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The Landau-Lifshitz Equations and the Damping Parameter.

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Sunto. – *In questo articolo si tratta l'effetto damping nei materiali ferromagnetici. In particolare, ci si occupa di determinare la sensibilità della soluzione ottenuta con il metodo LLG rispetto al parametro di damping* a*. Si analizza qui il comportamento della soluzione debole globale ad energia finita delle equazioni di Landau-Lifshitz quando il parametro di damping* a *tende verso* 0 *(underdamping) o verso* $+\infty$ *(overdamping).*

Summary. – *The present paper is particularly devoted to the damping effect in ferromagnetic materials. We are interested in determining the sensitivity of the LLG method solution to the phenomenological damping parameter* a*. We discuss the behaviour of the global weak solutions with finite energy of the Landau-Lifshitz equations when the damping parameter* a *tends either to* 0 *(underdamped case) or* $+\infty$ *(overdamped case).*

1. – Introduction.

The magnetization dynamics in thin magnetic films and microstructures is technologically relevant for, *e.g.*, magnetic recording applications at high bit densities. Over recent years the investigations of the magnetic switching behaviour of ferromagnetic elements has become more advanced due to improvements in numerical micromagnetic methods and high accuracy fabrication methods. Central problems are the calculation of the switching time and the stability of the switching process as a function of the time structure of the external field, and the detailed influence of magnetic damping. Most of the parameters required in micromagnetic equations are well characterized directly using conventional techniques [4]. However, the damping cannot be derived rigorously from basic principles, it is just added by a phenomenological term. In reality it is caused by a complex interaction of the conduction electrons in the magnet and its magnetization. For information we report the values of damping parameter of some particulate magnetic recording materials. $a = 0.051$ for CrO_2, $a = 0.066$ for γ-Fe_2O_3, $a = 0.13$ for Co-γ-Fe_2O_3, $a = 0.92$ for MP (metal particle), and $a = 0.007$ for permalloy. Damping has many effects for example micromagnetic experiments reveals that small values of a (≤ 0.1) lead to shorter switching times at small field strength. Damping has also a large effect on the dynamic character of domain walls in bubble garnet material with structure within the walls.

Let us now describe the mathematical problem. We consider $\Omega \subset \mathbb{R}^3$ a bounded and regular domain representing a ferromagnetic material. The boundary of Ω is denoted by $\partial\Omega$ and n is the unit outward normal to Ω. In $Q = \mathbb{R}^+ \times \Omega$, the magnetization field $M(t,x)$ of the ferromagnet satisfies the Landau-Lifshitz equations (LL)

$$(1) \qquad \begin{cases} \partial_t M = -\gamma M \times \mathcal{H}(M) - aM \times (M \times \mathcal{H}(M)) \text{ in } Q \\ M(0) = M_o \text{in } \Omega,\ M \times (A\nabla \cdot n) M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \end{cases}$$

coupled to the stray field equation satisfied by magnetic potential φ

$$(2) \qquad \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0\{\text{in } \mathbb{R}^+ \times \mathbb{R}^3.$$

The gyromagnetic parameter is denoted by $\gamma > 0$. The damping is given in terms of the paramater $a > 0$, which is and its implications are the main object of our attention. The effective magnetic field $\mathcal{H}$ is given by

$$(3) \qquad \mathcal{H}(M) = A\Delta M + \nabla\varphi + (\nabla_M \psi)(M)$$

where $A > 0$ is the exchange constant, $\psi : \mathbb{R}^3 \to \mathbb{R}^+$ is a regular function with bounded second derivatives representing the volume anisotropy energy. The LL equations are equivalent to the Landau-Lifshitz-Gilbert equations (LLG) given by (when $\gamma = 1$)

$$(4) \qquad \begin{cases} \partial_t M = aM \times \partial_t M - (1 + a^2) M \times \mathcal{H}(M) \text{ in } Q \\ M(0) = M_o \text{ in } \Omega,\ M \times (A\nabla \cdot n) M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega. \end{cases}$$

