda^s|$-a.e. $x \in \mathrm{spt}(\lambda^s)$, there exists a sequence of positive numbers*

$\{\rho_h\}_{h=1}^{\infty}$, *depending on x and decreasing to zero, such that* [2]

$$\lim_{h\to\infty}\left|\frac{d\lambda^s}{d|\lambda^s|}(x)-\frac{\lambda_{\rho_h}(x)}{\displaystyle\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)}\right|=0. \tag{13}$$

For simplicity we prove Theorem 1 by means of two lemmas that, in our opinion, are interesting on their own. The first lemma is a simple fact from measure theory and, in this form, can be found in [5], Theorem 2.3.

LEMMA 1. – *Let $\mu\in\mathcal{M}^+(\Omega)$ such that $\mu\perp\mathcal{L}^n$. Then, for μ-a.e. $x\in\operatorname{spt}(\mu)$ and, for every $\sigma\in(0,1)$, we have*

$$\sigma^n\le\limsup_{\rho\to 0}\frac{\mu(B(x,\sigma\rho))}{\mu(B(x,\rho))}\le 1.$$

PROOF. – Let us set

$$\Omega(\mu)=\left\{x\in\operatorname{spt}(\mu):\lim_{\rho\to 0}\frac{\mu(B(x,\rho))}{\rho^n}=\infty\right\}: \tag{14}$$

since $\mu(\Omega\setminus\Omega(\mu))=0$ and $\mathcal{L}^n(\Omega(\mu))=0$ (see [6] Theorem 2.22), we achieve the proof showing the wanted relation for every $x\in\Omega(\mu)$. Let us suppose, by contradiction, there exist $x_0\in\Omega(\mu)$ and $\sigma_0\in(0,1)$ such that

$$\sigma_0^n>\limsup_{\rho\to 0}\frac{\mu(B(x_0,\sigma_0\rho))}{\mu(B(x_0,\rho))}.$$

Then there exists ρ_0 such that, for every $0<\rho\le\rho_0$, $\mu(B(x_0,\sigma_0\rho))\le\sigma_0^n\mu(B(x_0,\rho))$. If we call $\omega(\rho)=\mu(B(x_0,\rho))$ we have that, for every $0<\rho\le\rho_0$, $\omega(\sigma_0\rho)\le\sigma_0^n\omega(\rho)$. Then, for every $h\in\mathbb{N}$, $\sigma_0^{-nh}\omega(\sigma_0^h\rho_0)\le\omega(\rho_0)$, thus

$$\frac{\mu(B(x_0,\sigma_0^h\rho_0))}{(\sigma_0^h\rho_0)^n}\rho_0^n\le\omega(\rho_0)=\mu(B(x_0,\rho_0))<\infty.$$

If now $h\to\infty$, then $\sigma_0^h\rho_0\to 0$ and the left hand side of the previous inequality tends to infinity: thus the contradiction is found. $\square$

In particular, fixed $\sigma\in(0,1)$, for μ-a.e. $x\in\operatorname{spt}(\mu)$, there exists a sequence $\{\rho_h\}_{h=1}^{\infty}$, depending on x and decreasing to zero, such that, for every $h\in\mathbb{N}$,

$$\mu(B(x,\sigma\rho_h))\ge\sigma^{n+1}\mu(B(x,\rho_h)). \tag{15}$$

(2) We point out that, for $|\lambda^s|$-a.e. $x\in\operatorname{spt}(\lambda^s)$, $\frac{d\lambda^s}{d|\lambda^s|}(x)\in S^{m-1}$ and $\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)\neq 0$.

LEMMA 2. – *Let $\lambda^s \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$ such that $\lambda^s \perp \mathcal{L}^n$ and $\mu << \mathcal{L}^n$. Then, for $|\lambda^s|$-a.e. $x \in \mathrm{spt}(\lambda^s)$, there exists a sequence of positive numbers $\{\rho_h\}_{h=1}^\infty$, depending on x and decreasing to zero, such that*

$$\lim_{h\to\infty} \frac{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d\mu(y)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = 0 \tag{16}$$

and

$$\lim_{h\to\infty} \frac{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)\left|\frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x)\right| d|\lambda^s|(y)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = 0. \tag{17}$$

PROOF. – Obviously $\lambda^s \perp \mu$. Thus, let us consider $\Omega(|\lambda^s|)$ defined as in (14), and the set

$$\Omega_0 = \Omega(|\lambda^s|) \cap \left\{ x \in \mathrm{spt}(\lambda^s) : \lim_{\rho\to 0} \frac{\mu(B(x,\rho))}{|\lambda^s|(B(x,\rho))} = 0 \right\}$$
$$\cap \left\{ x \in \mathrm{spt}(\lambda^s) : x \text{ is a Lebesgue's point for } \frac{d\lambda^s}{d|\lambda^s|} \right\}.$$

Clearly $|\lambda^s|(\Omega \setminus \Omega_0) = 0$: then the proof is achieved if, for every $x \in \Omega_0$, (16) and (17) hold. By the properties of the convolution kernel k, we can find $\sigma \in (0,1)$ and $c, M > 0$ such that, for every $x \in \mathbb{R}^n$, $k(x) \le M$ and, for every $x \in B(0,\sigma)$, $k(x) \ge c$ (remember that we suppose $k(0) \neq 0$). Thus, let us fix $x \in \Omega_0$ and, since $\Omega_0 \subseteq \Omega(|\lambda^s|)$, let us consider, with respect to $|\lambda^s|$, the sequence $\{\rho_h\}_{h=1}^\infty$ ($\rho_h < \sigma$) given by (15). Then we have

$$0 \le \lim_{h\to\infty} \frac{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d\mu(y)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = \lim_{h\to\infty} \frac{\int\limits_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right)d\mu(y)}{\int\limits_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right)d|\lambda^s|(y)}$$

$$\le \lim_{h\to\infty} \frac{M}{c} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\sigma\rho_h))} = \lim_{h\to\infty} \frac{M}{c} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\rho_h))} \cdot \frac{|\lambda^s|(B(x,\rho_h))}{|\lambda^s|(B(x,\sigma\rho_h))}$$

$$\le \lim_{h\to\infty} \frac{M}{c\sigma^{n+1}} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\rho_h))} = 0,$$

that proves (16). A similar computation allows to achieve (17). $\square$

PROOF (of Theorem 1). – *(i)*. See for instance [27] Theorem 2.

(ii). Let us fix $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and consider λ^s and $\mu = |\lambda^a(x)| \cdot \mathcal{L}^n$: since $\lambda^s \perp \mathcal{L}^n$ and $\mu << \mathcal{L}^n$ we can apply Lemma 2 to λ^s and μ. Thus, we can find $M \subseteq \Omega$ with $|\lambda^s|(M) = 0$ such that, for every $x \in \Omega \setminus M$, there exists a sequence $\{\rho_h\}_{h=1}^{\infty}$ satisfying the conditions (16) and (17) of Lemma 2. Then we simply end since, for every $x \in \Omega \setminus M$,

$$\left| \frac{\lambda_{\rho_h}(x)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} - \frac{d\lambda^s}{d|\lambda^s|}(x) \right|$$

$$\leq \frac{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)|\lambda^a(y)|dy}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} +$$

$$\frac{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)\left|\frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x)\right|d|\lambda^s|(y)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)},$$

that goes to zero as $h \to \infty$ by the conditions (16) and (17). □

We propose now two propositions.

PROPOSITION 1. – *Let $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $\lambda_1 \perp \lambda_2$. Then, for $|\lambda_1|$-a.e. $x \in \Omega$, $\frac{d\lambda_1}{d|\lambda_1|}(x) = \frac{d(\lambda_1+\lambda_2)}{d|\lambda_1+\lambda_2|}(x)$.*

PROOF. – Let $A \in \mathcal{B}(\Omega)$ such that $|\lambda_1|(\Omega \setminus A) = |\lambda_2|(A) = 0$: clearly we can prove the wanted equality only for $|\lambda_1|$-a.e. $x \in A$. Since $|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|$, $\lambda_1 << |\lambda_1|$ and $\lambda_1, \lambda_2 << |\lambda_1 + \lambda_2|$, we have that, for every $B \in \mathcal{B}(A)$,

$$\lambda_1(B) = \int\limits_B \frac{d\lambda_1}{d|\lambda_1|}(x)d|\lambda_1|(x),$$

and

$$\lambda_1(B) = \int\limits_B \frac{d\lambda_1}{d|\lambda_1 + \lambda_2|}(x)d|\lambda_1 + \lambda_2|(x) = \int\limits_B \frac{d\lambda_1}{d|\lambda_1 + \lambda_2|}(x)d|\lambda_1|(x).$$

Thus, for $|\lambda_1|$-a.e. $x \in A$, $\frac{d\lambda_1}{d|\lambda_1|}(x) = \frac{d\lambda_1}{d|\lambda_1+\lambda_2|}(x)$. We end noting that, with a similar argument, for $|\lambda_1|$-a.e. $x \in A$, $\frac{d\lambda_2}{d|\lambda_1+\lambda_2|}(x) = 0$. □

Before stating the following proposition, let us introduce some notations.

Given $\mu \in \mathcal{M}^+(\Omega)$ and a Borel function $\varphi : \Omega \to [0, \infty]$ we define the *essential supremum* on $B \in \mathcal{B}(\Omega)$ of φ with respect to μ as

$$\mu\text{- ess sup}_{x \in B} \ \varphi(x)$$

$$= \begin{cases} \inf\left\{ \sup\limits_{x \in B \setminus A} \varphi(x) : A \in \mathcal{B}(\Omega), A \subseteq B, \mu(A) = 0 \right\} & \text{if } \mu(B) \neq 0, \\ -\infty & \text{if } \mu(B) = 0, \end{cases}$$

pointing out that, if $\mu_1, \mu_2 \in \mathcal{M}^+(\Omega)$ with $\mu_1 << \mu_2$, then

$$\mu_1\text{- ess sup}_{x \in \Omega} \ \varphi(x) \leq \mu_2\text{- ess sup}_{x \in \Omega} \ \varphi(x). \tag{18}$$

At last, for every $a, b \in \mathbb{R}$, we set $a \vee b = \sup\{a, b\}$ while $a \wedge b = \inf\{a, b\}$.

PROPOSITION 2. – *Let* $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *such that* $\lambda_1 \perp \lambda_2$ *and let* $\varphi : \Omega \to [0, \infty]$ *be a Borel function. Then*

$$|\lambda_1 + \lambda_2|\text{- ess sup}_{x \in \Omega} \ \varphi(x) = \left[|\lambda_1|\text{- ess sup}_{x \in \Omega} \ \varphi(x) \right] \vee \left[|\lambda_2|\text{- ess sup}_{x \in \Omega} \ \varphi(x) \right].$$

PROOF. – Let $A \in \mathcal{B}(\Omega)$ such that $|\lambda_1|(\Omega \setminus A) = |\lambda_2|(A) = 0$. Since $|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|$, we have

$$\begin{aligned} |\lambda_1 + \lambda_2|\text{- ess sup}_{x \in \Omega} \ \varphi(x) &= \left[|\lambda_1 + \lambda_2|\text{- ess sup}_{x \in A} \ \varphi(x) \right] \vee \left[|\lambda_1 + \lambda_2|\text{- ess sup}_{x \in \Omega \setminus A} \ \varphi(x) \right] \\ &= \left[|\lambda_1|\text{- ess sup}_{x \in A} \ \varphi(x) \right] \vee \left[|\lambda_2|\text{- ess sup}_{x \in \Omega \setminus A} \ \varphi(x) \right] \\ &= \left[|\lambda_1|\text{- ess sup}_{x \in \Omega} \ \varphi(x) \right] \vee \left[|\lambda_2|\text{- ess sup}_{x \in \Omega} \ \varphi(x) \right]. \end{aligned}$$

□

2.2. *Some tools from convex analysis.*

Let $f : \mathbb{R}^m \to [0, \infty]$. We say that f is *level convex* if, for every $\xi, \eta \in \mathbb{R}^m$, $t \in (0, 1)$,

$$f(t\xi + (1 - t)\eta) \leq f(\xi) \vee f(\eta),$$

while we say that f is *sub-maximal* if, for every $\xi, \eta \in \mathbb{R}^m$, it is

$$f(\xi + \eta) \leq f(\xi) \vee f(\eta).$$

We define the *domain* of f as the set $\mathrm{dom}\,(f) = \{\xi \in \mathbb{R}^m : f(\xi) < \infty\}$, and when $\mathrm{dom}\,(f) \neq \emptyset$ we say that f is *proper*: in the following we will always deal with proper functions. The *sub-level set* of f to level $r \in [0, \infty]$ is defined as

$$E_f(r) = \{\xi \in \mathbb{R}^m : f(\xi) \leq r\},$$

and it is well known that f is level convex if and only if, for every $r \in [0, \infty]$, $E_f(r)$ is a convex set.

We can also define, for every $\xi \in \mathbb{R}^m$,

$$f^{\natural}(\xi) = \inf\left\{\liminf_{h\to\infty} f(t_h\xi_h) : \xi_h \to \xi, t_h \uparrow \infty\right\},$$

and

$$f^{\flat}(\xi) = \sup\left\{\limsup_{h\to\infty} f(t_h\xi) : t_h \downarrow 0\right\},$$

and let us note that, for every $\xi \in \mathbb{R}^m$, we can find suitable sequences $\xi_h \to \xi$ and $t_h \uparrow \infty$ (resp. $t_h \downarrow 0$) such that $f(t_h\xi_h) \to f^{\natural}(\xi)$ (resp. $f(t_h\xi) \to f^{\flat}(\xi)$) ([3]).

In the end we say that a function $f : \mathbb{R}^m \to [0, \infty]$ is *positively homogeneous* of degree 0 if, for every $\xi \in \mathbb{R}^m$, $t > 0$, it is $f(t\xi) = f(\xi)$, while we say that f is *demi-coercive* if there exist $a > 0$, $b \geq 0$ and $\eta \in \mathbb{R}^m$ such that, for every $\xi \in \mathbb{R}^m$, $a|\xi| \leq f(\xi) + \langle \eta, \xi\rangle + b$.

