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Existence and Uniqueness for an Integro-Differential Equation with Singular Kernel.

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Sunto. – *In questo articolo si studia un problema evolutivo per la viscoelasticità lineare, supponendo che il nucleo di memoria G' sia singolare. Si assume che G' presenti una singolarità iniziale in modo che non sia una funzione L^1 nel tempo, ma che la funzione G sia integrabile per $t = 0$. Applicando il metodo delle trasformate di Fourier, si dimostra un teorema di esistenza e unicità della soluzione debole, in un opportuno spazio funzionale, la cui definizione dipende esplicitamente dalle proprietà del nucleo di memoria.*

Summary. – *In this paper we study the evolutive problem of linear viscoelasticity with a singular kernel memory G'. We assume that G' presents an initial singularity, so that it is not a L^1 –function in time, whereas the relaxation function G is integrable at $t = 0$. By applying the Fourier transform method, we prove a theorem of existence and uniqueness of the weak solutions in a functional space whose definition is strictly related to the properties of the kernel memory.*

1. – Introduction.

The evolution of a viscoelastic solid, contained in a domain $\Omega \subset \mathbb{R}^3$ is governed by the equation

$$\ddot{u}(x, t) = \nabla \cdot T(x, t) + f(x, t) \qquad x \in \Omega, \quad t > 0 ,$$

where u is the displacement vector, T is the stress tensor and f is the body force.

We assume that the stress tensor satisfies the linear constitutive equation [6]

$$T(x, t) = G_\infty(x) \nabla u(x, t) - \int_0^\infty G'(x, s)[\nabla u(x, t) - \nabla u(x, t - s)]\, ds ,$$

where G_∞ and G' are symmetric fourth-order tensors.

The kernel memory G' is usually required to be a L^1-function in time ([1, 2]). However the behaviour of some kind of materials can be described through a model which involves a singular kernel (see [4, 5, 8] and references therein). In such a case, the tensor G' is assumed to present an initial singularity, so that $G' \notin L^1$.

In this paper we consider this case and suppose also that the relaxation function

$$G(x, t) = G_\infty(x) - \int_t^\infty G'(x, s)\, ds \tag{1.1}$$

is integrable at $t = 0$. To deal with the singularity of the kernel memory, we apply the Fourier transform method and introduce a definition of the weak solutions of the problem in the frequencies domain. In this way, we analyze the properties of a complex-valued tensor H, related to the Fourier transform of G'. The search of the weak solutions leads to the introduction of a functional space, whose definition depends explicitly on the properties of H.

In Section 2 we consider the transformed system and show that the original problem is related to the solution of an elliptic equation with fixed frequency. Under suitable hypotheses on the source f and on the initial history $u^0(x, s) = u(x, -s)$, $s > 0$, we prove in Section 3 an estimate which allows us to obtain the existence and uniqueness of the weak solution.

2. – Formulation of the problem.

Let us consider the equation

$$\ddot{u}(x, t) = \nabla \cdot \left[G_\infty(x)\, \nabla u(x, t) - \int_0^\infty G'(x, s)[\nabla u(x, t) - \nabla u^t(x, s)]\, ds \right] + f(x, t), \tag{2.2}$$

where $u^t(x, s) = u(x, t - s)$ denotes the history of the displacement u.

We assume that Ω is a regular bounded domain, with boundary $\partial\Omega$ C^1-smooth and locally on one side of $\partial\Omega$.

Moreover, we require that u satisfies the elastic boundary condition

$$\alpha(\sigma)\, u(\sigma, t) + T(\sigma, t)\, n(\sigma) = 0 \qquad \sigma \in \partial\Omega\,, \tag{2.3}$$

where n denotes the outward normal of the boundary $\partial\Omega$ and $\alpha \in L^\infty(\partial\Omega)$ is a scalar function such that

$$\alpha(\sigma) \geq \alpha_m > 0 \qquad \sigma \in \partial\Omega\,.$$

The initial history, defined as

$$u(x, t) = u^0(x, -t) \qquad t \leq 0$$

will be considered as a known function. In order to emphasize the initial history, the stress tensor T can be written in the form

$$T(x, t) = G_\infty(x)\, \nabla u(x, t) - \int_0^t G'(x, s)[\nabla u(x, t) - \nabla u^t(x, s)]\, ds$$
$$- \int_t^\infty G'(x, s)\, \nabla u(x, t)\, ds + T_0(x, t),$$

where

$$T_0(x, t) = \int_t^\infty G'(x, s)\, \nabla u^t(x, s)\, ds = \int_0^\infty G'(x, t+s)\, \nabla u^0(x, s)\, ds\,.$$