As usually the term $M \times \Delta M$ is understood in the weak sense $\nabla \cdot (M \times \nabla M)$. Note that the motion conserves $|M|$. The parameter a plays a crucial role in the existence theory of global weak solutions with finite energy see [16], [1]. Our aims is to understand the behaviour of the solutions of LLG equation (4) for small values of damping *i.e.*, $a \to 0$. If we set $a = 0$ in the LL equation (1) we get the so called gyromagnetic equation (GLL) that describes the undamped precession of the magnetization vector M about the effective field. When, $\mathcal{H}$ is reduced to the exchange term $A\Delta M$ equation (4) describe the symplectic flow of harmonic maps see [14], [13]. In the same time, we want to discuss the behaviour of the global solutions when $a \to \infty$. The precession is negligible compared with the damping parameter. Even if this question is academic, this behaviour is not clear as we shall show. We introduce a new time scaling for the solutions of LL equation (1) changing its dependency with respect to a. For this new equation the behaviour when the damping parameter is large corresponds to the study of the initial layer of the original solutions. The system in this case is said to be overdamped. The limit equation involves the damping equation (11) known as the heat flow for harmonic maps see [3], [2], [5], [6] for example.

Let $M_o \in \mathbb{H}^1(\Omega)$ satisfying the saturation condition $|M_o(x)|^2 = 1$ a.e in Ω and φ_o be the solution of the equation (2) associated with M_o. Let $\mathcal{E}_o$ be the initial energy

$$\mathcal{E}(M_o) = A|\nabla M_o|^2_{\mathbb{L}^2(\Omega)} + |\nabla\varphi_o|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega \psi(M_o)\, dx \tag{5}$$

It is easy to see that $\int_{\mathbb{R}^3} |\nabla\varphi_o|^2\, dx = -\int_\Omega M_o \cdot \nabla\varphi_o\, dx$ and then $0 \leq \mathcal{E}(M_o) < \infty$. Following Visintin [16] and Alouges-Soyeur [1], let (M^a, φ^a) be a global weak solution of LLG equation (4), with finite energy, associated with the initial data M_o. That is $M^a \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$, $\partial_t M^a \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$, $\nabla\varphi^a \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))$, M^a satisfies the saturation condition

$$|M^a(t,x)|^2 = 1 \text{ a.e} \tag{6}$$

and for all $t \geq 0$ the energy inequality

$$\mathcal{E}(M^a(t)) + 2\kappa(a)\int_0^t |\partial_t M^a(s)|^2_{\mathbb{L}^2(\Omega)}\, ds \leq \mathcal{E}(M_o). \tag{7}$$

The energy of the system at the time t is defined by (see for example [8])

$$\mathcal{E}(M^a(t)) = A|\nabla M^a(t)|^2_{\mathbb{L}^2(\Omega)} + |\nabla\varphi^a(t)|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega \psi(M^a(t))\, dx \tag{8}$$

and the constant $\kappa(a)$ is given by

$$\kappa(a) = \frac{a}{1+a^2}. \tag{9}$$

The solutions (M^a, φ^a) satisfy the LLG equation (4) and stray field equation in the sense of distributions. The coefficient $\kappa(a)$ gives the strenght of the decay of the energy. One observes that $\kappa(a) \to 0$ for either $a \to 0$ or $a \to +\infty$. We have in fact, $\frac{a}{2} \leq \kappa(a) \leq a$ for $a \leq 1$, $\frac{1}{2a} \leq \kappa(a) \leq \frac{1}{a}$ for $a > 1$. The loss of the bound of $\partial_t M^a$ may induce a lack of compactness of the sequence $(M^a)_a$ when either $a \to 0$ or $a \to +\infty$. The following uniform estimates hold for global weak solutions

LEMMA 1. – *There exists $C > 0$ which is independent of a such that*

$$\begin{cases} |M^a(t,x)|^2 = 1 \text{ a.e} \\ |\nabla M^a|_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))} + |\nabla\varphi^a|_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))} \leq C \\ \sqrt{\kappa(a)}|\partial_t M^a|_{L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))} \leq C. \end{cases} \tag{10}$$

The rest of paper is organized as follows. Section 2 is devoted to overdamped behaviour that is the damping equation. We give convergence results when $a \to +\infty$ and identify the limiting problem. The underdamped case which correspond to $a \to 0$ is studied in Section 3. We present in Section 4 our next results. We give direct proofs for the global existence, both for DLL equation (11) and for a particular case of GLL equation (see Section 3) corresponding to the case where $\mathcal{H} = \nabla\varphi$ (see [11] for the full LLG equation in that case). The first result is established via an approximated problem of the semilinear heat equation and the second one via an iterative scheme. We conclude in Section 5 by some remarks and perspectives.