We list below several propositions involving level convex and sub-maximal functions.

PROPOSITION 3. – *Let $f : \mathbb{R}^m \to [0, +\infty]$ be proper and a positively homogeneous of degree 0 function: then f is level convex if and only if f is sub-maximal.*

PROOF. – In order to prove the *if* part, let $\xi, \eta \in \mathbb{R}^m$ and $t \in (0, 1)$. We have that

$$f(t\xi + (1-t)\eta) \leq f(t\xi) \vee f((1-t)\eta) = f(\xi) \vee f(\eta).$$

([3]) The definition of $f^{\natural}$ is analogous with the one recession function (see [26]. Definition 3.17 and Theorem 3.21) that can be defined also as follows: for every $\xi \in \mathbb{R}^m$,

$$f^{\infty}(\xi) = \inf\left\{\liminf_{h\to\infty} \frac{f(t_h\xi_h)}{t_h} : \xi_h \to \xi, t_h \uparrow \infty\right\}.$$

The definition of $f^{\natural}$ is inspired by [12] equation (2.7).

In order to prove the *only if* part, let $\xi, \eta \in \mathbb{R}^m$. Then

$$f(\xi+\eta) = f\left(\frac{\xi+\eta}{2}\right) \leq f(\xi) \vee f(\eta),$$

and we end the proof. □

PROPOSITION 4. – *Let* $f : \mathbb{R}^m \to [0, \infty]$ *be a proper function. Then* $f^\natural$ *is l.s.c., positively homogeneous of degree* 0 *but not necessarily proper. Moreover, if* f *is level convex then* $f^\natural$ *is level convex too.*

PROOF. – Let us fix $\xi_h, \xi_0 \in \mathbb{R}^m$ such that $\xi_h \to \xi_0$. Then, for every $h \in \mathbb{N}$, there exist two sequences $\{t_h^j\}_{j=1}^\infty \subseteq \mathbb{R}$, and $\{\xi_h^j\}_{j=1}^\infty \subseteq \mathbb{R}^m$ such that $t_h^j \uparrow \infty$, $\xi_h^j \to \xi_h$ and $f(t_h^j \xi_h^j) \to f^\natural(\xi_h)$ as $j \to \infty$. Thus, for every $h \in \mathbb{N}$, there exists j_h such that, $t_h^{j_h} \geq h$, $|\xi_h^{j_h} - \xi_h| \leq \frac{1}{h}$ and $f\left(t_h^{j_h} \xi_h^{j_h}\right) \leq f^\natural(\xi_h) + \frac{1}{h}$ and, unless to extract a subsequence, we can suppose that $t_h^{j_h} \uparrow \infty$ and $\xi_h^{j_h} \to \xi_0$ as $h \to \infty$. Then

$$f^\natural(\xi_0) \leq \liminf_{h\to\infty} f\left(t_h^{j_h} \xi_h^{j_h}\right) \leq \liminf_{h\to\infty} \left(f^\natural(\xi_h) + \frac{1}{h}\right) = \liminf_{h\to\infty} f^\natural(\xi_h),$$

and the lower semicontinuity is proved.

The proof of the positive homogeneity of degree 0 is very simple and it can be omitted.

Let us prove now the level convexity, that is, that for every $r \in [0, \infty)$ the set $\{\xi : f^\natural(\xi) \leq r\}$ is convex. If $r < \inf\{f(\xi) : \xi \in \mathbb{R}^m\}$ there is nothing to prove. Thus, fixed $r \geq \inf\{f(\xi) : \xi \in \mathbb{R}^m\}$ and $E_f(r) = \{\xi : f(\xi) \leq r\}$ we have

$$\{\xi : f^\natural(\xi) \leq r\} = \left\{\xi : \exists t_h \uparrow \infty, \xi_h \to \xi, \text{ such that } \lim_{h\to\infty} f(t_h \xi_h) \leq r\right\}$$

$$= \bigcap_{\varepsilon>0} \left\{\xi : \exists t_h \uparrow \infty, \xi_h \to \xi \text{ such that } \forall h \in \mathbb{N}, f(t_h \xi_h) \leq r + \varepsilon\right\}$$

$$= \bigcap_{\varepsilon>0} \left\{\xi : \exists \tau_h \downarrow 0, \zeta_h \in E_f(r+\varepsilon) \text{ such that } \tau_h \zeta_h \to \xi\right\} = \bigcap_{\varepsilon>0} E_f^\infty(r+\varepsilon).$$

For every $\varepsilon > 0$, $E_f(r+\varepsilon) \neq \emptyset$ and by definition we have that $E_f^\infty(r+\varepsilon)$ is the so called *horizon cone* of $E_f(r+\varepsilon)$ (see [26] Definition 3.3): since the horizon cone of a convex set is convex (see [26] Theorem 3.6) we end the proof. □

PROPOSITION 5. – *Let* $f : \mathbb{R}^m \to [0, \infty]$ *be a proper, l.s.c. and positively homogeneous of degree* 0 *function. Then* $f = f^\natural$.

PROOF. – Considering $\xi_h = \xi$ and $t_h \uparrow \infty$ we obtain $f^\natural(\xi) \leq f(\xi)$. In order to prove the converse inequality we use the lower semicontinuity of f: indeed let

$\xi_h \to \xi$ and $t_h \uparrow \infty$ such that $f(t_h\xi_h) \to f^\natural(\xi)$, then

$$f(\xi) \leq \liminf_{h\to\infty} f(\xi_h) = \liminf_{h\to\infty} f(t_h\xi_h) = f^\natural(\xi),$$

and we achieve the proof. $\square$

PROPOSITION 6. – *Let* $f : \mathbb{R}^m \to [0,\infty]$ *be a proper, l.s.c. and sub-maximal function. Then* $f^\flat$ *is l.s.c., positively homogeneous of degree* 0, *level convex and, for every* $\xi \in \mathbb{R}^m$,

$$f^\flat(\xi) = \sup_{t>0} f(t\xi).$$

PROOF. – We follow Proposition 2.2 in [12]. Clearly $f^\flat$ is positively homogeneous of degree 0. Fixed now $\xi \in \mathbb{R}^m$, for every $r > 0$, we set

$$a(r) = \sup\{f(t\xi) : 0 < t \leq r\}.$$

We have that a is non decreasing and $f^\flat(\xi) = \lim_{r\to 0} a(r)$. However, by the sub-maximality of f we have, for every $r > 0$,

$$a(2r) = \sup_{0<t\leq 2r} f(t\xi) = \sup_{0<t\leq r} f(2t\xi) \leq \sup_{0<t\leq r} f(t\xi) = a(r),$$

that implies that a is constant on $(0,\infty)$. Therefore

$$f^\flat(\xi) = \sup_{t>0} f(t\xi),$$

thus, in particular, being $f^\flat$ the supremum of a family of l.s.c. and sub-maximal functions, it is also l.s.c. and sub-maximal. Finally, using Proposition 3, the level convexity of $f^\flat$ follows too. $\square$

PROPOSITION 7. – *Let* $f : \mathbb{R}^m \to [0,\infty]$ *be a proper, l.s.c. and level convex function such that, for every* $\xi \in \mathbb{R}^m \setminus \{0\}$, $f^\natural(\xi) = \infty$. *Then there exists a function* $\theta : [0,\infty) \to [0,\infty)$ *such that, for every* $\xi \in \mathbb{R}^m$,

$$f(\xi) \geq \theta(|\xi|) \quad \text{and} \quad \lim_{t\to\infty} \theta(t) = \infty. \tag{19}$$

PROOF. – It is sufficient to prove that, for every $h \in \mathbb{N}$, there exists $r_h > 0$ such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $|\xi| > r_h$, we have $f(\xi) > h$. Let us suppose by contradiction that there exists $h_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, we can find $\xi_k \in \mathbb{R}^m$, $|\xi_k| > k$, with $f(\xi_k) \leq h_0$. Clearly $|\xi_k| \to \infty$ and, unless to extract a (not relabelled) subsequence, we have also $\frac{\xi_k}{|\xi_k|} \to v \in S^{m-1}$. Then

$$\infty = f^\natural(v) \leq \liminf_{k\to\infty} f\left(|\xi_k|\frac{\xi_k}{|\xi_k|}\right) \leq h_0 < \infty$$

which yields a contradiction. $\square$

PROPOSITION 8. – *Let* $f : \mathbb{R}^m \to [0, \infty]$ *be a proper, l.s.c. and sub-maximal function such that, for every* $\xi \in \mathbb{R}^m \setminus \{0\}$, $f^\flat(\xi) = \infty$. *Then there exists a function* $\gamma : (0, \infty) \to [0, \infty)$ *such that, for every* $\xi \in \mathbb{R}^m \setminus \{0\}$,

$$f(\xi) \geq \gamma(|\xi|) \quad \text{and} \quad \lim_{t\to 0} \gamma(t) = \infty. \tag{20}$$

PROOF. – It suffices to prove that, for every $h \in \mathbb{N}$, there exists $\varepsilon_h > 0$ such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $|\xi| < \varepsilon_h$, we have $f(\xi) > h$. Let us suppose by contradiction that there exists $h_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, we can find $\xi_k \in \mathbb{R}^m \setminus \{0\}$, $|\xi_k| \leq \frac{1}{k}$, with $f(\xi_k) \leq h_0$. Then $|\xi_k| \to 0$ and unless to extract a (not relabelled) subsequence, $\frac{\xi_k}{|\xi_k|} \to \upsilon \in S^{m-1}$. Let $\{t_h\}_{h=1}^\infty \subseteq (0, \infty)$ such that $t_h \downarrow 0$ and

$$\lim_{h\to\infty} f(t_h \upsilon) = \infty.$$

For every $h, k \in \mathbb{N}$, there exists $j_{h,k} \in \mathbb{N}$ such that $\frac{t_h}{|\xi_k|} \leq j_{h,k} < \frac{t_h}{|\xi_k|} + 1$ and since $t_h \frac{\xi_k}{|\xi_k|} \to t_h \upsilon$ as $k \to \infty$, then also $j_{h,k}\xi_k \to t_h \upsilon$ as $k \to \infty$. By the lower semi-continuity and the sub-maximality of f, we have

$$f(t_h \upsilon) \leq \liminf_{k\to\infty} f(j_{h,k}\xi_k) = \liminf_{k\to\infty} f\left(\sum_{i=1}^{j_{h,k}} \xi_k\right) \leq \liminf_{k\to\infty} f(\xi_k).$$

Then, for every $h \in \mathbb{N}$,

$$f(t_h \upsilon) \leq \liminf_{k\to\infty} f(\xi_k) \leq h_0 < \infty.$$

Taking the limit as $h \to \infty$, we find a contradiction and the proof is achieved. $\square$

PROPOSITION 9. – *Let* $f : \mathbb{R}^m \to [0, \infty]$ *be a proper, l.s.c. and level convex function. Let us define the function* $\hat{f} : \mathbb{R}^m \times \mathbb{R} \to [0, \infty]$ *in this way:*

$$\hat{f}(\xi, \tau) = \begin{cases} f(\frac{\xi}{\tau}) & \text{if } \tau > 0, \\ f^\natural(\xi) & \text{if } \tau = 0, \\ \infty & \text{if } \tau < 0. \end{cases} \tag{21}$$

Then $\hat{f}$ *is proper, l.s.c., positively homogeneous of degree* 0 *and level convex on* $\mathbb{R}^{m+1}$.

PROOF. – The functional $\hat{f}$ is clearly proper and positive homogeneous of degree 0. In order to prove the lower semicontinuity we work in the following way. Let us fix $(\xi_h, \tau_h) \to (\xi_0, \tau_0)$: since $\hat{f}$ is lower semicontinuous both on $\mathbb{R}^m \times (0, \infty)$ (because of the lower semicontinuity of f) and on $\mathbb{R}^m \times (-\infty, 0)$, the lower semicontinuity inequality has to be proved only in the case in which $\tau_0 = 0$. If this is the case, we can find I_1, I_2, I_3 disjoint subsets of $\mathbb{N}$ such that $I_1 \cup I_2 \cup I_3 = \mathbb{N}$ and such that, if $h \in I_1$ then $\tau_h > 0$, if $h \in I_2$ then $\tau_h = 0$ and if $h \in I_3$ then $\tau_h < 0$.

Since

$$\liminf_{h\to\infty}\hat{f}(\xi_h,\tau_h)=\inf\left\{\liminf_{h\to\infty,h\in I_i}\hat{f}(\xi_h,\tau_h):i\in\{1,2,3\},\#(I_i)=\infty\right\},$$

it suffices to prove the lower semicontinuity inequality of $\hat{f}$ only in the three cases in which $I_i=\mathbb{N}$, $i\in\{1,2,3\}$. However if, for every $h\in\mathbb{N}$, $\tau_h>0$, by the definition of $f^\natural$,

$$\hat{f}(\xi_0,0)=f^\natural(\xi_0)\le\liminf_{h\to\infty}f\left(\frac{\xi_h}{\tau_h}\right)=\liminf_{h\to\infty}\hat{f}(\xi_h,\tau_h),$$

if, for every $h\in\mathbb{N}$, $\tau_h=0$, by the lower semicontinuity of $f^\natural$ (see Proposition 4),

$$\hat{f}(\xi_0,0)=f^\natural(\xi_0)\le\liminf_{h\to\infty}f^\natural(\xi_h)=\liminf_{h\to\infty}\hat{f}(\xi_h,\tau_h),$$

and finally if, for every $h\in\mathbb{N}$, $\tau_h<0$ the inequality is trivially satisfied: thus the lower semicontinuity of $\hat{f}$ is achieved.

Let us show now that $\hat{f}$ is level convex, that is, for every $r\in[0,\infty)$, the set

$$\big\{(\xi,\tau):\hat{f}(\xi,\tau)\le r\big\}$$

$$=\big\{(\xi,\tau):\tau>0,f(\xi\tau^{-1})\le r\big\}\cup\big\{(\xi,0):f^\natural(\xi)\le r\big\}=A\cup B$$

is convex. We know that $A\cup B$ is closed and, since $f^\natural$ is l.s.c. and level convex, that B is a convex and closed set. We note also that A is convex since $A=\emptyset$ or, if $A\ne\emptyset$, we have that the convex set $E_f(r)=\{\xi:f(\xi)\le r\}\ne\emptyset$ and $A=\{(t\xi,t):\xi\in E_f(r),t>0\}$ that is convex.