In this way the equation (2.2) yields

$$\text{(2.4)}\qquad \ddot{u}(x, t) = \nabla\cdot\Bigg[G_\infty(x)\, \nabla u(x, t) - \int_0^t G'(x, s)[\nabla u(x, t) - \nabla u^t(x, s)]\, ds$$
$$- \int_t^\infty G'(x, s)\, \nabla u(x, t)\, ds \Bigg] + \nabla\cdot T_0(x, t) + f(x, t)$$

and we restrict our attention to the search of a solution $u(x, t)$, with initial data

$$\text{(2.5)}\qquad u(x, 0) = u^0(x, 0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) = \frac{\partial u^0}{\partial t}(x, 0) = \dot{u}_0(x), \qquad x \in \Omega\,.$$

In what follows, we denote by $\hat{f}$ the Fourier transform of f with respect to the variable t, namely

$$\hat{f}(x, \omega) = \int_{-\infty}^{\infty} f(x, t)\, e^{-i\omega t}\, dt\,.$$

Moreover, each function f, defined for $t \geqslant 0$, will be identified with its causal extension, which coincides with f for $t \geqslant 0$ and vanishes for $t < 0$. For causal functions, let f_s, f_c be respectively the sine and cosine transforms, i.e.

$$f_s(x, \omega) = \int_0^\infty f(x, t) \sin \omega t\, dt, \qquad f_c(x, \omega) = \int_0^\infty f(x, t) \cos \omega t\, dt\,.$$

The Fourier transform of the stress tensor T is

$$\widehat{T}(x, \omega) = G_\infty(x)\, \nabla \widehat{u}(x, \omega) - \int_0^\infty \int_0^t G'(x, s)[\nabla u(x, t) - \nabla u^t(x, s)]\, ds\, e^{-i\omega t}\, dt$$

$$- \int_0^\infty \int_t^\infty G'(x, s)\, ds \nabla u(x, t)\, e^{-i\omega t}\, dt + \widehat{T}_0(x, \omega)$$

and a direct check proves the identity

$$\widehat{T}(x, \omega) = G_\infty(x)\, \nabla \widehat{u}(x, \omega) - \int_0^\infty \int_t^\infty G'(x, s)\, ds \nabla u(x, t)\, e^{-i\omega t}\, dt + \widehat{T}_0(x, \omega)$$

$$- \int_0^\infty G'(x, s) \left[\int_s^\infty \nabla u(x, t)\, e^{-i\omega t}\, dt - \int_0^\infty \nabla u(x, t)\, e^{-i\omega (t+s)}\, dt \right] ds$$

$$= [H(x, \omega) + G_\infty(x)]\, \nabla \widehat{u}(x, \omega) + \widehat{T}_0(x, \omega),$$

where

$$H(x, \omega) = -\int_0^\infty G'(x, s)(1 - e^{-i\omega s})\, ds\, .$$

We recall here the thermodynamic restrictions on the relaxation function, related to the tensor H ([3]). As a consequence of the Second Law, the sine transform $G_s'(x, \omega)$ of the causal extension of $G'(t)$ is negative definite for each $\omega > 0$, so that the tensor

$$\omega \Im H(x, \omega) = -\omega G_s'(x, \omega)$$

is positive definite for any $\omega \in \mathbb{R}$ and $x \in \Omega$.

Henceforth we denote by Sym the set of symmetric tensors and by $|\cdot|$ the modulus of a tensor in Sym.

Since the system is a solid, G_∞ is uniformly positive defined, i.e. there exists a constant $g_{\infty m} > 0$ such that

$$\inf_{x \in \Omega} A \cdot G_\infty A \geqslant g_{\infty m} |A|^2,$$

for each $A \in Sym$.