2. – The overdamped case.

We discuss in this section the behaviour of the global weak solutions of the LLG equations (4) when $a \to +\infty$. We call damping equation (DLL) the following equation

$$\begin{cases} \partial_t M = -aM \times (M \times \mathcal{H}(M)) \text{ in } Q \\ M(0) = M_o \text{ in } \Omega,\ M \times (A\nabla \cdot n)M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{11}$$

It is important to precise the sense of the right hand side of (11) when M is a weak solution of the problem that is $M \in L^\infty(\mathbb{R}^+; H^1(\Omega))$, $\partial_t M \in L^2(\mathbb{R}^+; L^2(\Omega))$ and $|M(t,x)|^2 = 1$. We have then $M \times (M \times \Delta M) \in L^2(\mathbb{R}^+; L^2(\Omega))$ since $\partial_t M \in L^2(\mathbb{R}^+; L^2(\Omega))$. Moreover we may write $M \times \Delta M = \nabla \cdot (M \times \nabla M)$ then for all test function ϕ we have

$$\int_Q M \times (M \times \Delta M) \cdot \phi \, dx\, dt = \int_Q M \times \nabla M \cdot \nabla(M \times \phi)\, dx\, dt.$$

Hence, the weak form of (11) is $M \times \partial_t M = aM \times \mathcal{H}(M)$.

Let $(a_n)_n$ be a sequence such that $a_n \to +\infty$ when $n \to +\infty$. We denote by (M^n, φ^n) the solutions of LLG equations (4) associated with a_n and the initial data M_o. Using lemma 1 and LLG equations (4) it is easy to obtain the convergence result

$$M^n \times \mathcal{H}(M^n) \to 0 \text{ strongly in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)). \tag{12}$$

Since, for a subsequence still denoted (M^n, φ^n) we have that $M^n \rightharpoonup M$ weakly-$\star$ in $L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ but not strongly in $L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{L}^2(\Omega))$, then $M^n \times \nabla M^n \rightharpoonup M \times \nabla M + \xi$ and $M^n \times \nabla\varphi^n \rightharpoonup M \times \nabla\varphi + \mu$ weakly-$\star$ in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ with $\xi \neq 0$ and $\mu \neq 0$ in general. It is not clear to show that $\xi = 0$ and $\mu = 0$ or not and in this case how to characterize ξ, μ in term of M.

We shall introduce a usefull time scaling of the solutions to obtain the strong convergence of the sequence of solutions. Let us consider LLG equations (4) satisfied by (M^a, φ^a). We set

$$\beta(a) = \nu\kappa(a) \tag{13}$$

where $\nu > 0$ is a fixed constant. We introduce the couple $(m^a(t), \phi^a(t))$ by setting for all $t \geq 0$ and a.e $x \in \Omega$

$$m^a(t,x) = M^a(\beta(a)t, x),\ \ \phi^a(t,x) = \varphi^a(\beta(a)t, x). \tag{14}$$

We easily verify that (m^a, ϕ^a) satisfies the new LLG equation labelled (NLLG)

$$\begin{cases} \partial_t m^a = a(m^a \times \partial_t m^a - \nu m^a \times \mathcal{H}(m^a)) \\ m^a(0) = M_o \text{ in } \Omega,\ m^a \times (A\nabla \cdot n)m^a = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{15}$$

coupled to the stray field equation (2) satisfied by ϕ^a. The solutions (m^a, ϕ^a) is such that the saturation condition (6) is satisfied by m^a and the following energy inequality holds

$$\mathcal{E}(m^a(t)) + 2\nu \int_0^t |\partial_t m^a(s)|^2_{\mathbb{L}^2(\Omega)}\, ds \leq \mathcal{E}(M_o). \tag{16}$$

The LL equation associated to the NLLG equation (15) takes the form

$$\partial_t m^a = \nu\Big(-\frac{a}{1+a^2} m^a \times \mathcal{H}(m^a) - \frac{a^2}{1+a^2} m^a \times (m^a \times \mathcal{H}(m^a))\Big). \tag{17}$$