If $A=\emptyset$ then $A\cup B=B$ that is convex. If $A\ne\emptyset$ we are going to prove that $A\cup B=\mathrm{cl}(A)$ that is convex since A is convex. Clearly $\mathrm{cl}(A)\subseteq A\cup B$: to prove the converse we only need to prove that $B\subseteq\mathrm{cl}(A)$.

In order to prove this let us fix $(\xi_0,0)\in B$ and show that there exist two sequences $\zeta_h\to\xi_0$ and $\tau_h\downarrow 0$ such that, for every $h\in\mathbb{N}$, $(\zeta_h,\tau_h)\in A$, that is, $f(\zeta_h\tau_h^{-1})\le r$. Since $A\ne\emptyset$ there exists $\xi_1\in\mathbb{R}^m$ such that $f(\xi_1)\le r$: we claim that $\{\xi_1+t\xi_0:t\ge 0\}\subseteq E_f(r)$.

If $\xi_0=0$ there is nothing to prove; instead, supposing $\xi_0\ne 0$, let fix $t\ge 0$ and consider $\xi_h\to\xi_0$ ($\xi_h\ne 0$), $t_h\uparrow\infty$ ($|t_h\xi_h|>t|\xi_0|$) such that $f(t_h\xi_h)\to f^\natural(\xi_0)$: then, for every $h\in\mathbb{N}$, by the level convexity of f,

$$f\left(\xi_1+t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}\right)\le f(t_h\xi_h)\vee f(\xi_1),$$

since the point $\xi_1+t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}$ belongs to the segment joining ξ_1 and $t_h\xi_h$. However

$$\lim_{h\to\infty}\left(\xi_1+t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}\right)=\xi_1+t\xi_0,$$

and then, by the lower semicontinuity of f,

$$f(\xi_1 + t\xi_0) \leq \liminf_{h\to\infty} (f(t_h\xi_h) \vee f(\xi_1)) \leq r,$$

that proves the claim.

At last setting, for every $h \in \mathbb{N}$, $\zeta_h = \frac{\xi_1 + h\xi_0}{h}$ and $\tau_h = \frac{1}{h}$, we have $\hat{f}(\zeta_h, \tau_h) = f(\zeta_h \tau_h^{-1}) \leq r$, that is $(\zeta_h, \tau_h) \in A$, and moreover

$$\lim_{h\to\infty} (\zeta_h, \tau_h) = (\xi_0, 0) \in \mathrm{cl}(A),$$

that ends the proof. □

PROPOSITION 10. – *Let $f : \mathbb{R} \to [0, \infty]$ be a continuous and sub-maximal function and let $s = \sup\{f(\xi) : \xi \in \mathbb{R}\}$. Then, for every $\xi \in [0, \infty)$, $f(\xi) = s$ or, for every $\xi \in (-\infty, 0]$, $f(\xi) = s$.*

PROOF. – First of all we prove that $f(0) = s$. Indeed, if this is not true then $f(0)$ is finite, there exists $\varepsilon > 0$ such that $f(0) + 2\varepsilon \leq s$ and $\xi_0 \in \mathbb{R}$ such that $f(\xi_0) \geq f(0) + \varepsilon$. Then, for every $h \in \mathbb{N}$, by the sub-maximality of f, also $f\left(\frac{\xi_0}{h}\right) \geq f(0) + \varepsilon$. But $\frac{\xi_0}{h} \to 0$ as $h \to \infty$ thus, by continuity of f, $f(0) = \lim_{h\to\infty} f\left(\frac{\xi_0}{h}\right) \geq f(0) + \varepsilon$ that is a contradiction.

Let us suppose now, again by contradiction, that there exist $\xi_1, \xi_2 > 0$ such that $f(\xi_1) \vee f(-\xi_2) < s$: by continuity of f, we can suppose also that $\xi_1, \xi_2 \in \mathbb{Q}$. Writing $\frac{\xi_1}{\xi_2} = \frac{k}{h}$, where $h, k \in \mathbb{N}$, we have $h\xi_1 - k\xi_2 = 0$ and

$$s = f(0) = f(h\xi_1 - k\xi_2) \leq f(\xi_1) \vee f(-\xi_2) < s :$$

having found a contradiction, we achieve the proof. □

Let us point out that if f is only l.s.c. and sub-maximal on $\mathbb{R}$, then the thesis of Proposition 10 is false: to verify this fact one can consider for instance the function f defined, for every $\xi \in \mathbb{R} \setminus \mathbb{Z}$, as $f(\xi) = M > 0$ and, for every $\xi \in \mathbb{Z}$, as $f(\xi) = 0$.

The proofs of the following two propositions are simple and then omitted.

PROPOSITION 11. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and demi-coercive function, that is, there exist $a > 0$, $b \geq 0$ and $\eta \in \mathbb{R}^m$ such that, for every $\xi \in \mathbb{R}^m$, $f(\xi) \geq a|\xi| - \langle \eta, \xi \rangle - b$. Then, for every $\xi \in \mathbb{R}^m$, the following properties hold:*

(i) $f^\infty(\xi) \leq f^\natural(\xi)$;
(ii) *if* $f^\natural(\xi) < \infty$ *then* $f^\infty(\xi) = 0$;
(iii) $f^\infty(\xi) \geq a|\xi| - \langle \eta, \xi \rangle$.

PROPOSITION 12. – *Let $f : \mathbb{R} \to [0, \infty]$ be a proper and l.s.c. function. Then f is level convex if and only if f belongs to one of the three following classes:*

(i) f is non decreasing on $\mathbb{R}$;

(ii) f is non increasing on $\mathbb{R}$;

(iii) there exists $x_0 \in \mathbb{R}$ such that f is non increasing on $(-\infty, x_0]$ and non decreasing on $[x_0, \infty)$.

In particular if $f : \mathbb{R}^m \to [0, \infty]$ is a proper, l.s.c. and level convex function, then $f(0) = \inf\{f(\xi) : \xi \in \mathbb{R}^m\}$ if and only if, for every $v \in S^{m-1}$, the function $t \mapsto f(vt)$ is non decreasing on $(0, \infty)$.

The two propositions below describe some properties of the composition of a level convex function with an increasing one.

PROPOSITION 13. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c and level convex function, $s = \sup\{f(\xi) : \xi \in \mathbb{R}^m\}$ and $\Theta : [0, s] \to [0, \bar{s}]$ be a continuous, increasing function with $\bar{s} = \sup\{\Theta(t) : t \in [0, s]\}$. Then the composition $\Theta \circ f : \mathbb{R}^m \to [0, \bar{s}]$ is proper, l.s.c. and level convex. Moreover $(\Theta \circ f)^\natural = \Theta \circ f^\natural$.*

PROOF. – Clearly $\Theta \circ f$ is proper and l.s.c.. Let $\xi, \eta \in \mathbb{R}^m$ and $t \in (0, 1)$, then

$$\begin{aligned}(\Theta \circ f)(t\xi + (1-t)\eta) &= \Theta(f(t\xi + (1-t)\eta)) \\ &\leq \Theta(f(\xi) \vee f(\eta)) = \Theta(f(\xi)) \vee \Theta(f(\eta)),\end{aligned}$$

that is $\Theta \circ f$ is level convex too. In order to prove that $(\Theta \circ f)^\natural = \Theta \circ f^\natural$, we point out that Θ is bijective and $\Theta^{-1} : [0, \bar{s}] \to [0, s]$ is still continuous and strictly increasing. Let us fix $\xi_0 \in \mathbb{R}^m$ and let $t_h \uparrow \infty$, $\xi_h \to \xi_0$ such that $(\Theta \circ f)(t_h \xi_h) \to$ $\to (\Theta \circ f)^\natural(\xi_0)$. Then, using the continuity of Θ^{-1},

$$\begin{aligned}(\Theta \circ f)^\natural(\xi_0) &= \lim_{h\to\infty} \Theta(f(t_h\xi_h)) = \Theta \circ \Theta^{-1}\left(\lim_{h\to\infty} \Theta(f(t_h\xi_h))\right) \\ &= \Theta\left(\lim_{h\to\infty} f(t_h\xi_h)\right) \geq \Theta \circ f^\natural(\xi_0).\end{aligned}$$

Conversely let $\bar{t}_h \uparrow \infty$, $\bar{\xi}_h \to \xi_0$ such that $f\left(\bar{t}_h\bar{\xi}_h\right) \to f^\natural(\xi_0)$. Then, using the continuity of Θ,

$$\begin{aligned}(\Theta \circ f^\natural)(\xi_0) &= \Theta\big(f^\natural(\xi_0)\big) = \Theta\left(\lim_{h\to\infty} f\left(\bar{t}_h\bar{\xi}_h\right)\right) \\ &= \lim_{h\to\infty} \Theta\big(f\left(\bar{t}_h\bar{\xi}_h\right)\big) \geq (\Theta \circ f)^\natural(\xi_0)\end{aligned}$$

and the equality is finally achieved. $\square$

PROPOSITION 14. – *Let* $f : \mathbb{R}^m \to [0, \infty]$ *be a proper, l.s.c and level convex function,* $s = \sup\{f(\xi) : \xi \in \mathbb{R}^m\}$ *and* $K_f = \{\xi \in \mathbb{R}^m : f^\natural(\xi) < s\}$*. Let us suppose that* $\mathrm{cl}(K_f)$ *does not contain any straight line. Then there exists a continuous and strictly increasing function* $\Theta : [0, s] \to [0, \infty]$ *such that the composition* $\Theta \circ f : \mathbb{R}^m \to [0, \infty]$ *is demi-coercive.*

PROOF. – First of all let us note that, by the properties of $f^\natural$, $\mathrm{cl}(K_f)$ is a closed and convex cone of $\mathbb{R}^m$. Moreover, it is quite simple to prove that there exists a closed and convex cone C with non empty interior such that it does not contain any straight line and $\mathrm{cl}(K_f) \subseteq C$.

We claim that there exists $\eta_0 \in \mathbb{R}^m$, such that

$$\mathrm{cl}(K_f) \setminus \{0\} \subseteq C \setminus \{0\} \subseteq \{\xi \in \mathbb{R}^m : \langle \eta_0, \xi \rangle > 0\}. \tag{22}$$

Let us first note that if a convex cone $K \subseteq \mathbb{R}^m$ has empty interior, then its polar cone

$$K^* = \{\eta \in \mathbb{R}^m : \langle \eta, \xi \rangle \leq 0 \ \forall \xi \in K\}$$

contains at least one straight line: indeed K spans a vector subspace E of $\mathbb{R}^m$ with dimension less then m, and then the orthogonal of E contains at least a straight line, which is then included in K^*. Now consider the closed convex cone C of (22): it is well known that $C^{**} = C$ (see [26] Corollary 6.21), and what precedes implies that C^* has non empty interior. Taking η in the interior of C^*, clearly $\eta_0 = -\eta$ satisfies (22): indeed, if $\xi \in C \setminus \{0\}$, then $\langle \eta, \xi \rangle \leq 0$, and if we suppose $\langle \eta, \xi \rangle = 0$, then we can find $\eta' \in C^*$ in the neighborhood of η such that $\langle \eta', \xi \rangle > 0$, which is a contradiction; as a consequence $\langle \eta, \xi \rangle < 0$, so that $\langle \eta_0, \xi \rangle > 0$.

Clearly (22) implies that there exists $\varepsilon > 0$ such that

$$\mathrm{cl}(K_f) \setminus \{0\} \subseteq \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle > \varepsilon \right\},$$

where, without loss of generality, we can also assume $|\eta_0| = 1$.

Let us suppose at first $s < \infty$. For every $h \in \mathbb{N}$, we set

$$C_h = \left\{\xi : f(\xi) \leq s\left(1 - \frac{1}{h}\right)\right\}$$

and

$$D_h = C_h \cap \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon \right\}.$$

Obviously $\{C_h\}_{h=1}^\infty$ and $\{D_h\}_{h=1}^\infty$ are two non decreasing sequences of sets. Moreover we have that every D_h is bounded. Indeed, if by contradiction there exists $h_0 \in \mathbb{N}$ such that D_{h_0} is unbounded, we can find a sequence $\{\xi_j\}_{j=1}^\infty \subseteq D_{h_0}$

such that $|\xi_j| \to \infty$ and $\frac{\xi_j}{|\xi_j|} \to \xi_0 \in S^{m-1}$. Then

$$s\left(1 - \frac{1}{h_0}\right) \geq \liminf_{j\to\infty} f\left(|\xi_j|\frac{\xi_j}{|\xi_j|}\right) \geq f^{\natural}(\xi_0),$$

but since $\xi_0 \in \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon\right\}$ it should be $f^{\natural}(\xi_0) = s$ and the contradiction is found.

Let us define now

$$\theta(0) = 2\sup\{|\xi| : \xi \in D_1\}, \quad (\theta(0) = 0 \text{ if } D_1 = \emptyset),$$

and, for every $h \in \mathbb{N}$, $t \in \left(s\left(1 - \frac{1}{h}\right), s\left(1 - \frac{1}{h+1}\right)\right]$,

$$\theta(t) = 2\sup\{|\xi| : \xi \in D_{h+1}\}, \quad (\theta(t) = 0 \text{ if } D_{h+1} = \emptyset).$$

Then $\theta : [0, s) \to [0, \infty)$ is clearly non decreasing and $\sup\{\theta(t) : t \in [0, s)\} = \infty$: thus we can define also $\theta(s) = \infty$.