As we have already pointed out in the introduction, we assume that the relaxation function G, defined by (1.1) presents an integrable singularity as $t = 0$. In particular, we require the existence of a constant $\omega_0 > 0$ such that G satisfies the conditions

$$\sup_{x \in \Omega} |\Re H(x, \omega)[-\omega G_s'(x, \omega)]^{-1}| \leqslant C_1, \tag{2.6}$$

$$\sup_{x \in \Omega} |[-\omega G_s'(x, \omega)]^{-1}| \leqslant C_2, \tag{2.7}$$

for any $\omega > \omega_0$. Such assumption allows to know the behaviour of the kernel memory for large values of ω.

An example of kernel G' which satisfies the hypotheses above mentioned is given by a series of exponential functions, namely

$$G'(t) = -\sum_{n=1}^{\infty} e^{-n^p t}, \qquad t > 0,$$

where p is a fixed parameter such that $p \in \left(\frac{1}{2}, 1\right)$. This condition ensures that G presents an integrable singularity at $t = 0$. Moreover, since

$$\widehat{G'}(\omega) = -\sum_{n=1}^{\infty} \frac{1}{n^p + i\omega},$$

the following relations can be easily verified

$$G_s'(\omega) = -\omega \sum_{n=1}^{\infty} \frac{1}{n^{2p} + \omega^2},$$

$$\Re H(\omega) = \omega^2 \sum_{n=1}^{\infty} \frac{1}{n^p(n^{2p} + \omega^2)}.$$

Therefore G satisfies (2.6) and (2.7).

The assumptions on the kernel memory allow us to give the definition of weak solutions of the problem. To this purpose, we introduce the functional space

$$\widehat{\mathcal{H}}(\mathbb{R}, \Omega) = \Big\{ \widehat{u} \in L^2(\mathbb{R}, H^1(\Omega)) : \omega \widehat{u} \in L^2(\mathbb{R}, L^2(\Omega)), \int_{-\infty}^{\infty} \int_{\Omega} -\omega G_s'(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)\, dx\, d\omega < \infty \Big\}$$

where $*$ denotes the complex conjugate.

DEFINITION 2.1. – *A function u is a weak solution of the problem* (2.4)-(2.5), *with boundary condition* (2.3) *if $\widehat{u} \in \widehat{\mathcal{H}}(\mathbb{R}, \Omega)$, $u_0 \in H^1(\Omega)$, $\dot{u}_0 \in L^2(\Omega)$, $\widehat{f} \in L^2(\mathbb{R}, L^2(\Omega))$, $\widehat{T}_0 \in L^2(\mathbb{R}, L^2(\Omega))$ and the equality*

$$(2.8) \quad \int_{-\infty}^{\infty} \int_{\Omega} [-\omega^2 \widehat{u}(x,\omega) \widehat{\phi}(x,\omega) + [H(x,\omega) + G_\infty(x)] \nabla \widehat{u}(x,\omega) \cdot \nabla \widehat{\phi}(x,\omega)]\, dx d\omega = 0$$

$$\int_{-\infty}^{\infty} \int_{\Omega} [\widehat{f}(x, \omega) \widehat{\phi}(x, \omega) - \widehat{T}_0(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)]\, dx d\omega +$$

$$\int_{-\infty}^{\infty} \int_{\Omega} [\dot{u}_0(x) + i\omega u_0(x)] \widehat{\phi}(x, \omega)\, dx d\omega - \int_{-\infty}^{\infty} \int_{\partial\Omega} \alpha(\sigma) \widehat{u}(\sigma, \omega) \widehat{\phi}(\sigma, \omega)\, d\sigma d\omega$$

holds for each $\widehat{\phi} \in \widehat{\mathcal{H}}(\mathbb{R}, \Omega)$.