As for lemma 1 we have the bounds

LEMMA 2. – *There exists $C > 0$ which is independent of a such that*

$$\begin{cases} |\nabla m^a|_{L^\infty(\mathbb{R}^+;\, \mathbb{L}^2(\Omega))} + |\nabla \phi^a|_{L^\infty(\mathbb{R}^+;\, \mathbb{L}^2(\mathbb{R}^3))} \leq C \\ |\partial_t m^a|_{L^2(\mathbb{R}^+;\, \mathbb{L}^2(\Omega))} \leq C. \end{cases} \tag{18}$$

We denote by (m^n, ϕ^n) the solutions associated with a_n. Lemma 2 implies the following convergences

LEMMA 3. – *There exists a subsequence still denoted (m^n, ϕ^n) such that it holds that*

$$\begin{cases} m^n \rightharpoonup m \text{ weakly}-\star \text{ in } L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \\ \partial_t m^n \rightharpoonup \partial_t m \text{ weakly in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \\ m^n \to m \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^+;\, \mathbb{L}^2(\Omega)) \\ \nabla \phi^n \rightharpoonup \nabla \phi \text{ weakly}-\star \text{ in } L^\infty(\mathbb{R}^+;\, \mathbb{L}^2(\mathbb{R}^3)). \end{cases} \tag{19}$$

Moreover, using once more lemma 2, we get the result

LEMMA 4. – *The sequence* (m^n, ϕ^n) *satisfies the strong convergence*

$$m^n \times \partial_t m^n - \nu m^n \times \mathcal{H}(m^n) \to 0 \ \text{ strongly in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \tag{20}$$

Now, we are able to pass to the limit in the weak formulation of the NLLG equation (15) associated with a_n. Let g be a regular test function defined in $\mathbb{R}^+ \times \Omega$. Then m^n satisfies the weak formulation

$$\begin{cases} \dfrac{1}{a_n}\Big(-\displaystyle\int_Q m^n \cdot \partial_t g \; dxdt + \int_\Omega M_o \cdot g(0) \; dx\Big) = \\ \displaystyle\int_Q m^n \times \partial_t m^n \cdot g \; dxdt + A\int_Q m^n \times \nabla m^n \cdot \nabla g \; dxdt - \\ \displaystyle\int_Q m^n \times \nabla \phi^n \cdot g \; dxdt - \int_Q m^n \times (\nabla_M \psi)(m^n) \cdot g \; dxdt \end{cases} \tag{21}$$

Passing to the limit in the weak formulation by using lemma 3 and lemma 4 it follows that m satisfies the equation

$$\begin{cases} \displaystyle\int_Q m \times \partial_t m \cdot g \; dxdt + A\int_Q m \times \nabla m \cdot \nabla g \; dxdt - \\ \displaystyle\int_Q m \times \nabla \phi \cdot g \; dxdt - \int_Q m \times (\nabla_M \psi)(m) \cdot g \; dxdt = 0 \end{cases} \tag{22}$$

where ϕ satisfies, in the sense of distributions, the stray field equation

$$\nabla \cdot (\nabla \phi + \chi(\Omega)\, m) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3. \tag{23}$$

It remains to precise the sense of the initial data verified by m. Since we have $m^n \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ and $\partial_t m^n \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ then m satisfies the same properties and $m \in H^1(0, T; \mathbb{L}^2(\Omega))$ for all $T > 0$. Hence $m(0)$ is well defined in $\mathbb{L}^2(\Omega)$. Using the inequality

$$|m^n(0) - m(0)|^2_{\mathbb{L}^2(\Omega)} \leq 2\int_0^t |m^n(s) - m(s)|_{\mathbb{L}^2(\Omega)} |\partial_t m^n(s) - \partial_t m(s)|_{\mathbb{L}^2(\Omega)} \; ds + |m^n(t) - m(t)|^2_{\mathbb{L}^2(\Omega)} \tag{24}$$

we deduce that

$$|m^n(0) - m(0)|^2_{\mathbb{L}^2(\Omega)} \leq cT|m^n - m|_{L^2(0,T;\mathbb{L}^2(\Omega))} + |m^n - m|^2_{L^2(0,T;\mathbb{L}^2(\Omega))}.$$

Finally, since we have $m^n(0) = M_o$ and m^n converges strongly in $L^2(0, T; \mathbb{L}^2(\Omega))$ to m, we get that $m(0) = M_o$ in $\mathbb{L}^2(\Omega)$. We proved the following result