Let us fix now $\xi_0 \in \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon\right\}$: if $f(\xi_0) = 0$ then $\xi_0 \in D_1$ and

$$(\theta \circ f)(\xi_0) = \theta(0) = 2\sup\{|\xi| : \xi \in D_1\} \geq 2|\xi_0|;$$

if $0 < f(\xi_0) < s$ then there exists $h_0 \in \mathbb{N}$ such that

$$f(\xi_0) \in \left(s\left(1 - \frac{1}{h_0}\right), s\left(1 - \frac{1}{h_0 + 1}\right)\right]$$

and then $\xi_0 \in D_{h_0+1}$ and

$$(\theta \circ f)(\xi_0) = 2\sup\{|\xi| : \xi \in D_{h_0+1}\} \geq 2|\xi_0|;$$

if at last $f(\xi_0) = s$ then

$$(\theta \circ f)(\xi_0) = \infty \geq 2|\xi_0|.$$

Now we can prove easily that $(\theta \circ f)$ is demi-coercive. Indeed, fixed $\xi \in \mathbb{R}^m \setminus \{0\}$, if $\left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon$ then

$$(\theta \circ f)(\xi) + \langle \xi, \eta_0 \rangle \geq 2|\xi| - |\xi| = |\xi|,$$

while, if $\left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle > \varepsilon$, then

$$(\theta \circ f)(\xi) + \langle \xi, \eta_0 \rangle \geq \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle |\xi| > \varepsilon|\xi|.$$

Thus, since trivially $(\theta \circ f)(0) \geq 0$, it holds that, for every $\xi \in \mathbb{R}^m$, $(\theta \circ f)(\xi) + \langle \xi, \eta_0 \rangle \geq \varepsilon|\xi|$.

We achieve the proof of the case $s < \infty$ choosing any function $\Theta : [0, s] \to [0, \infty]$

which is continuous, strictly increasing and such that, for every $t \in [0, s]$, $\theta(t) \leq \Theta(t)$: in this way $\Theta \circ f$ is proper, l.s.c. and demi-coercive. The construction of the function Θ is simple and it can be omitted.

In order to treat the case $s = \infty$, we can use the same argument once $s\left(1 - \frac{1}{h}\right)$ is changed with h and the definition given for $\theta(0)$ is used to define θ on $[0, 1]$. $\square$

The two propositions below are based on the following simple equalities: if $t_h, l \in \mathbb{R}$, then

$$\liminf_{h\to\infty} (t_h \vee l) = \left(\liminf_{h\to\infty} t_h\right) \vee l \quad \text{and} \quad \limsup_{h\to\infty} (t_h \vee l) = \left(\limsup_{h\to\infty} t_h\right) \vee l.$$

PROPOSITION 15. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function and let $l_j, l \in \mathbb{R}$ such that $l_j \downarrow l$. If, for every $j \in \mathbb{N}$, $f \vee l_j$ is l.s.c. then $f \vee l$ is l.s.c. too.*

PROOF. – Let $\xi_h, \xi_0 \in \mathbb{R}^m$ such that $\xi_h \to \xi_0$. Then, for every $j \in \mathbb{N}$,

$$(f \vee l)(\xi_0) \leq (f \vee l_j)(\xi_0) \leq \liminf_{h\to\infty} (f \vee l_j)(\xi_h) = \left(\liminf_{h\to\infty} f(\xi_h)\right) \vee l_j,$$

and, letting $j \to \infty$, we obtain

$$(f \vee l)(\xi_0) \leq \left(\liminf_{h\to\infty} f(\xi_h)\right) \vee l = \liminf_{h\to\infty} (f(\xi_h) \vee l),$$

that completes the proof. $\square$

PROPOSITION 16. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function and let $l \in \mathbb{R}$. Then $(f \vee l)^\flat = f^\flat \vee l$.*

PROOF. – Let us fix $\xi_0 \in \mathbb{R}^m$. If we consider $t_h \downarrow 0$ such that $f(t_h\xi_0) \to f^\flat(\xi_0)$, then

$$(f \vee l)^\flat(\xi_0) \geq \limsup_{h\to\infty} (f \vee l)(t_h\xi_0) = \left(\limsup_{h\to\infty} f(t_h\xi_0)\right) \vee l = f^\flat(\xi_0) \vee l.$$

Conversely let $\bar{t}_h \downarrow 0$ such that $(f \vee l)(\bar{t}_h\xi_0) \to (f \vee l)^\flat(\xi_0)$. Then

$$f^\flat(\xi_0) \vee l \geq \left(\limsup_{h\to\infty} f(\bar{t}_h\xi_0)\right) \vee l = \limsup_{h\to\infty} (f \vee l)(\bar{t}_h\xi_0) = (f \vee l)^\flat(\xi_0),$$

and the proof is achieved. $\square$

At last we state now a Jensen type inequality involving level convex functions and whose simple proof can be found for instance in [9], Theorem 1.2.

THEOREM 2. – *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a l.s.c., proper and level convex function and let $\mu \in \mathcal{M}^+(\Omega)$, $\mu(\Omega) = 1$. Let $\varphi \in L^1_\mu(\Omega, \mathbb{R}^m)$ be a given function. Then*

$$f\left(\int_\Omega \varphi(x) d\mu(x)\right) \leq \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} f(\varphi(x)).$$

3. – Level convex functionals.

In this section we state and prove the lower semicontinuity results about the functional F given by (7). Before stating the main theorem, let us define, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$, the functional

$$\mathcal{F}(\lambda, \Omega) = \inf\Big\{ \liminf_{h\to\infty} F(\lambda_h, \Omega_h) : \lambda_h \in \mathcal{M}_{\text{loc}}(\Omega_h, \mathbb{R}^m),$$

$$\lambda_h << \mathcal{L}^n, \lambda_h \to \lambda \; w^*\text{-}\mathcal{M}_{\text{loc}}(\updownarrow \Omega, \mathbb{R}^m)\Big\}. \tag{23}$$

THEOREM 3. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function. Then the two following conditions are equivalent:*

(i) f is l.s.c. and level convex on $\mathbb{R}^m$,

(ii) for every $\Omega \subseteq \mathbb{R}^n$ open set, the functional F is l.s.c. on $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ convergence.

Moreover in these hypotheses, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$,

$$F(\lambda, \Omega) = \mathcal{F}(\lambda, \Omega) = \lim_{\rho\to 0} F(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho). \tag{24}$$

and F is level convex on $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$.

PROOF. – [Proof of Theorem 3 $(ii) \Rightarrow (i)$] Let $\xi_h, \xi_0 \in \mathbb{R}^m$, $\xi_h \to \xi_0$ and let $\lambda_h = \xi_h \cdot \mathcal{L}^n$, $\lambda_0 = \xi_0 \cdot \mathcal{L}^n$. We have $\lambda_h \to \lambda_0$ in w^*-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and then

$$f(\xi_0) = F(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} F(\lambda_h, \mathbb{R}^n) = \liminf_{h\to\infty} f(\xi_h),$$

that is f is l.s.c.

Let us consider now $\xi, \eta \in \mathbb{R}^m$, $t \in (0, 1)$ and a sequence $\{B_h\}_{h=1}^\infty \subseteq \mathcal{B}(\mathbb{R}^n)$ such that $1_{B_h}(x) \to t1_{\mathbb{R}^n}(x)$ in w^*-$L^\infty(\mathbb{R}^n)$ (see [1] Proposition 4.2 Remark 4.3). If $\lambda_h = \xi 1_{B_h}(x) \cdot \mathcal{L}^n + \eta 1_{\mathbb{R}^n \setminus B_h}(x) \cdot \mathcal{L}^n$ then $\lambda_h \to \lambda_0 = (t\xi + (1-t)\eta) \cdot \mathcal{L}^n$ in w^*-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ thus

$$f(t\xi + (1-t)\eta) = F(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} F(\lambda_h, \mathbb{R}^n) = f(\xi) \vee f(\eta),$$

that is f is level convex. $\square$

The proof of the implication $(i) \Rightarrow (ii)$ of Theorem 3 is more difficult and for this reason it is developed in three steps. In the first one we prove that, for every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\mathcal{F}(\lambda, \Omega)$ can be computed considering only the convolutions of λ as in formula (24) (see Theorem 4). This fact implies, in a very simple way, the second step that is the proof that $\mathcal{F}$ is l.s.c. on $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ convergence (see Proposition 17) and it is level convex on $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (see Proposition 18). At last, in the third step we prove the equality $F = \mathcal{F}$ (see Theorem 5) that completes the proof of Theorem 3.

For an analogy between the integral and supremal cases compare Lemma 3 and Theorem 4 below with Lemma 1 and Theorem 1 in [27] (see also [22]).

LEMMA 3. – *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Let $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\lambda_h \in \mathcal{M}_{\mathrm{loc}}(\Omega_h, \mathbb{R}^m)$, $\lambda_h << \mathcal{L}^n$ such that $\lambda_h \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$. Then, for every $\rho > 0$, we have*

$$F\big(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho\big) \leq \liminf_{h\to\infty} F(\lambda_h, \Omega_h).$$

PROOF. – Let us fix $\rho > 0$ and $K \subset\subset \Omega_\rho$. We have that $(\lambda_h)_\rho \to \lambda_\rho$ pointwise on K as $h \to \infty$. Indeed if $x \in K$ and h is large enough, we have $K \subset\subset \Omega_{h,\rho}$ and

$$|(\lambda_h)_\rho(x) - \lambda_\rho(x)| = \left| \int_{B(x,\rho)} k_\rho(x-y)\,d\lambda_h(y) - \int_{B(x,\rho)} k_\rho(x-y)\,d\lambda(y) \right|$$

that converges to zero as $h \to \infty$ by definition of w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ convergence.

Let us fix $x \in K$ and apply Theorem 2 and (18) with $\mu = k_\rho(x-y) \cdot \cdot \mathcal{L}^n << \mathcal{L}^n$, in order to get

$$f(\lambda_\rho(x)) \leq \liminf_{h\to\infty} f\big((\lambda_h)_\rho(x)\big) \leq \liminf_{h\to\infty} \operatorname*{ess\,sup}_{y\in B(x,\rho)} f(\lambda_h^a(y)) \leq \liminf_{h\to\infty} F(\lambda_h, \Omega_h).$$

Then

$$\sup_{x\in K} f(\lambda_\rho(x)) \leq \liminf_{h\to\infty} F(\lambda_h, \Omega_h),$$

and we can conclude by taking $K \uparrow \Omega_\rho$ and noting that, since the composition $f \circ \lambda_\rho$ is l.s.c. on Ω_ρ, the supremum considered in the previous inequality agrees with the essential supremum with respect to $\mathcal{L}^n$. $\square$

By means of Lemma 3 we find a representation formula for $\mathcal{F}$ via convolutions, proving in this way the first step of the proof of Theorem 3.

THEOREM 4. – *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, we have*

$$\mathcal{F}(\lambda, \Omega) = \lim_{\rho\to 0} F(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho) \tag{25}$$

In particular the limit in the right hand side exists.

PROOF. – For every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\lambda_\rho \cdot \mathcal{L}^n \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ as $\rho \to 0$ and $\lambda_\rho \cdot \mathcal{L}^n << \mathcal{L}^n$ so that, by Lemma 3,

$$\limsup_{\rho\to 0} F(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho) \le \mathcal{F}(\lambda, \Omega) \le \liminf_{\rho\to 0} F(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho),$$

and this end the proof. □

By means of Theorem 3 we are able to prove the two following propositions that complete the second step of the proof of Theorem 3.

PROPOSITION 17. – *Let* $f : \mathbb{R}^m \to [0, +\infty]$ *be a proper, l.s.c. and level convex function. Then* $\mathcal{F}$ *is l.s.c. on* $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *with respect to the* w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *convergence.*

PROOF. – Let $\lambda, \lambda_h \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be such that $\lambda_h \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, by Theorem 4, there exists a sequence $\rho_h \downarrow 0$ such that, for every $h \in \mathbb{N}$,

$$F\big((\lambda_h)_{\rho_h} \cdot \mathcal{L}^n, \Omega_{\rho_h}\big) \le \mathcal{F}(\lambda_h, \Omega) + \frac{1}{h}. \tag{26}$$

Now claim that $(\lambda_h)_{\rho_h} \cdot \mathcal{L}^n \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$. Indeed, fixed $\varphi \in C_c(\Omega)$, if h is large enough such that $\mathrm{spt}(\varphi) + B(0, \rho_h) \subset \Omega_{\rho_h}$, by using the usual properties of the convolutions (see [6] equation (2.3) page 42), we have

$$\left| \int_\Omega \varphi(x)(\lambda_h)_{\rho_h}(x)dx - \int_\Omega \varphi(x)d\lambda(x) \right|$$

$$\le \left| \int_\Omega \varphi(x)(\lambda_h)_{\rho_h}(x)dx - \int_\Omega \varphi(x)d\lambda_h(x) \right| + \left| \int_\Omega \varphi(x)d\lambda_h(x) - \int_\Omega \varphi(x)d\lambda(x) \right|$$

$$\le \int_\Omega |\varphi_{\rho_h}(x) - \varphi(x)|d\lambda_h(x) + \left| \int_\Omega \varphi(x)d\lambda_h(x) - \int_\Omega \varphi(x)d\lambda(x) \right|.$$

Now the first term goes to zero as $h \to \infty$ since $\varphi_{\rho_h} \to \varphi$ uniformly on Ω when $h \to \infty$ and there exists a constant $M > 0$ such that, for every $h \in \mathbb{N}$, $|\lambda_h|(\mathrm{spt}(\varphi) + B(0, \rho_h)) \le M$; the second term goes to zero by definition of $\{\lambda_h\}_{h=1}^\infty$.

Then using the definition of $\mathcal{F}$ and (26) we obtain its lower semicontinuity since

$$\mathcal{F}(\lambda, \Omega) \le \liminf_{h\to\infty} F\big((\lambda_h)_{\rho_h} \cdot \mathcal{L}^n, \Omega_{\rho_h}\big) \le \liminf_{h\to\infty} \mathcal{F}(\lambda_h, \Omega),$$

which ends the proof. □

PROPOSITION 18. – *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and for every $t \in (0,1)$,*

$$\mathcal{F}(t\lambda_1 + (1-t)\lambda_2, \Omega) \le \mathcal{F}(\lambda_1, \Omega) \vee \mathcal{F}(\lambda_2, \Omega),$$

that is, $\mathcal{F}$ is level convex on the linear space $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.