From the definition of the space $\widehat{\mathcal{H}}(\mathbb{R}, \Omega)$, we can prove that all the integrals in (2.8) converge. Concerning the second term, observe that, in view of (2.6), for $|\omega| > \omega_0$ the following inequality holds

$$|H(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)| \leq |\Re H(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)| +$$
$$|\Im H(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)| \leq (C_1 + \omega_0^{-1}) |\omega G_s'(\omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)|,$$

whereas, if $|\omega| \leq \omega_0$, there exists $\delta > 0$ such that

$$|H(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)| \leq \delta |\nabla \widehat{u}(x, \omega)| |\nabla \widehat{\phi}(x, \omega)|.$$

Therefore

$$\int_{-\infty}^{\infty} \int_{\Omega} |H(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}(x, \omega)| dx d\omega < \infty .$$

The choice $\widehat{\phi}(x, \omega) = \widehat{\phi}_1(x)\, \widehat{\phi}_2(\omega)$ with $\widehat{\phi}_1 \in H^1(\Omega)$ and $\widehat{\phi}_2$ smooth with compact support, yields

$$\int_{-\infty}^{\infty} \int_{\Omega} [-\omega^2 \widehat{u}(x, \omega) \widehat{\phi}_1(x) + [H(x, \omega) + G_\infty(x)] \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{\phi}_1(x)] dx\, \widehat{\phi}_2(\omega) d\omega =$$
$$\int_{-\infty}^{\infty} \int_{\Omega} [\widehat{f}(x, \omega)\, \widehat{\phi}_1(x) - \widehat{T}_0(x, \omega) \cdot \nabla \widehat{\phi}_1(x)]\, dx\, \widehat{\phi}_2(\omega)\, d\omega +$$
$$\int_{-\infty}^{\infty} \int_{\Omega} [\dot{u}_0(x) + i\omega u_0(x)]\, \widehat{\phi}_1(x)\, dx\, \widehat{\phi}_2(\omega)\, d\omega -$$
$$\int_{-\infty}^{\infty} \int_{\partial\Omega} \alpha(\sigma)\, \widehat{u}(\sigma, \omega)\, \widehat{\phi}_1(\sigma)\, d\sigma\, \widehat{\phi}_2(\omega)\, d\omega .$$

By the arbitrariness of $\widehat{\phi}_2$ we conclude that $\widehat{u}(\cdot, \omega)$ is a weak solution of the transformed problem

$$-\omega^2 \widehat{u}(\omega) - \dot{u}_0 - i\omega u_0 = \nabla \cdot [(H(\omega) + G_\infty(x)) \nabla \widehat{u}(\omega) + \widehat{T}_0(\omega)] + \widehat{f}(\omega) \tag{2.9}$$

$$\alpha(\sigma)\, \widehat{u}(\sigma, \omega) + \widehat{T}(\sigma, \omega) \cdot n(\sigma) = 0 \qquad \sigma \in \partial\Omega \tag{2.10}$$

for almost every $\omega \in \mathbb{R}$.

3. – Existence and uniqueness.

In this section we prove the existence and uniqueness for the weak solution of the problem (2.4)-(2.5).

Since we can change the unknowns in order to have zero initial data, it is not restrictive to assume $u_0 = 0$ and $\dot{u}_0 = 0$.

We denote by $\|\cdot\|$ and $\|\cdot\|_{\partial\Omega}$ the norms in $L^2(\Omega)$ and $L^2(\partial\Omega)$ respectively and recall the following classical inequalities ([7, 9])

$$\|\widehat{u}(\omega)\|^2 \leqslant K_1 \|\nabla \widehat{u}(\omega)\|^2 + K_2 \|\widehat{u}(\omega)\|^2_{\partial\Omega}, \tag{3.1}$$

$$\|\widehat{u}(\omega)\|^2_{\partial\Omega} \leqslant \|\nabla \widehat{u}(\omega)\|^2 + K_3 \|\widehat{u}(\omega)\|^2, \tag{3.2}$$

where K_1, K_2, K_3 are positive constants depending on Ω.

PROPOSITION 3.1. – *For each* $\omega \in \mathbb{R}$, *the problem* (2.9)-(2.10) *with* $u_0 = 0$ *and* $\dot{u}_0 = 0$, *admits a unique weak solution* $\widehat{u}(\cdot, \omega) \in H^1(\Omega)$, *which satisfies the inequality*

$$\|\omega \widehat{u}(\omega)\|^2 + \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 + \|\widehat{u}(\omega)\|^2_{\partial\Omega} \leqslant \lambda[\|\widehat{f}(\omega)\|^2 + \|(1+|w|)\,\widehat{T}_0(\omega)\|^2], \tag{3.3}$$

where λ is a suitable positive constant.