THEOREM 1. – *Let* (m, ϕ) *be the limit of a subsequence of* (m^n, ϕ^n). *Then* (m, ϕ) *is a global weak solution of the damping equation*

$$\begin{cases} m \times \partial_t m = \nu m \times \mathcal{H}(m) \text{ in } \mathbb{R}^+ \times \Omega \\ m(0) = M_o \text{ in } \Omega, \; m \times (A\nabla \cdot n) m = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \\ \nabla \cdot (\nabla\phi + \chi(\Omega) m) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3. \end{cases} \tag{25}$$

Moreover m *satisfies the saturation condition* (6) *and for all* $t \geq 0$, *the energy estimate*

$$\mathcal{E}(m(t)) + 2\nu \int_0^t |\partial_t m(s)|^2_{\mathbb{L}^2(\Omega)} \, ds \leq \mathcal{E}(M_o). \tag{26}$$

PROOF. – The strong convergence of the subsequence m^n implies the saturation condition for m. To prove the energy inequality we proceed as follows. We consider the energy estimate (16) satisfied by m^n. We multiply the inequality by $\theta(t)^2$ where θ is a regular function, and integrate over $(0, T)$. Using the convexity of the L^2 norm and the weak convergences we get

$$A|\nabla(m\theta)|^2_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq \liminf A|\nabla(m^n\theta)|^2_{L^2(0,T;\mathbb{L}^2(\Omega))},$$

and

$$|\nabla(\phi\theta)|^2_{L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq \liminf |\nabla(\phi^n\theta)|^2_{L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))}.$$

Next the strong convergence of m^n gives

$$|\psi(m)\theta^2|^2_{L^1(0,T;\mathbb{L}^1(\Omega))} = \lim |\psi(m^n)\theta^2|^2_{L^1(0,T;\mathbb{L}^1(\Omega))}.$$

Consider the damping term we have $\int_\Delta |\theta(t)\partial_t m^n(s)|^2_{\mathbb{L}^2(\Omega)} \, dsdt$ where $\Delta = \{(t, s), 0 < t < T, 0 < s < t\}$. We conclude by using the weak convergence in $L^2(\Delta; \mathbb{L}^2(\Omega))$ of the sequence $\theta(s)\partial_t m^n(t)$. The proof of the theorem is then complete.

REMARK 1. – *The term* $m \times \mathcal{H}(m)$ *is understood in its weak form* $\nabla \cdot (m \times \nabla m) + m \times \nabla\phi + m \times (\nabla_M \psi)(m)$. *From equation* (25) *it follows that* $m \times \mathcal{H}(m) \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *and then* $m \times (m \times \mathcal{H}(m))$ *is well defined in the same space. Next,* $m \times \partial_t m \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *and then* $m \times (m \times \partial_t m)$ *belongs to the same space. Finally since we have* $|m(t, x)|^2 = 1$ *then* $\partial_t m = -m \times (m \times \partial_t m)$. *Consequently from DLL equation* (25), m *satisfies in* $L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *the equation*

$$\partial_t m = -\nu m \times (m \times \mathcal{H}(m)). \tag{27}$$

REMARK 2. – *We conclude that at high damping the magnetization rotates more or less directly towards the effective field direction without any precession.*

3. – The underdamped case.

We discuss in this section the behaviour of the solutions of the LLG equations (4) when $a \to 0$. We call gyromagnetic equation (GLL) the LL equations (1) where we set $a = 0$.

Let $(a_n)_n$ be a sequence of $\mathbb{R}^+$ such that $a_n \to 0$ when $n \to +\infty$. We denote by (M^n, φ^n) a global weak solution of the LLG equations (4) satisfying the energy inequality (7). We have the estimates

LEMMA 5. – *There exists $C > 0$, independent of n such that the sequence (M^n, φ^n) satisfies the estimates*

$$
\begin{cases} |M^n(t,x)|^2 = 1 \text{ a.e} \\ |\nabla M^n|_{L^\infty(\mathbb{R}^+;\,\mathbb{L}^2(\Omega))} + |\nabla \varphi^n|_{L^\infty(\mathbb{R}^+;\,\mathbb{L}^2(\mathbb{R}^3))} \le C \\ \sqrt{a_n}|\partial_t M^n|_{L^2(\mathbb{R}^+;\,\mathbb{L}^2(\Omega))} \le C. \end{cases} \tag{28}
$$