PROOF. – Fixed $\rho > 0$, by the level convexity of f, we have, for every $x \in \Omega_\rho$,

$$\begin{aligned} f((t\lambda_1 + (1-t)\lambda_2)_\rho(x)) &= f(t(\lambda_1)_\rho(x) + (1-t)(\lambda_2)_\rho(x)) \\ &\le f((\lambda_1)_\rho(x)) \vee f((\lambda_2)_\rho(x)), \end{aligned}$$

and so

$$F((t\lambda_1 + (1-t)\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho) \le F((\lambda_1)_\rho \cdot \mathcal{L}^n, \Omega_\rho) \vee F((\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho).$$

Passing to the limit as $\rho \to 0$ we complete the proof applying Theorem 4. □

We prove now the equality $F = \mathcal{F}$. Theorems 4 and 5 together with Propositions 17 and 18 complete the proof of Theorem 3.

THEOREM 5. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $F(\lambda, \Omega) = \mathcal{F}(\lambda, \Omega)$.*

PROOF. – Let us suppose at first that f is positively homogeneous of degree 0.

Let us fix $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and show at first that $F(\lambda, \Omega) \le \mathcal{F}(\lambda, \Omega)$. By Theorem 1 (i) there exists $N \subseteq \Omega$ such that $\mathcal{L}^n(N) = 0$ and, for every $x \in \Omega \setminus N$, $\lambda_\rho(x) \to \lambda^a(x)$ as $\rho \to 0$. Then, thanks to Theorem 4, for every $x \in \Omega \setminus N$,

$$f(\lambda^a(x)) \le \liminf_{\rho \to 0} f(\lambda_\rho(x)) \le \liminf_{\rho \to 0} \sup_{x \in \Omega_\rho} f(\lambda_\rho(x)) = \mathcal{F}(\lambda, \Omega),$$

and then

$$\operatorname*{ess\,sup}_{x \in \Omega} f(\lambda^a(x)) \le \sup_{x \in \Omega \setminus N} f(\lambda^a(x)) \le \mathcal{F}(\lambda, \Omega).$$

Thus it remains to show that

$$|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) \le \mathcal{F}(\lambda, \Omega).$$

By Theorem 1(ii), there exists $M \subseteq \Omega$ with $|\lambda^s|(M) = 0$ and such that, for every $x \in \Omega \setminus M$, there exists a sequence $\{\rho_h\}_{h=1}^\infty$, depending on x and decreasing to zero, such that

$$\lim_{h \to \infty} \left| \frac{d\lambda^s}{d|\lambda^s|}(x) - \frac{\lambda_{\rho_h}(x)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)\, d|\lambda^s|(y)} \right| = 0. \tag{27}$$

Then, using Proposition 5 and Theorem 4, for every $x \in \Omega \setminus M$,

$$f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) = f\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) \leq \liminf_{h\to\infty} f\left(\frac{\lambda_{\rho_h}(x)}{\int\limits_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)}\right)$$

$$= \liminf_{h\to\infty} f\left(\lambda_{\rho_h}(x)\right) \leq \lim_{h\to\infty} F(\lambda_{\rho_h}\cdot\mathcal{L}^n, \Omega_{\rho_h}) = \mathcal{F}(\lambda, \Omega).$$

In conclusion,

$$|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) \leq \sup_{x\in\Omega\setminus M} f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) \leq \mathcal{F}(\lambda, \Omega),$$

and we achieve that $F(\lambda, \Omega) \leq \mathcal{F}(\lambda, \Omega)$.

In order to prove the converse inequality let fix $\rho > 0$. By Theorem 2, equation (18) and Propositions 3, 5 we have, for every $x \in \Omega_\rho$,

$$f(\lambda_\rho(x)) = f\left(\int\limits_{B(x,\rho)} k_\rho(x-y)d\lambda(y)\right)$$

$$= f\left(\int\limits_{B(x,\rho)} k_\rho(x-y)\lambda^a(y)dy + \int\limits_{B(x,\rho)} k_\rho(x-y)\frac{d\lambda^s}{d|\lambda^s|}(y)d|\lambda^s|(y)\right)$$

$$\leq f\left(\int\limits_{B(x,\rho)} k_\rho(x-y)\lambda^a(y)dy\right) \vee f\left(\int\limits_{B(x,\rho)} k_\rho(x-y)\frac{d\lambda^s}{d|\lambda^s|}(y)d|\lambda^s|(y)\right)$$

$$\leq \left[\operatorname*{ess\,sup}_{y\in B(x,\rho)} f(\lambda^a(y))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{y\in B(x,\rho)} f\left(\frac{d\lambda^s}{d|\lambda^s|}(y)\int\limits_{B(x,\rho)} k_\rho(x-z)d|\lambda^s|(z)\right)\right]$$

$$\leq \left[\operatorname*{ess\,sup}_{x\in\Omega} f(\lambda^a(x))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)\right] = F(\lambda, \Omega),$$

thus

$$\sup_{x\in\Omega_\rho} f(\lambda_\rho(x)) \leq F(\lambda, \Omega).$$

The wanted inequality $\mathcal{F}(\lambda, \Omega) \leq F(\lambda, \Omega)$ is achieved as $\rho \to 0$, invoking again Theorem 4.

In the general case, let us consider the function $\hat{f}$ defined in (21): by Proposition 9 we know that $\hat{f}$ is l.s.c., positively homogeneous of degree 0 and level convex. Let now fix $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and consider $(\lambda, \mathcal{L}^n) \in \mathcal{M}(\Omega, \mathbb{R}^{m+1})$: its decomposition with respect to $\mathcal{L}^n$ is clearly given by

$$(\lambda, \mathcal{L}^n) = (\lambda^a(x), 1) \cdot \mathcal{L}^n + \left(\frac{d\lambda^s}{d|\lambda^s|}(x), 0\right) \cdot |\lambda^s|.$$

We also have that the function $(\lambda, \mathcal{L}^n)_\rho : \Omega_\rho \to \mathbb{R}^{m+1}$ is given by $(\lambda, \mathcal{L}^n)_\rho(x) = (\lambda_\rho(x), 1)$. Then applying the first step to $\hat{f}$ and $(\lambda, \mathcal{L}^n)$ we have

$$\hat{\mathcal{F}}((\lambda, \mathcal{L}^n), \Omega) = \hat{F}((\lambda, \mathcal{L}^n), \Omega),$$

where $\hat{\mathcal{F}}$ and $\hat{F}$ denote the functionals (23) and (7) built up considering $\hat{f}$ and defined on the space $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^{m+1})$. But now we have

$$\begin{aligned}\hat{\mathcal{F}}((\lambda, \mathcal{L}^n), \Omega) &= \lim_{\rho \to 0} \sup_{x \in \Omega_\rho} \hat{f}((\lambda, \mathcal{L}^n)_\rho(x)) = \lim_{\rho \to 0} \sup_{x \in \Omega_\rho} \hat{f}(\lambda_\rho(x), 1)\\ &= \lim_{\rho \to 0} \sup_{x \in \Omega_\rho} f(\lambda_\rho(x)) = \mathcal{F}(\lambda, \Omega),\end{aligned}$$

and, again by Proposition 5,

$$\begin{aligned}\hat{F}((\lambda, \mathcal{L}^n), \Omega) &= \left[\operatorname*{ess\,sup}_{x \in \Omega} \ \hat{f}(\lambda^a(x), 1)\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} \ \hat{f}\left(\frac{d\lambda^s}{d|\lambda^s|}(x), 0\right)\right]\\ &= \left[\operatorname*{ess\,sup}_{x \in \Omega} \ f(\lambda^a(x))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} \ f^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)\right] = F(\lambda, \Omega).\end{aligned}$$

This achieves the proof. □

4. – Non level convex functionals.

In this section we state and prove the results about the functional $\mathbb{F}$ given by (8). First of all we need to prove that $\mathbb{F}$ is really a generalization of F. This can be proved noting that, when ϕ is positively homogeneous of degree 0, we have

$$\bigvee_{x \in A_\lambda} \phi(\lambda^\#(x)) = |\lambda^\#|\text{-}\operatorname*{ess\,sup}_{x \in A_\lambda} \ \phi\left(\frac{d\lambda^\#}{d|\lambda^\#|}(x)\right).$$

Then, when $\phi \equiv f^\natural$, applying Propositions 1 and 2 we obtain, for every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\mathbb{F}(\lambda, \Omega) = F(\lambda, \Omega)$. Let us note also that the value of ϕ for $\xi = 0$ does not enter in the computation of $\mathbb{F}$ thus ϕ could be defined only on $\mathbb{R}^m \setminus \{0\}$: moreover, to simplify several arguments we prefer to define ϕ on the whole space $\mathbb{R}^m$ adding the condition $\phi(0) = 0$.

Having in mind the theorems proved for the integral case, as for instance Theorem 2.3 in [13] and Theorem 3.3 in [12], we propose some analogous results about $\mathbb{F}$.

4.1. *Necessary conditions.*

THEOREM 6. – *Let $f, \phi : \mathbb{R}^m \to [0, \infty]$ be proper and Borel functions with $\phi(0) = 0$. Let us suppose that, for every $\Omega \subseteq \mathbb{R}^n$ open set, the functional $\mathbb{F}$ is l.s.c. on $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ convergence. Then f is l.s.c. and level convex on $\mathbb{R}^m$ and, defined $l = \inf\{f(\xi) : \xi \in \mathbb{R}^m\}$, $\phi \vee l$ is l.s.c. and sub-maximal on $\mathbb{R}^m$. Moreover, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $f^\natural(\xi) = (\phi \vee l)^\flat(\xi)$.*

PROOF. – The fact that f is l.s.c. and level convex can be proved using the same proof of the implication $(ii) \Rightarrow (i)$ of Theorem 3. In order to prove the other parts of the theorem let us consider $\{\eta_j\}_{j=1}^\infty \subseteq \mathbb{R}^m$ such that $f(\eta_j) \downarrow l$.

Let us study now the function ϕ. Let $x_0 \in \mathbb{R}^n$ and $\xi_h, \xi_0 \in \mathbb{R}^m \setminus \{0\}$ such that $\xi_h \to \xi_0$ and, fixed $j \in \mathbb{N}$, let

$$\lambda_h = \xi_h \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = \xi_0 \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n.$$

Then $\lambda_h \to \lambda_0$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\phi(\xi_0) \vee f(\eta_j) = \mathbb{F}(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbb{F}(\lambda_h, \mathbb{R}^n) = \liminf_{h\to\infty} (\phi(\xi_h) \vee f(\eta_j)).$$

This implies that, for every $j \in \mathbb{N}$, $\phi \vee f(\eta_j)$ is l.s.c. on $\mathbb{R}^m$ since also the lower semicontinuity in $\xi = 0$ trivially follows from $\phi(0) = 0$: by Proposition 15, we obtain that the same holds for $\phi \vee l$.

Let $\xi, \eta \in \mathbb{R}^m \setminus \{0\}$ such that $\xi \neq -\eta$ and $x_h, x_0 \in \mathbb{R}^n$, $x_h \neq x_0$, such that $x_h \to x_0$. Moreover, fixed $j \in \mathbb{N}$, set

$$\lambda_h = \xi \cdot \delta_{x_h} + \eta \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = (\xi + \eta) \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n.$$

Since $\lambda_h \to \lambda_0$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ we have

$$\begin{aligned}\phi(\xi + \eta) \vee f(\eta_j) &= \mathbb{F}(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbb{F}(\lambda_h, \mathbb{R}^n)\\ &= \liminf_{h\to\infty} \phi(\xi) \vee \phi(\eta) \vee f(\eta_j) = \phi(\xi) \vee \phi(\eta) \vee f(\eta_j).\end{aligned}$$

Then, since $\phi(0) = 0$, we have that, for every $j \in \mathbb{N}$, $\phi \vee f(\eta_j)$ is sub-maximal on $\mathbb{R}^m$ that implies, as $j \to \infty$, that $\phi \vee l$ satisfies the same property.

Let us prove now the inequality $f^\natural \leq (\phi \vee l)^\flat$. Let $\xi_0 \in \mathbb{R}^m \setminus \{0\}$ and let $\xi_h \to \xi_0$, $t_h \uparrow \infty$ such that $f(t_h \xi_h) \to f^\natural(\xi_0)$. Thus we have that there exists a sequence of positive numbers $\{r_h\}_{h=1}^\infty$ such that, for every $h \in \mathbb{N}$, $t_h \mathcal{L}^n(B(0, r_h)) = 1$. Fixed $j \in \mathbb{N}$ and setting

$$\lambda_h = t_h \left(\xi_h - \frac{\eta_j}{t_h} \right) 1_{B(0,r_h)}(x) \cdot \mathcal{L}^n + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = \xi_0 \cdot \delta_0 + \eta_j \cdot \mathcal{L}^n,$$

we have $\lambda_h \to \lambda_0$ in w^*-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and then

$$\phi(\xi_0) \vee f(\eta_j) = \mathbb{F}(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbb{F}(\lambda_h, \mathbb{R}^n)$$

$$= \lim_{h\to\infty} f(t_h \xi_h) \vee f(\eta_j) = f^{\natural}(\xi_0) \vee f(\eta_j).$$

This fact proves, as $j \to \infty$, that, for every $\xi_0 \in \mathbb{R}^m \setminus \{0\}$, $(\phi \vee l)(\xi_0) \leq f^{\natural}(\xi_0)$. If now we consider a sequence $s_h \downarrow 0$ such that $(\phi \vee l)^{\flat}(\xi_0) = \lim_{h\to\infty} (\phi \vee l)(s_h \xi_0)$, since $f^{\natural}$ is positively homogeneous of degree 0, we have

$$(\phi \vee l)^{\flat}(\xi_0) = \lim_{h\to\infty} (\phi \vee l)(s_h \xi_0) \leq \liminf_{h\to\infty} f^{\natural}(s_h \xi_0) = f^{\natural}(\xi_0),$$

and we find the desired inequality.