PROOF. – The existence and uniqueness of the weak solution of the problem (2.9)-(2.10) can be proved like in [2], therefore we need only to verify the inequality (3.3). To this aim, observe that if $\widehat{u}(\cdot, \omega)$ is a weak solution, then it satisfies the relation

$$\int_\Omega \{-\omega^2 |\widehat{u}(x, \omega)|^2 + [H(x, \omega) + G_\infty(x)]\, \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)\}\, dx = \int_\Omega [\widehat{f}(x, \omega)\, \widehat{u}^*(x, \omega) - \widehat{T}_0(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)]\, dx - \int_{\partial\Omega} \alpha(\sigma)\, |\widehat{u}(\sigma, \omega)|^2 d\sigma\,. \tag{3.4}$$

Since

$$\lim_{\omega \to 0} \Re H(x, \omega) = 0\,,$$

there exists a constant $\omega_1 > 0$ such that if $|\omega| < \omega_1$

$$|\Re H(x, \omega)| < \frac{1}{2} g_{\infty m}.$$

Therefore, from the real part of the equation (3.4), we obtain

$$\|\omega \widehat{u}(\omega)\|^2 = \int_\Omega \Re[H(x, \omega) + G_\infty(x)]\, \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)\, dx +$$

$$\int_{\partial\Omega} \alpha(\sigma, \omega)\, |\widehat{u}(\sigma, \omega)|^2 d\sigma - \Re \int_\Omega [\widehat{f}(x, \omega)\, \widehat{u}^*(x, \omega) - \widehat{T}_0(x, \omega)\, \nabla \widehat{u}^*(x, \omega)]\, dx \geqslant$$

$$\frac{1}{2} g_{\infty m} \|\nabla \widehat{u}(\omega)\|^2 + \alpha_m \|\widehat{u}(\omega)\|^2_{\partial\Omega} - \Re \int_\Omega [\widehat{f}(x, \omega)\, \widehat{u}^*(x, \omega) - \widehat{T}_0(x, \omega)\, \nabla \widehat{u}^*(x, \omega)]\, dx\,.$$

Keeping (3.1) into account, the previous inequality yields

$$\frac{1}{2} g_{\infty m} \|\nabla \widehat{u}(\omega)\|^2 + \alpha_m \|\widehat{u}(\omega)\|^2_{\partial\Omega} \leqslant K_1 \omega^2 \|\nabla \widehat{u}(\omega)\|^2 + K_2 \omega^2 \|\widehat{u}(\omega)\|^2_{\partial\Omega} + \|\widehat{f}(\omega)\| \|\widehat{u}(\omega)\| + \|\widehat{T}_0(\omega)\| \|\nabla \widehat{u}(\omega)\| .$$

Thus, if $|\omega| < \min\left\{\omega_1, \sqrt{\frac{g_{\infty m}}{4K_1}}, \sqrt{\frac{\alpha_m}{2K_2}}\right\}$, we have

$$\frac{1}{4} g_{\infty m} \|\nabla \widehat{u}(\omega)\|^2 + \frac{\alpha_m}{2} \|\widehat{u}(\omega)\|^2_{\partial\Omega} \leqslant \|\widehat{f}(\omega)\| \|\widehat{u}(\omega)\| + \|\widehat{T}_0(\omega)\| \|\nabla \widehat{u}(\omega)\| .$$

Since

$$\lim_{\omega \to 0} \omega G_s'(\omega) = 0 ,$$

we conclude that there exist λ_1, ω_2 such that the estimate

$$\|\omega \widehat{u}(\omega)\|^2 + \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 + \|\widehat{u}(\omega)\|^2_{\partial\Omega} \leqslant \lambda_1 [\|\widehat{f}(\omega)\|^2 + \|\widehat{T}_0(\omega)\|^2] \tag{3.5}$$

holds for each $|\omega| < \omega_2$.