We have also

LEMMA 6. – *The sequence (M^n) is compact in $L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{L}^2(\Omega))$.*

PROOF. – From lemma 5, the sequence (M^n) is uniformly bounded in $L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ and $a_n \partial_t M^n \to 0$ strongly in $L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$. Then $M^n \times a_n \partial_t M^n \to 0$ strongly in $L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$. Moreover, $M^n \times \mathcal{H}(M^n)$ is uniformly bounded in $L^\infty(\mathbb{R}^+; \mathbb{H}^{-1}(\Omega))$. Here, we used the weak form $M^n \times \Delta M^n = \nabla \cdot (M^n \times \nabla M^n)$. Hence, using the LLG equations (4), $\partial_t M^n = M^n \times a_n \partial_t M^n + a_n^2 M^n \times \mathcal{H}(M^n) + M^n \times \mathcal{H}(M^n)$ we deduce that $\partial_t M^n$ is uniformly bounded in $\sqrt{a_n} L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) + a_n^2 L^\infty(\mathbb{R}^+; \mathbb{H}^{-1}(\Omega)) + L^\infty(\mathbb{R}^+; \mathbb{H}^{-1}(\Omega))$. Consequently, for $T > 0$ fixed, we deduce that $\partial_t M^n$ is uniformly bounded in $L^2(0, T; \mathbb{H}^{-1}(\Omega))$. Using the compactness of the embedding of $\mathbb{H}^1(\Omega)$ into $\mathbb{L}^2(\Omega)$ and Aubin's compactness lemma, it follows that the sequence (M^n) is a compact sequence in $L^2(0, T; \mathbb{L}^2(\Omega))$.

Lemmas 5 and 6 imply the following convergence results

LEMMA 7. – *There exists a subsequence still denoted (M^n, φ^n) such that*

$$
\begin{cases} M^n \rightharpoonup M \quad \text{weakly} - \star \ \text{in } L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \\ \partial_t M^n \rightharpoonup \partial_t M \ \text{weakly in } L^2(\mathbb{R}^+; \mathbb{H}^{-1}(\Omega)) \\ a_n \partial_t M^n \to 0 \ \text{strongly in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \\ M^n \to M \ \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \\ \nabla \varphi^n \rightharpoonup \nabla \varphi \ \text{weakly} - \star \ \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)). \end{cases} \tag{29}
$$

Moreover M satisfies the saturation condition (6).

Now, we are able to pass to the limit in the weak formulation of the LLG equations (4)

$$
(30)\quad \begin{cases} -\int_Q M^n \cdot \partial_t g \, dx\, dt + \int_\Omega M_o \cdot g(0)\, dx = \sqrt{a_n}\int_Q M^n \times \sqrt{a_n}\partial_t M^n \cdot g\, dx\, dt \\ +(1+a_n^2)\Big\{ \int_Q M^n \times \nabla M^n \cdot \nabla g\, dx\, dt - \int_Q M^n \times \nabla \varphi^n \cdot g\, dx\, dt \\ -\int_Q M^n \times \nabla_M \psi(M^n) \cdot g\, dx\, dt \Big\}. \end{cases}
$$

We obtain the result

THEOREM 2. – *Let (M, φ) be the limit of a subsequence of (M^n, φ^n). Then, (M, φ) is a global weak solution of the gyromagnetic equations (GLL)*

$$
(31)\quad \begin{cases} \partial_t M = -M \times \mathcal{H}(M) \ in \ \mathbb{R}^+ \times \Omega \\ M(0) = M_o \ in \ \Omega, \ M \times (A\nabla \cdot n)M = 0 \ on \ \partial\Omega \\ \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \ in \ \mathbb{R}^+ \times \mathbb{R}^3. \end{cases}
$$

Moreover, M satisfies the saturation condition (6) *and, for all $t \geq 0$, the energy inequality*

$$
(32)\quad \mathcal{E}(M(t)) \leq \mathcal{E}(M_o).
$$

PROOF. – Passing to the limit in the weak formulation (30) we deduce that M satisfies the equation

$$
(33)\quad \begin{cases} -\int_Q M \cdot \partial_t g\, dx\, dt + \int_\Omega M_o \cdot g(0)\, dx = \\ \int_Q M \times \nabla M \cdot \nabla g\, dx\, dt - \int_Q M \times \nabla\varphi \cdot g\, dx\, dt \\ -\int_Q M \times \nabla_M \psi(M) \cdot g\, dx\, dt \end{cases}
$$