We end proving $f^{\natural} \geq (\phi \vee l)^{\flat}$. Let us consider $\xi_0 \in \mathbb{R}^m \setminus \{0\}$ and let $t_h \downarrow 0$, $t_h < 1$, such that $\phi(t_h \xi_0) \to \phi^{\flat}(\xi_0)$. Let us fix $\Omega = Q^n = (0,1)^n$ and define, for every $k \in \mathbb{N}$,

$$G_k = \left\{ x \in Q^n : x_i = \frac{q}{k+1}, q \in \{1, \ldots, k\}, i \in \{1, \ldots, n\} \right\}:$$

note that $\#(G_k) = k^n$. Fixed now $M \in \mathbb{N}$, we claim that there exists a sequence $\{k_h\}_{h=1}^{\infty} \subseteq \mathbb{N}$, depending on M and such that $k_h \to \infty$ and, for every $h \in \mathbb{N}$, $M \leq t_h k_h^n < 2^n M$. Obviously this happen if, for every $h \in \mathbb{N}$, there exists $k_h \in \mathbb{N}$ such that

$$\left(M t_h^{-1}\right)^{\frac{1}{n}} \leq k_h < 2\left(M t_h^{-1}\right)^{\frac{1}{n}},$$

and since $\left(M t_h^{-1}\right)^{\frac{1}{n}} > 1$ and $\left(M t_h^{-1}\right)^{\frac{1}{n}} \to \infty$ as $h \to \infty$, k_h can be found. Then, unless to extract a (not relabelled) subsequence, we have that $t_h k_h^n \to s_M \in [M, 2^n M]$.

Fixed $j \in \mathbb{N}$, set the following elements of $\mathcal{M}_{\text{loc}}(Q^n, \mathbb{R}^m)$

$$\lambda_h = \sum_{x \in G_{k_h}} \frac{t_h k_h^n}{k_h^n} \xi_0 \cdot \delta_x + \eta_j \cdot \mathcal{L}^n \quad \text{and} \quad \lambda_0 = s_M \xi_0 \cdot \mathcal{L}^n + \eta_j \cdot \mathcal{L}^n.$$

It is a quite standard result that $\lambda_h \to \lambda_0$ in w^*-$\mathcal{M}_{\text{loc}}(Q^n, \mathbb{R}^n)$, thus

$$f(s_M \xi_0 + \eta_j) = \mathbb{F}(\lambda_0, Q^n) \leq \liminf_{h\to\infty} \mathbb{F}(\lambda_h, Q^n)$$

$$= \liminf_{h\to\infty} \phi(t_h \xi_0) \vee f(\eta_j) = \phi^{\flat}(\xi_0) \vee f(\eta_j).$$

Then, letting $j \to \infty$ and using Proposition 16, we have

$$f\left(s_M\left(\xi_0 + \frac{\eta_j}{s_M}\right)\right) = f(s_M \xi_0 + \eta_j) \leq \phi^{\flat}(\xi_0) \vee l = (\phi \vee l)^{\flat}(\xi_0),$$

and since this relation holds for every $M \in \mathbb{N}$, taking the limit as $M \to \infty$ also $s_M \to \infty$ and then we obtain $f^{\natural}(\xi_0) \leq (\phi \vee l)^{\flat}(\xi_0)$. $\square$

4.2. *Sufficient conditions.*

In this section we find sufficient conditions for the lower semicontinuity of $\mathbb{F}$: we treat here the more general case in which f and ϕ depends on $x \in \Omega$ too. The following lemma is based on Lemma 3.6 in [12]: its simple proof is omitted and left to the reader.

LEMMA 4. – *Let $k \in \mathbb{N}$ and*

$$\mathcal{M}^k = \{\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m) : \#(\mathrm{spt}(\lambda)) \leq k\}.$$

Then $\mathcal{M}^k$ is sequentially closed with respect to the w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ topology. Moreover if $l \in [0, \infty)$ and $\phi : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a proper and Borel function such that $\phi \vee l$ is l.s.c. on $\Omega \times \mathbb{R}^m$ and, for every $x \in \Omega$, $\phi(x, \cdot) \vee l$ is sub-maximal on $\mathbb{R}^m$ and $\phi(x, 0) = 0$, then the functional

$$\Phi(\lambda, \Omega) = \bigvee_{x \in A_\lambda} \Big(\phi(x, \lambda(x)) \vee l\Big),$$

is l.s.c on $\mathcal{M}^k$ with respect to the w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$ topology.

In order to prove the main result we need a weak version of Theorem 4.1 in [1].

THEOREM 7. – *Let $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $f(x, \cdot)$ is l.s.c. and level convex on $\mathbb{R}^m$. Then the functional*

$$\operatorname*{ess\,sup}_{x \in \Omega} \; f(x, u(x))$$

is l.s.c. on $L^\infty(\Omega, \mathbb{R}^m)$ with respect to the w^*-$L^\infty(\Omega, \mathbb{R}^m)$ convergence.

We can now prove the main result of the section. Note that, when we are dealing with $f : \Omega \times \mathbb{R}^m \to [0, \infty]$, then $f^\natural(x, \xi)$ means $(f(x, \cdot))^\natural(\xi)$.

THEOREM 8. – *Let $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $f(x, \cdot)$ is l.s.c. and level convex on $\mathbb{R}^m$ and there exists a function $\theta : [0, \infty) \to [0, \infty)$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^m$,*

$$f(x, \xi) \geq \theta(|\xi|) \quad \text{and} \quad \lim_{t \to \infty} \theta(t) = \infty. \tag{28}$$

Moreover, setting $l = \inf\{f(x, \xi) : (x, \xi) \in \Omega \times \mathbb{R}^m\}$, let $\phi : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function such that $\phi \vee l$ is l.s.c. on $\Omega \times \mathbb{R}^m$, for every $x \in \Omega$, $\phi(x, \cdot) \vee l$ is sub-maximal on $\mathbb{R}^m$ and $\phi(x, 0) = 0$, and there exists a functions $\gamma : (0, \infty) \to [0, \infty)$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(x, \xi) \vee l \geq \gamma(|\xi|) \quad \text{and} \quad \lim_{t \to 0} \gamma(t) = \infty. \tag{29}$$

Then the functional

$$\mathbb{F}(\lambda,\Omega) = \left[\operatorname*{ess\,sup}_{x\in\Omega} f(x,\lambda^a(x))\right]$$

$$\vee\left[|\lambda^c|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} f^{\natural}\left(x,\frac{d\lambda^c}{d|\lambda^c|}(x)\right)\right]\vee\left[\bigvee_{x\in A_\lambda}\phi(x,\lambda^{\#}(x))\right]. \tag{30}$$

is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.

PROOF. – Let us suppose at first that $\Omega\subset\mathbb{R}^n$ is bounded and let us prove the lower semicontinuity of $\mathbb{F}$ on $\mathcal{M}(\Omega,\mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}(\Omega,\mathbb{R}^m)$ convergence. Consider $\lambda,\lambda_h\in\mathcal{M}(\Omega,\mathbb{R}^m)$ such that $\lambda_h\to\lambda$ in w^*-$\mathcal{M}(\Omega,\mathbb{R}^m)$: then there exists $L>0$ such that $|\lambda_h|(\Omega),|\lambda|(\Omega)\le L$. Without loss of generality we can suppose

$$\liminf_{h\to\infty}\mathbb{F}(\lambda_h,\Omega)=\lim_{h\to\infty}\mathbb{F}(\lambda_h,\Omega),$$

and that there exists a constant $M>0$ such that, for every $h\in\mathbb{N}$, $\mathbb{F}(\lambda_h,\Omega)\le M$. This fact and the coercivity conditions (28) and (29) imply that there exists $M'>0$ such that, for every $h\in\mathbb{N}$,

(i) $\lambda_h^c=0$,
(ii) for $\mathcal{L}^n$-a.e. $x\in\Omega$, $|\lambda_h^a(x)|\le M'$,
(iii) for every $x\in A_{\lambda_h}$, $|\lambda_h^{\#}(x)|\ge\frac{1}{M'}$.

By (ii) we obtain that $\{\lambda_h^a(x)\}_{h=1}^\infty\subseteq L^\infty(\Omega,\mathbb{R}^m)$ and $\|\lambda_h^a(x)\|_{L^\infty}\le M'$. Then there exists a (not relabelled) subsequence and a function $u\in L^\infty(\Omega,\mathbb{R}^m)$, such that

$$\lambda_h^a(x)\to u(x)\quad\text{in}\quad w^*\text{-}L^\infty(\Omega,\mathbb{R}^n). \tag{31}$$

Moreover, since, for every $h\in\mathbb{N}$, $|\lambda_h|(\Omega)\le L$, we have also $|\lambda_h^{\#}|(\Omega)\le L$ thus, writing now $\lambda_h^{\#}=\sum\limits_{x\in A_{\lambda_h}}\lambda_h^{\#}(x)\cdot\delta_x$ and using (iii), it follows, for every $h\in\mathbb{N}$,

$$L\ge\left|\lambda_h^{\#}\right|(\Omega)=\sum_{x\in A_{\lambda_h}}\left|\lambda_h^{\#}(x)\right|\ge\frac{1}{M'}\#(A_{\lambda_h}).$$

Then there exists $k\in\mathbb{N}$ such that, for every $h\in\mathbb{N}$, $\#(A_{\lambda_h})\le k$ that implies that, for every $h\in\mathbb{N}$, $\lambda_h^{\#}\in\mathcal{M}^k$. Thus there exists a (not relabelled) subsequence and a measure $\upsilon\in\mathcal{M}(\Omega,\mathbb{R}^m)$ such that

$$\lambda_h^{\#}\to\upsilon\quad\text{in}\quad w^*\text{-}\mathcal{M}(\Omega,\mathbb{R}^m). \tag{32}$$

By Lemma 4 we know that $\upsilon\in\mathcal{M}^k$.

We claim now that the condition $\lambda_h\to\lambda$ in w^*-$\mathcal{M}(\Omega,\mathbb{R}^m)$, together with (31) and (32), gives that $\lambda=u\cdot\mathcal{L}^n+\upsilon$, that is $\lambda^a=u$ and $\lambda^{\#}=\upsilon$. Indeed, for every

$\varphi \in C_0(\Omega)$,

$$\int_\Omega \varphi(x) d\lambda_h(x) = \int_\Omega \varphi(x)\lambda_h^a(x)dx + \int_\Omega \varphi(x) d\lambda_h^\#(x),$$

and, by (31) (noting that $C_0(\Omega) \subseteq L^1(\Omega)$ since $\mathcal{L}^n(\Omega) < \infty$) and (32) we have that $\lambda_h \to u \cdot \mathcal{L}^n + \upsilon$ in w^*-$\mathcal{M}(\Omega, \mathbb{R}^m)$. The uniqueness of the limit allows to achieve the claim.

By Theorem 7, we have

$$\operatorname*{ess\,sup}_{x\in\Omega} \; f(x, \lambda^a(x)) \le \liminf_{h\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} \; f(x, \lambda_h^a(x)),$$

while, by Lemma 4, we have

$$\bigvee_{x \in A_\lambda} \left(\phi(x, \lambda^\#(x)) \vee l\right) \le \liminf_{h\to\infty} \bigvee_{x\in A_{\lambda_h}} \left(\phi(x, \lambda_h^\#(x)) \vee l\right).$$

Then

$$\liminf_{h\to\infty} \mathbb{F}(\lambda_h, \Omega) = \liminf_{h\to\infty} \left[\operatorname*{ess\,sup}_{x\in\Omega} \; f(x, \lambda_h^a(x))\right] \vee \left[\bigvee_{x\in A_{\lambda_h}} \left(\phi(x, \lambda_h^\#(x)) \vee l\right)\right]$$

$$\ge \left[\operatorname*{ess\,sup}_{x\in\Omega} \; f(x, \lambda^a(x))\right] \vee \left[\bigvee_{x\in A_\lambda} \left(\phi(x, \lambda^\#(x)) \vee l\right)\right] = \mathbb{F}(\lambda, \Omega),$$

and we get the lower semicontinuity.

In the general case let us consider $\lambda, \lambda_h \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $\lambda_h \to \lambda$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, for every $\Omega' \subset\subset \Omega$ open set, $\lambda_h \to \lambda$ in w^*-$\mathcal{M}(\Omega', \mathbb{R}^m)$ and then

$$\mathbb{F}(\lambda, \Omega') \le \liminf_{h\to\infty} \mathbb{F}(\lambda_h, \Omega') \le \liminf_{h\to\infty} \mathbb{F}(\lambda_h, \Omega).$$

At least, since

$$\sup\{\mathbb{F}(\lambda, \Omega') : \Omega' \subset\subset \Omega\} = \mathbb{F}(\lambda, \Omega),$$

the proof is finally achieved. □

A simplified version of the previous theorem is given by the following result easily comparable with Theorem 6.

THEOREM 9. – *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and level convex function, $l = \inf\{f(\xi) : \xi \in \mathbb{R}^m\}$ and $\phi : \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function such that $\phi \vee l$ is a l.s.c and sub-maximal function with $\phi(0) = 0$. Let us suppose that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $f^\natural(\xi) = (\phi \vee l)^\flat(\xi) = \infty$. Then the functional $\mathbb{F}$ is l.s.c. on $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with respect to the w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ convergence.*

PROOF. – The proof easily follows from Theorem 8 and Propositions 7 and 8. $\square$

Our conjecture is that in Theorem 9 the condition $f^{\natural} = (\phi \vee l)^{\flat} = \infty$ could be replaced with the weaker condition $f^{\natural} = (\phi \vee l)^{\flat}$ that we know both, thanks to Theorem 6, to be necessary and, by Theorem 4.4 in [2], to be sufficient at least in dimension one.

5. – Applications to the BV setting.

As already stated, this section is devoted to some simple applications of the results obtained in Sections 3 and 4 to the BV setting.

A Borel function $u : \Omega \to \mathbb{R}$ is said to be a function of (resp. *locally*) *bounded variation* on Ω, or briefly $u \in BV(\Omega)$ (resp. $u \in BV_{\mathrm{loc}}(\Omega)$), if $u \in L^1(\Omega)$ (resp. $u \in L^1_{\mathrm{loc}}(\Omega)$) and there exists $Du \in \mathcal{M}(\Omega, \mathbb{R}^n)$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$) such that, for every $\varphi \in C_c^{\infty}(\Omega)$,

$$\int_{\Omega} \varphi(x) dDu(x) = -\int_{\Omega} u(x) \nabla \varphi(x) dx.$$

Since $Du \in \mathcal{M}(\Omega, \mathbb{R}^n)$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$), following (10), it can be decomposed with respect to $\mathcal{L}^n$ as

$$Du = \nabla u \cdot \mathcal{L}^n + D^s u,$$

where $\nabla u \in L^1(\Omega, \mathbb{R}^n)$ (resp. $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$), $D^s u \in \mathcal{M}(\Omega, \mathbb{R}^n)$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$) and $D^s u \perp \mathcal{L}^n$.