Let us examine the case of large values of ω. Suppose $\omega > \omega_0$ and multiply the imaginary part of (3.4) by ω, thus obtaining the inequality

$$\int_\Omega -\omega G_s'(x, \omega) \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)\, dx \leqslant \int_\Omega \big[|\omega \widehat{f}(x, \omega)\, \widehat{u}^*(x, \omega)| + |\omega \widehat{T}_0(x, \omega) \nabla \widehat{u}^*(x, \omega)| \big]\, dx \leqslant \|\omega \widehat{u}(\omega)\| \|\widehat{f}(\omega)\| + \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\| \|[-\omega^{-1} G_s'(\omega)]^{-1/2} \widehat{T}_0(\omega)\| .$$

Therefore

$$\|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 \leqslant c \|\omega \widehat{u}(\omega)\|^2 + \frac{1}{c} \|\widehat{f}(\omega)\|^2 + \|[-\omega^{-1} G_s'(\omega)]^{-1/2} \widehat{T}_0(\omega)\|^2,$$

where c is an arbitrary positive constant and the condition (2.7) yields

$$\|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 \leqslant c \|\omega \widehat{u}(\omega)\|^2 + \frac{1}{c} \|\widehat{f}(\omega)\|^2 + C_2 \|\omega \widehat{T}_0(\omega)\|^2. \tag{3.6}$$

Moreover the real part of (3.4) leads to

$$\|\omega \widehat{u}(\omega)\|^2 \leqslant \int_\Omega [\Re H(x, \omega) + G_\infty(x)] \nabla \widehat{u}(x, \omega) \cdot \nabla \widehat{u}^*(x, \omega)\, dx + \|\omega^{-1} \widehat{f}(\omega)\| \|\omega \widehat{u}(\omega)\| + \|[-\omega G_s'(\omega)]^{-1/2} \widehat{T}_0(\omega)\| \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\| + \alpha_M \|\widehat{u}(\omega)\|^2_{\partial\Omega},$$

where

$$\alpha_M = \sup_{x \in \partial\Omega} |\alpha(\sigma)|$$

and in view of the inequalities (2.7) and (3.2) we have

$$(3.7) \qquad \|\omega \hat{u}(\omega)\|^2 \leqslant$$

$$\sup_{x\in\Omega}\left(|[\Re H(x,\,\omega)+G_\infty(x)][-\omega G_s'(x,\,\omega)]^{-1}| + \alpha_M K_3\, |[-\omega G_s'(x,\,\omega)]^{-1}| + \frac{1}{2}\right)\cdot$$

$$\|[-\omega G_s'(\omega)]^{1/2}\nabla\hat{u}(\omega)\|^2 + \|\omega^{-1}\hat{f}(\omega)\|\,\|\omega\hat{u}(\omega)\| + \frac{C_2}{2}\|\hat{T}_0(\omega)\|^2 + \alpha_M\omega^{-2}\|\omega\hat{u}(\omega)\|^2.$$

The hypotheses on the kernel ensure that there exists a positive constant β such that

$$\sup_{x\in\Omega} |[\Re H(x,\,\omega) + G_\infty(x)][\omega G_s'(x,\,\omega)]^{-1}\ | + \alpha_M K_3 |\omega G_s'(x,\,\omega)|^{-1} + \frac{1}{2} < \beta\,.$$

Therefore, (3.7) yields the inequality

$$\|\omega\hat{u}(\omega)\|^2 \leqslant \beta\|[\omega G_s'(\omega)]^{1/2}\nabla\hat{u}(\omega)\|^2 + \|\omega^{-1}\hat{f}(\omega)\|\,\|\omega\hat{u}(\omega)\| +$$
$$\frac{C_2}{2}\|\hat{T}_0(\omega)\|^2 + \alpha_M\omega^{-2}\|\omega\hat{u}(\omega)\|^2$$

and, keeping the equation (3.6) into account, we obtain

$$\|\omega\hat{u}(\omega)\|^2 \leqslant \beta\left[c\|\omega\hat{u}(\omega)\|^2 + \frac{1}{c}\|\hat{f}(\omega)\|^2 + C_2\|\omega\hat{T}_0(\omega)\|^2\right] +$$
$$\frac{1}{4}\|\omega\hat{u}(\omega)\|^2 + \|\omega^{-1}\hat{f}(\omega)\|^2 + \frac{C_2}{2}\|\hat{T}_0(\omega)\|^2 + \alpha_M\omega^{-2}\|\omega\hat{u}(\omega)\|^2.$$