where φ satisfies the stray field equation (2) associated to M. Since $M \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ and $\partial_t M \in L^2(\mathbb{R}^+; \mathbb{H}^{-1}(\Omega))$ then $M \in C^0([0,T]; \mathbb{L}^2(\Omega))$ for all $T > 0$ fixed. It follows that $M(0)$ is well defined in $\mathbb{L}^2(\Omega)$. Integrating by part in the equation satisfied by M we deduce that M satisfies the gyromagnetic equation (31) with the initial and boundary condition. It remains to prove the energy inequality. The sequence M^n satisfies (7) and then we have $\mathcal{E}(M^n(t)) \leq$

$\mathcal{E}(M_o)$ for all $t \geq 0$. Using the convexity of the $L^2(0,T;\mathbb{L}^2(\Omega))$-norm and the strong convergence of M^n, we get the wished result. This complete the proof of the theorem.

REMARK 3. – *From equation* (31) *we can say that for lower values of* α, *the magnetic damping vanishes and the precession continues for ever. In other words, the magnetization precesses several times around the effective field direction before it reach equilibrium.*

4. – The DLL and GLL equations.

We adress in this section the question of a direct proof of global weak solutions of the GLL equation and the DLL equations. Let us first consider the DLL equations

$$(34)\qquad \begin{cases} \partial_t M = -\nu M \times (M \times \mathcal{H}(M)) \text{ in } \mathbb{R}^+ \times \Omega \\ M(0) = M_o \text{ in } \Omega,\ M \times (A\nabla \cdot n)M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \\ \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \end{cases}$$

where $M_o \in \mathbb{H}^1(\Omega)$, $|M_o(x)|^2 = 1$ a.e and for simplicity $\mathcal{H}(M) = A\Delta M + \nabla\varphi$. Since the weak solutions satisfy the saturation condition $|M(t,x)|^2 = 1$ a.e then M satisfies formally

$$M \times \partial_t M = -\nu M \times (M \cdot \mathcal{H}(M)M - \mathcal{H}(M)) = \nu M \times \mathcal{H}(M)$$

that is

$$M \times \partial_t M = \nu M \times \mathcal{H}(M).$$

We are interested in the global weak solution of the following weak form of the DLL equations

$$(35)\qquad \begin{cases} M \times \partial_t M = \nu M \times \mathcal{H}(M) \text{ in } \mathbb{R}^+ \times \Omega \\ M(0) = M_o \text{ in } \Omega,\ M \times (A\nabla \cdot n)M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \\ \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \end{cases}$$

We will say that M is a global weak solution of the DLL equation (35) if M satisfies the saturation condition, $M \in L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))$, $\partial_t M \in L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))$, $\nabla\varphi \in L^\infty(\mathbb{R}^+;\mathbb{L}^2(\mathbb{R}^3))$, has a finite energy and satisfies the weak formulation

$$(36)\qquad \int_Q \partial_t M \cdot M \times \phi \, dx\, dt + \nu A \int_Q \nabla M \cdot M \times \nabla\phi \, dx\, dt = \nu \int_Q \nabla\varphi \cdot M \times \phi \, dx\, dt,$$

for all test functions ϕ.

In this proof we shall employ the method proposed by Hamdache [7]. We introduce the approximated solutions U^ε of the semilinear heat equation.

$$\begin{cases} \partial_t U^\varepsilon - \nu A \Delta U^\varepsilon + \lambda U^\varepsilon = \lambda U^\varepsilon + \nu \nabla \varphi^\varepsilon - \dfrac{1}{\varepsilon} \Gamma(U^\varepsilon) \\ U^\varepsilon(0) = M_o \text{ in } \Omega, \ (A\nabla \cdot n) U^\varepsilon = 0 \text{ in } \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{37}$$

where $\lambda > 0$ is fixed, $\Gamma(U) = \nabla(\gamma(|U|))$ and $\gamma(U) = |(1 + |U|^2)^{1/2} - 2^{1/2}|^2$. We have $\Gamma(U) \times U = 0$. We set $R^\varepsilon(U) = \lambda U + \nu \nabla \varphi - \frac{1}{\varepsilon} \Gamma(|U|)$ for $U \in \mathbb{L}^2(\Omega)$ and φ is the solution of the stray field equation associated with U. Thus, there exists $C_\varepsilon > 0$ such that for all $U, V \in \mathbb{L}^2(\Omega)$ we have

$$|R^\varepsilon(U) - R^\varepsilon(V)|_{\mathbb{L}^2(\Omega)} \leq C_\varepsilon |U - V|_{\mathbb{L}^2(\Omega)}. \tag{38}$$