For the main properties of these functions we refer to [6, 19].

5.1. *Level convex functionals: semicontinuity, relaxation and Dirichlet problem.*

The following result easily follows by Theorem 3 and it provides an extension of the supremal functional (5) on $BV_{\mathrm{loc}}(\Omega)$. This theorem was proved for the first time by Gori in [22].

THEOREM 10. – *Let $f : \mathbb{R}^n \to [0, +\infty]$ be a proper, l.s.c and level convex function. Let us consider the following functional defined, for every $u \in BV_{\mathrm{loc}}(\Omega)$, as*

$$F(u, \Omega) = \operatorname*{ess\,sup}_{x \in \Omega} f(\nabla u(x)) \vee |D_s u|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} f^{\natural}\left(\frac{dD_s u}{d|D_s u|}(x)\right). \tag{33}$$

Then F is l.s.c. on $BV_{\mathrm{loc}}(\Omega)$ with respect to the w^*-$BV_{\mathrm{loc}}(\Omega)$ convergence and, for every $u \in BV_{\mathrm{loc}}(\Omega)$,

$$F(u,\Omega) = \mathcal{F}(u,\Omega) = \lim_{\rho\to 0} F(u_\rho,\Omega_\rho), \tag{34}$$

where $\mathcal{F}$ is given by

$$\mathcal{F}(u,\Omega) = \inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega_h) : u_h \in W^{1,1}_{\mathrm{loc}}(\Omega_h), \quad u_h \to u \; w^*\text{-}BV_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m) \right\},$$

while u_ρ denotes the convolution of u.

PROOF. – Since, when $u_h, u \in BV_{\mathrm{loc}}(\Omega)$ and $u_h \to u$ in w^*-$BV_{\mathrm{loc}}(\Omega)$, we have $Du_h \to Du$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ and since, for every $u \in BV_{\mathrm{loc}}(\Omega)$, $u_\rho \to u$ in w^*-$BV_{\mathrm{loc}}(\updownarrow \Omega)$ (and then $\nabla u_\rho \cdot \mathcal{L}^n = (Du)_\rho \to Du$ in w^*-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^n)$) we end applying directly Theorem 3. $\square$

Equalities (34) say that, in particular, for every $u \in BV(\Omega)$,

$$F(u,\Omega) \le R[w^*\text{-}BV]\Big(F_{|C^\infty(\overline{\Omega})}\Big)(u,\Omega),$$

where

$$R[w^*\text{-}BV]\Big(F_{|C^\infty(\overline{\Omega})}\Big)(u,\Omega)$$
$$= \inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : u_h \in C^\infty(\overline{\Omega}), u_h \to u \; w^*\text{-}BV(\Omega,\mathbb{R}^m) \right\},$$

is the relaxed functional of $F_{|C^\infty(\overline{\Omega})}$ on $BV(\Omega)$.

The next proposition shows a particular case in which also the opposite inequality hold.

PROPOSITION 19. – *Let $\Omega = B(0,1) \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to [0,\infty]$ be a proper, l.s.c. and level convex function such that $f(0) = 0$. Then, for every $u \in BV(\Omega)$,*

$$F(u,\Omega) = R[w^*\text{-}BV]\Big(F_{|C^\infty(\overline{\Omega})}\Big)(u,\Omega).$$

PROOF. – Let us fix $u \in BV(\Omega)$, consider the convolution $u_\rho : B(0,1-\rho) \to \mathbb{R}$ (with $\rho < \frac{1}{4}$) and define, for every $x \in B(0,1)$, $\hat{u}_\rho(x) = u_\rho((1-2\rho)x)$: clearly $\hat{u}_\rho \in C^\infty(\overline{B(0,1)})$.

We claim at first that $\hat{u}_\rho \to u$ in w^*-$BV(B(0,1))$. Indeed we simply have that, by means of the properties of the convolutions and the change of variables formula, there exists a constant $C > 0$ such that, for every $0<\rho<\frac{1}{4}$,

$$\int_{B(0,1)}|\hat{u}_\rho(x)|dx+\int_{B(0,1)}|\nabla\hat{u}_\rho(x)|dx$$

$$\leq\frac{1}{(1-2\rho)^n}\int_{B(0,1)}|u(x)|dx+\frac{1}{(1-2\rho)^{n-1}}|Du|(B(0,1))dx\leq C$$

Thus, by the compactness theorem on $BV(\Omega)$, there exists $v\in BV(B(0,1))$ such that $\hat{u}_\rho\to v$ in w^*-$BV(B(0,1))$. We prove that, for $\mathcal{L}^n$-a.e. $x\in B(0,1)$, $u(x)=v(x)$ showing that, for $\mathcal{L}^n$-a.e. $x\in B(0,1)$, $\hat{u}_\rho(x)\to u(x)$. Indeed, if $M=\sup\{k(x):x\in\mathbb{R}^n\}$ we have

$$|\hat{u}_\rho(x)-u(x)|=\left|\int_{B((1-2\rho)x,\rho)}k_\rho((1-2\rho)x-y)u(y)dy-u(x)\right|$$

$$\leq\int_{B((1-2\rho)x,\rho)}k_\rho((1-2\rho)x-y)|u(y)-u(x)|dy\leq\frac{M}{\rho^n}\int_{B(x,3\rho)\cap B(0,1)}|u(y)-u(x)|dy.$$

Now if ρ is small enough (that is $3\rho\leq 1-|x|$), $B(x,3\rho)\cap B(0,1)=B(x,3\rho)$ and so

$$|\hat{u}_\rho(x)-u(x)|\leq\frac{3^nM}{(3\rho)^n}\int_{B(x,3\rho)}|u(y)-u(x)|dy,$$

that goes, for $\mathcal{L}^n$-a.e. $x\in B(0,1)$, to zero as $\rho\to 0$ by the Lebesgue's points Theorem.

Once the claim is proved, applying Proposition 12 to f, we have that, for every $x\in B(0,1)$,

$$f\big(\nabla\hat{u}_\rho(x)\big)=f\big((1-2\rho)\nabla u_\rho((1-2\rho)x)\big)$$

$$\leq f\big(\nabla u_\rho((1-2\rho)x)\big)\leq\sup_{x\in B(0,1-\rho)}f\big(\nabla u_\rho(x)\big),$$

and then

$$\sup_{x\in B(0,1)}f\big(\nabla\hat{u}_\rho(x)\big)\leq\sup_{x\in B(0,1-\rho)}f\big(\nabla u_\rho(x)\big).$$

This allows to achieve the thesis since

$$F(u,\Omega)=\lim_{\rho\to 0}F(u_\rho,\Omega_\rho)\geq\liminf_{\rho\to 0}F(\hat{u}_\rho,\Omega)\geq R[w^*\text{-}BV]\Big(F_{|C^\infty(\overline{\Omega})}\Big)(u,\Omega)$$

and being the opposite inequality satisfied. $\square$

REMARK 1. – It is quite simple to understand that the above proposition holds also if we set Ω to be a bounded convex set. We have written down the proof for the unit ball in order to simplify many technical details.

Following the work of Anzellotti, Buttazzo and Dal Maso [7] in which the same problem is considered for the integral functionals, we shall consider a generalized Dirichlet problem for supremal functionals defined on $BV(\Omega)$. We note that the formulation we proposed is similar to the one used in the nonparametric Plateau's problem (see [20]).

We have to underline that functional (35) of Theorem 11 does not arise from the relaxation on $BV(\Omega)$, with respect to the w^*-$BV(\Omega)$ convergence, of the functional (33) restricted on $C^1_\varphi(\overline{\Omega})$ (that is the space of the function belonging to $C^1(\overline{\Omega})$ with fixed boundary data φ) but it is simply suggested by the analogy with the integral setting. However, because of Theorem 11, it can be considered a good candidate to represent just the functional $R[w^*\text{-}BV(\Omega)]\left(F_{|C^1_\varphi(\overline{\Omega})}\right)$.

In the following, given a function $u \in BV(\Omega)$, we will always consider the restriction of u to $\partial\Omega$ in the trace sense (see for instance Theorem 1 pg. 177 [19]).

THEOREM 11. – *Let $\Omega \subset\subset \Omega'$ be two open and bounded subsets of $\mathbb{R}^n$ with Lipschitz boundaries and $f : \mathbb{R}^n \to [0, \infty]$ be a proper, l.s.c. and level convex function. Let us consider $w \in W^{1,1}(\Omega' \setminus \Omega)$ and set*

$$\varphi = w_{|\partial\Omega} \in L^1_{\mathcal{H}^{n-1}}(\partial\Omega) \quad \text{and} \quad L = \operatorname*{ess\,sup}_{x \in \Omega' \setminus \Omega} f(\nabla w(x)).$$

For every $u \in BV(\Omega)$, let us define the functional

$$F_\varphi(u, \Omega) = F(u, \Omega) \vee \mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x \in \partial\Omega} f^\natural\Big((\varphi(x) - u(x))\upsilon(x)\Big) \vee L, \tag{35}$$

Where F is given by (33) *and $\upsilon : \partial\Omega \to \mathbb{R}^n$ is the vector field of the outer normal vectors to $\partial\Omega$.*

Then F_φ is l.s.c. on $BV(\Omega)$ with respect to the w^*-$BV(\Omega)$ convergence. Moreover if $s = \sup\{f(\xi) : \xi \in \mathbb{R}^n\}$ and $K_f = \{\xi \in \mathbb{R}^n : f^\natural(\xi) < s\}$ is such that $\mathrm{cl}(K_f)$ *does not contain any straight line, then the problem*

$$\min\big\{F_\varphi(u, \Omega) : u \in BV(\Omega)\big\}, \tag{36}$$

admits at least a solution.

PROOF. – Let $u, u_h \in BV(\Omega)$ such that $u_h \to u$ in w^*-$BV(\Omega)$ and define

$$\hat{u}(x) = \begin{cases} u(x) \text{ if } x \in \Omega, \\ w(x) \text{ if } x \in \Omega' \setminus \Omega, \end{cases} \quad \text{and} \quad \hat{u}_h(x) = \begin{cases} u_h(x) \text{ if } x \in \Omega, \\ w(x) \text{ if } x \in \Omega' \setminus \Omega. \end{cases}$$

It is easy to see that $\hat{u}, \hat{u}_h \in BV(\Omega')$ and $\hat{u}_h \to \hat{u}$ in w^*-$BV(\Omega')$. Moreover, since

$$D\hat{u} \llcorner \partial\Omega = (\varphi - u)\upsilon \cdot \mathcal{H}^{n-1} \llcorner \partial\Omega \quad \text{and} \quad D\hat{u}_h \llcorner \partial\Omega = (\varphi - u_h)\upsilon \cdot \mathcal{H}^{n-1} \llcorner \partial\Omega$$

(see [20] and [7] Theorem 3.1), we have also

$$F_\varphi(u,\Omega) = F(\hat{u},\Omega') \quad \text{and} \quad F_\varphi(u_h,\Omega) = F(\hat{u}_h,\Omega').$$

Then, by means of Theorem 10, it is

$$F_\varphi(u,\Omega) = F(\hat{u},\Omega') \le \liminf_{h\to\infty} F(\hat{u}_h,\Omega') = \liminf_{h\to\infty} F_\varphi(u_h,\Omega)$$

and the first part of the theorem is proved.

Before proving the second part of the theorem we need a remark. Let $\Theta : [0,s] \to [0,\infty]$ be a continuous and strictly increasing function: it is clear that $\bar{u} \in BV(\Omega)$ is a minimum point of $F_\varphi(u,\Omega)$ if and only if it is a minimum point of $\Theta\big(F_\varphi(u,\Omega)\big)$. However, a simple computation gives that, for every $u \in BV(\Omega)$,

$$\begin{aligned} \Theta\big(F_\varphi(u,\Omega)\big) = & \left[\operatorname*{ess\,sup}_{x\in\Omega} \ (\Theta \circ f)(\nabla u(x))\right] \\ & \vee \left[|D^s u|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \ (\Theta \circ f^{\natural})\left(\frac{dD^s u}{d|D^s u|}(x)\right)\right] \\ & \vee \left[\mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x\in\partial\Omega} \ (\Theta \circ f^{\natural})\Big((\varphi(x) - u(x))\upsilon(x)\Big)\right] \ \vee \ \tilde{L}, \end{aligned}$$

where

$$\tilde{L} = \operatorname*{ess\,sup}_{x\in\Omega'\setminus\Omega} \ (\Theta \circ f)(\nabla w(x)).$$

Thus, by Proposition 13, we have that $\Theta\big(F_\varphi(u,\Omega)\big)$ is in fact the functional (35) related to $\Theta \circ f : \mathbb{R}^n \to [0,\infty]$. Then, invoking now Proposition 14, we can suppose, without loss of generality, that f is demi-coercive, that is, there exist $a > 0, b \ge 0$ and $\eta \in \mathbb{R}^n$ such that, for every $\xi \in \mathbb{R}^n$,

$$f(\xi) \ge a|\xi| - \langle \eta, \xi \rangle - b.$$

If, for every $u \in BV(\Omega)$, $F_\varphi(u,\Omega) = \infty$ there is nothing to prove. Thus let us consider any $u \in BV(\Omega)$ such that $F_\varphi(u,\Omega) < \infty$: we claim that, setting $c = \frac{1}{3\mathcal{L}^n(\Omega)} \wedge \frac{1}{3\mathcal{H}^{n-1}(\partial\Omega)}$,

$$\begin{aligned} F_\varphi(u,\Omega) \ge c \Bigg\{ & \int_\Omega f(\nabla u(x))dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x) \\ & + \int_{\partial\Omega} f^\infty\Big((\varphi(x) - u(x))\upsilon(x)\Big) \mathcal{H}^{n-1}(x) \Bigg\}. \end{aligned} \tag{37}$$

Indeed clearly it holds

$$\operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \geq \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega f(\nabla u(x))dx. \tag{38}$$

Moreover since

$$|D^s u|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} f^\natural\left(\frac{dD^s u}{d|D^s u|}(x)\right) < \infty,$$

it is that, for $|D^s u|$-a.e. $x \in \Omega, f^\natural\left(\frac{dD^s u}{d|D^s u|}(x)\right) < \infty$ and then, by Proposition 11(ii), we have also that, for $|D^s u|$-a.e. $x \in \Omega, f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) = 0$: then in particular

$$\frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x) = 0. \tag{39}$$

At last using Proposition 11(i) we have

$$\begin{aligned} &\mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x\in\partial\Omega} f^\natural\Big((\varphi(x) - u(x))\nu(x)\Big) \\ &\geq \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} f^\infty\Big((\varphi(x) - u(x))\nu(x)\Big)\mathcal{H}^{n-1}(x). \end{aligned} \tag{40}$$

Putting together (38), (39) and (40) we obtain that, for every $u \in BV(\Omega)$ such that $F_\varphi(u, \Omega) < \infty$,

$$\begin{aligned} F_\varphi(u,\Omega) &\geq \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega f(\nabla u(x))dx \vee \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x) \\ &\quad \vee \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} f^\infty\Big((\varphi(x) - u(x))\nu(x)\Big)\mathcal{H}^{n-1}(x) \\ &\geq c\left\{\int_\Omega f(\nabla u(x))dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x)\right. \\ &\quad \left. + \int_{\partial\Omega} f^\infty\Big((\varphi(x) - u(x))\nu(x)\Big)\mathcal{H}^{n-1}(x)\right\}, \end{aligned}$$

and (37) is proved.