The choices $c = \frac{1}{4\beta}$ and $|\omega| > \max\{\omega_0,\, 2\sqrt{\alpha_M}\}$ give

$$(3.8) \qquad \frac{1}{4}\|\omega\hat{u}(\omega)\|^2 \leqslant 4\beta^2\|\hat{f}(\omega)\|^2 +$$
$$\beta C_2\|[\omega\hat{T}_0(\omega)\|^2 + \|\omega^{-1}\hat{f}(\omega)\|^2 + \frac{C_2}{2}\|\hat{T}_0(\omega)\|^2.$$

Finally, substitution in (3.6) leads to the estimate

$$(3.9) \qquad \|[-\omega G_s'(\omega)]^{1/2}\nabla\hat{u}(\omega)\|^2 \leqslant 8\beta\|\hat{f}(\omega)\|^2 + 2C_2\|\omega\hat{T}_0(\omega)\|^2 +$$
$$\frac{1}{\beta}\|\omega^{-1}\hat{f}(\omega)\|^2 + \frac{C_2}{2\beta}\|\hat{T}_0(\omega)\|^2.$$

From the inequalities (2.7) and (3.2), we deduce

$$\|\widehat{u}\|^2_{\partial\Omega} \leq C_2 \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 + K_3 \|\widehat{u}\|^2. \tag{3.10}$$

Therefore by virtue of (3.8), (3.9) and (3.10), we obtain that there exist $\lambda_2, \omega_3 > 0$ such that the inequality

$$\|\omega\widehat{u}(\omega)\|^2 + \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 + \|\widehat{u}(\omega)\|^2_{\partial\Omega} \leq \lambda_2 [\|\widehat{f}(\omega)\|^2 + \|\omega \widehat{T}_0(\omega)\|^2] \tag{3.11}$$

holds for any $|\omega| > \omega_3$.

The inequalities (3.5) and (3.11) show that (3.3) is satisfied for $|\omega| < \omega_2$ and $|\omega| > \omega_3$. In order to conclude the proof, observe that since the operator $A(\omega)$, defined by the system (2.9)-(2.10) as $A(\omega)(v) = \nabla \cdot [(H(\omega) + G_\infty) \nabla v] + \omega^2 v$, $v \in H^1(\Omega)$, is continuous with respect to the parameter ω, the inverse operator $A^{-1}(\omega)$ is continuous too (see Lemma 44.1 of [10]). Therefore the inequality (3.3) holds also for values of ω in the compact set $[\omega_2, \omega_3]$. ■

By integrating the estimate (3.3) in $(-\infty, \infty)$, we obtain the inequality

$$\int_{-\infty}^{\infty} \left\{ \|\omega\widehat{u}(\omega)\|^2 + \|[-\omega G_s'(\omega)]^{1/2} \nabla \widehat{u}(\omega)\|^2 + \|\widehat{u}(\omega)\|^2_{\partial\Omega} \right\} d\omega \leq$$

$$\lambda \int_{-\infty}^{\infty} \left\{ \|\widehat{f}(\omega)\|^2 + \|(1 + |\omega|)\, \widehat{T}_0\|^2 \right\} d\omega ,$$

where the right-hand side is finite, provided that the functions $\widehat{f}$ and $(1 + |\omega|)\, \widehat{T}_0$ belong to $L^2(\mathbb{R}, L^2(\Omega))$. In such a case $\widehat{u}(x, \omega) \in \widehat{\mathcal{H}}(\mathbb{R}, \Omega)$ and the following theorem is proved.

THEOREM 3.1. – *If* G *satisfies the assumptions* (2.6)-(2.7), $f \in L^2(\mathbb{R}^+, L^2(\Omega))$ *and* $T_0 \in H^1(\mathbb{R}^+, L^2(\Omega))$, *the evolutive problem* (2.4)-(2.5), *with zero initial data and boundary condition* (2.3), *admits a unique weak solution in the sense of Definition* 2.1.

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