By means of the inequality (37) and Proposition 11(iii) ([4]) we find

$$F_\varphi(u,\Omega) \geq ac\int_\Omega |\nabla u(x)|dx - c\int_\Omega \langle \eta, \nabla u(x)\rangle dx - bc\mathcal{L}^n(\Omega) + ac|D^s u|(\Omega)$$

$$-c\int_\Omega \left\langle \eta, \frac{dD^s u}{d|D^s u|}(x)\right\rangle d|D^s u|(x) + ac\int_{\partial\Omega} |\varphi(x) - u(x)|d\mathcal{H}^{n-1}(x)$$

$$-c\int_{\partial\Omega} \langle \eta, (\varphi(x) - u(x))\upsilon(x)\rangle d\mathcal{H}^{n-1}(x),$$

and then, using the fact that ([5]), for every $u \in BV(\Omega)$,

$$\int_\Omega \langle \eta, \nabla u(x)\rangle dx + \int_\Omega \left\langle \eta, \frac{dD^s u}{d|D^s u|}(x)\right\rangle d|D^s u|(x) = \int_{\partial\Omega} \langle \eta, u(x)\upsilon(x)\rangle d\mathcal{H}^{n-1}(x)$$

and that there exists ([6]) a constant $A = A(\Omega) > 0$ such that, for every $u \in BV(\Omega)$,

$$\int_\Omega |u(x)|dx \leq A\left\{ |Du|(\Omega) + \int_{\partial\Omega} |u(x)|d\mathcal{H}^{n-1}(x)\right\},$$

it follows

$$F_\varphi(u,\Omega) \geq \frac{ac}{2}\left(1 \wedge \frac{1}{A}\right)\left\{ |Du|(\Omega) + \int_\Omega |u(x)|dx + \int_{\partial\Omega} |u(x)|d\mathcal{H}^{n-1}(x)\right\} - dc, \tag{41}$$

where

$$d = \int_{\partial\Omega} |\varphi(x)|d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \langle \eta, \varphi(x)\upsilon(x)\rangle d\mathcal{H}^{n-1}(x).$$

If now we consider a minimizing sequence $\{u_h\}_{h=1}^\infty$ such that, for every $h \in \mathbb{N}$, $F_\varphi(u_h,\Omega) < \infty$ we can prove its compactness only noting that (41) allows just to apply the compactness theorem on $BV(\Omega)$: this fact, together with the lower semicontinuity of the functional F_φ, guarantees the existence of a minimum for (36). □

REMARK 2. – If there exists $u \in BV(\Omega)$ such that $F_\varphi(u,\Omega) < \infty$, then the same holds for every solution $\bar{u}$ of (36) too. Then, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$(\varphi(x) - \bar{u}(x))\upsilon(x) \in K_f = \{\xi \in \mathbb{R}^n : f^\natural(\xi) < \infty\}.$$

([4]) Compare with Theorems 2.7 and 3.2 of [7].
([5]) See Theorem 1 pg. 177 [19] and [20].
([6]) See [20] and equation (3.4) of [7].

Since $f^\natural$ is level convex and positively homogeneous of degree 0 (see Proposition 4), K_f is clearly a convex cone and then it follows that, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : \upsilon(x) \notin K_f\}$, $\varphi(x) - \bar{u}(x) \leq 0$ and, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : -\upsilon(x) \notin K_f\}$, $\varphi(x) - \bar{u}(x) \geq 0$. Therefore we conclude that, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : \upsilon(x) \notin K_f \cup (-K_f)\}$, $\varphi(x) = \bar{u}(x)$.

REMARK 3. – It is clear that, arguing as in the proof of Theorem 11, we cannot avoid the presence of the constant L. Nevertheless we can easily find suitable classes of examples in which $L = 0$ and then (36) really generalizes (6). Consider for instance the case in which $\Omega = B(0,1) \subseteq \mathbb{R}^2$ and $f : \mathbb{R}^2 \to [0,\infty)$ is defined as $f(\xi_1,\xi_2) = (\xi_1 \vee 0) + (\xi_2 \vee 0)$. If φ is the restriction to S^1 of a function $w \in C^1(B(0,1+\varepsilon))$ (where $\varepsilon > 0$) such that, for every $x \in B(0,1+\varepsilon) \setminus B(0,1)$, $\nabla w(x) \in \{(\xi_1,\xi_2) : \xi_1 \leq 0, \xi_2 \leq 0\}$, then $L = 0$.

5.2. *Non level convex functionals.*

In this section we find a simple application of Theorem 8 to the BV setting where obviously only the one dimensional case is considered: the result here presented should be compared with the ones obtained by Alicandro, Braides and Cicalese [2] in which an analogous problem is deeply analyzed.

For this let $(a,b) \subseteq \mathbb{R}$, $f, w, \phi : (a,b) \times \mathbb{R} \to [0,\infty]$ be proper and Borel functions. We want to minimize the following functional

$$L(u,K) = \left[\operatorname*{ess\,sup}_{x\in(a,b)\setminus K} f(x,u'(x))\right]$$

$$\vee \left[\bigvee_{x\in K} \phi(x, u(x+) - u(x-))\right] \vee \left[\operatorname*{ess\,sup}_{x\in(a,b)\setminus K} w(x,u(x))\right], \tag{42}$$

on the class

$$\mathcal{A} = \{(u,K) : K = \{a = t_0 < \ldots < t_m = b\} \subseteq (a,b), m \in \mathbb{N},$$

$$\forall i \in \{1,\ldots,m\}\ u \in W^{1,\infty}(t_{i-1},t_i)\}.$$

Here $u(x+)$ and $u(x-)$ are the right and the left limit of u in x that always exist for every $x \in K$ when $(u,K) \in \mathcal{A}$. Let us note the analogy between this problem and the classical Mumford-Shah image segmentation one (see [6] Chapter 6).

In order to solve this problem let's note at first that if $(u,K) \in \mathcal{A}$ then $u \in SBV(a,b)$ (for more details on this space of functions see [6] Chapter 4). For this we give a weak formulation of the functional L extending its definition on the space $SBV(a,b)$ in the obvious way: for every $u \in SBV(a,b)$, decomposing its distributional derivative as $Du = u' \cdot \mathcal{L}^1 + D^{\#}u \in \mathcal{M}(a,b)$, where $u' \in L^1(a,b)$ and $D^{\#}u$ is purely atomic, we define

$$\mathbb{L}(u) = \left[\operatorname*{ess\,sup}_{x\in(a,b)} f(x,u'(x))\right] \vee \left[\bigvee_{x\in A_{Du}} \phi(x, D^{\#}u(x))\right] \vee \left[\operatorname*{ess\,sup}_{x\in(a,b)} w(x,u(x))\right]. \tag{43}$$

Clearly, when $(u,K) \in \mathcal{A}$, $L(u,K) = \mathbb{L}(u)$, thus

$$\inf_{u\in SBV(a,b)} \mathbb{L}(u) \le \inf_{(u,K)\in\mathcal{A}} L(u,K).$$

The following theorem shows that, with suitable hypotheses on f, w, ϕ, the functional L admits a minimum on $\mathcal{A}$. Note that, in order to obtain the compactness we need a particular hypothesis on ϕ, that, as Proposition 10 shows, is more natural it seems.

THEOREM 12. – *Let $f, w, \phi : (a,b) \times \mathbb{R} \to [0,\infty]$ be proper and Borel functions. Let us suppose that, for $\mathcal{L}^1$-a.e. $x \in (a,b)$, $f(x,\cdot), w(x,\cdot)$ are l.s.c. and level convex on $\mathbb{R}$, and there exists a function $\theta : [0,\infty) \to [0,\infty)$ such that, for every $x \in (a,b)$ and $\xi \in \mathbb{R}$,*

$$f(x,\xi) \wedge w(x,\xi) \ge \theta(|\xi|) \quad \text{and} \quad \lim_{t\to\infty} \theta(t) = \infty.$$

Moreover, setting $l = \inf\{f(x,\xi) : (x,\xi) \in (a,b)\times\mathbb{R}\}$, let us suppose that $\phi \vee l$ is l.s.c. on $(a,b)\times\mathbb{R}$, for every $x \in (a,b)$, $\phi(x,\cdot) \vee l$ is sub-maximal on $\mathbb{R}$ and $\phi(x,0) = 0$, and there exists a function $\gamma : (0,\infty) \to [0,\infty)$ such that,

$$\begin{cases} \phi(x,\xi) \ge \gamma(\xi) \text{ if } x \in (a,b), \xi > 0, \\ \phi(x,\xi) = \infty \text{ if } x \in (a,b), \xi < 0, \end{cases} \quad \text{and} \quad \lim_{t\to 0} \gamma(t) = \infty.$$

Then the problem

$$\min\{L(u,K) : (u,K) \in \mathcal{A}\}$$

admits at least a solution.

PROOF. – We start considering the functional $\mathbb{L}$ defined by (43) and proving, by using the direct methods, that it has a minimum on $SBV(a,b)$. For this let us consider a minimizing sequence $\{u_h\}_{h=1}^{\infty} \subseteq SBV(a,b)$: we can suppose that there exists a constant $M > 0$ such that, for every $h \in \mathbb{N}$, $\mathbb{L}(u_h,(a,b)) \le M$ (if $\mathbb{L} \equiv \infty$ there is nothing to prove). We want to prove at first the compactness of this sequence.

Using the coercivity property of f, w and ϕ, we can find a constant $M' > 0$ such that, for every $u \in SBV(a,b)$ such that $\mathbb{L}(u,(a,b)) \le M$ (in particular, for every u_h), we have

(i) for $\mathcal{L}^1$-a.e. $x \in (a,b)$, $|u'(x)| \leq M'$,
(ii) for $\mathcal{L}^1$-a.e. $x \in (a,b)$, $|u(x)| \leq M'$,
(iii) for every $x \in A_{Du}$ (the set of the atoms of Du), $D^{\#}u(x) \geq \frac{1}{M'}$. This condition implies, in particular, that on every jump point the function u increases.

Then, thanks to Theorem 4.8 in [6], we may prove that the sequence $\{u_h\}_{h=1}^{\infty}$ is compact if we are able to prove that there exists $C > 0$ such that, for every $h \in \mathbb{N}$, $\#(A_{u_h}) \leq C$. In order to prove this we consider the formula, referred to a good representative of u_h, given by

$$u_h(b-) - u_h(a+) = \int_a^b u'(x)dx + \sum_{x \in A_{u_h}} D^{\#}u_h(x)$$

that implies, by means of (iii),

$$\#(A_{u_h})\frac{1}{M'} \leq \sum_{x \in A_{u_h}} D^{\#}u_h(x) \leq |u_h(b-) - u_h(a+)|$$

$$+\int_a^b |u_h'(x)|dx \leq 2M' + (b-a)M' = C.$$

Thus there exists a (not relabelled) subsequence and $\bar{u} \in SBV(a,b) \cap L^{\infty}(a,b)$ such that

$$u_h \to \bar{u} \text{ in } w^*\text{-}BV(a,b), \quad \text{and} \quad u_h \to \bar{u} \text{ in } w^*\text{-}L^{\infty}(a,b).$$

Let's show now the lower semicontinuity of $\mathbb{L}$ on this sequence. Obviously $Du_h \to D\bar{u}$ in w^*-$\mathcal{M}(a,b)$ and, by Theorem 8, for the functional

$$\left[\operatorname*{ess\,sup}_{x \in (a,b)} f(x, u'(x))\right] \vee \left[\bigvee_{x \in A_{Du}} \phi(x, D^{\#}u(x))\right],$$

the lower semicontinuity holds. By Theorem 7 also the functional

$$\operatorname*{ess\,sup}_{x \in (a,b)} w(x, u(x)),$$

is lower semicontinuous on the considered sequence. Then

$$\mathbb{L}(\bar{u}, (a,b)) \leq \liminf_{h \to \infty} \mathbb{L}(u_h, (a,b)),$$

and so $\mathbb{L}$ admits a minimum on $SBV(a,b)$ given by $\bar{u}$. Let's note now that since (i), (ii) and (iii) hold for $\bar{u}$ too, clearly $\#(A_{D\bar{u}}) \leq C$ and, for $\mathcal{L}^1$-a.e. $x \in (a,b)$, $|u'(x)| \leq M'$: then $(\bar{u}, A_{D\bar{u}}) \in \mathcal{A}$ and the thesis is achieved. $\square$

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