Consequently, R^ε is a Lipchitz perturbation in $\mathbb{L}^2(\Omega)$ of the operator $(-A\Delta + \lambda)$ with Neumann boundary condition then, by the classical existence theory of solutions to the semilinear heat equation we get the result see Mizohata [10], Pazy [12] for example

LEMMA 8. – *Let $M_o \in \mathbb{H}^1(\Omega)$ and $T > 0$ be fixed. Then there exists a global weak solution U^ε of* (37) *such that $U^\varepsilon \in C^0([0,T]; \mathbb{H}^1(\Omega))$. Moreover if $M_o \in \mathbb{H}^2(\Omega)$ with $(A\nabla \cdot n) M_o = 0$ on $\mathbb{R}^+ \times \partial\Omega$ that is M_o belongs to the domain of the Laplacian operator with the Neumann boundary condition and $|M_o(x)|^2 = 1$ a.e then $U^\varepsilon \in C^0([0,T]; \mathbb{H}^2(\Omega)) \cap C^1(]0,T[; \mathbb{H}^1(\Omega))$ and satisfies for all $t \in [0,T]$, the energy inequality*

$$\nu \mathcal{E}(U^\varepsilon(t)) + \frac{1}{\varepsilon} |\gamma(|U^\varepsilon(t)|)|_{\mathbb{L}^1(\Omega)} + 2 \int_0^t |\partial_t U^\varepsilon(s)|^2_{\mathbb{L}^2(\Omega)} \, ds \leq \nu \mathcal{E}(M_o) \tag{39}$$

where $\mathcal{E}(U^\varepsilon(t)) = A |\nabla U^\varepsilon(t)|^2_{\mathbb{L}^2(\Omega)} + |\nabla \varphi^\varepsilon(t)|^2_{\mathbb{L}^2(\mathbb{R}^3)}$ for $t \geq 0$. Notice that $\gamma(|M_o|) = 0$ and $|U^\varepsilon(t)|_{\mathbb{L}^2(\Omega)} \leq C_T$ for $t \in [0,T]$ where C_T is independent of ε.

Let (ε_n) be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. We denote by U^n and φ^n the solution associated to the sequence ε_n. We deduce the convergence results

LEMMA 9. – *Assume M_o as in lemma 8. Then there exists a subsequence still denoted (U^n, φ^n) such that*

$$\begin{cases} U^n \rightharpoonup M \textit{ weakly}-\star \quad \textit{in } L^\infty(0,T; \mathbb{H}^1(\Omega)), \\ \partial_t U^n \rightharpoonup \partial_t M \textit{ weakly in } L^2(0,T; \mathbb{L}^2(\Omega)), \\ U^n \to M \textit{ strongly in } L^2(0,T; \mathbb{L}^2(\Omega)), \\ \gamma(|U^n|) \to 0 \textit{ strongly in } L^\infty(0,T; \mathbb{L}^1(\Omega)), \\ |M(t,x)|^2 = 1 \textit{ a.e}, \ \nabla \varphi^n \rightharpoonup \nabla \varphi \textit{ weakly}-\star \quad \textit{in } L^\infty(0,T; \mathbb{L}^2(\mathbb{R}^3)). \end{cases} \tag{40}$$

We are able to prove the global existence of weak solutions to DLL equations (35). We have the result

THEOREM 3. – *Let $M_o \in \mathbb{H}^1(\Omega)$ satisfying the saturation condition $|M_o(x)|^2 = 1$ a.e. Then there exists a global weak solution (M, φ) to the DLL equation* (35) *with finite energy.*

PROOF. – First, assume that M_o is more regular in order that the energy inequality satisfied by U^n holds. Then by lemma 9, let (M, φ) be the limit of a subsequence of (U^n, φ^n). Let ϕ be a regular test function, then multiplying (37) by $U^n \times \phi$ and integrate by parts we get the weak formulation (observes that $\Gamma^n(U^n) \cdot U^n \times \phi = 0$)

$$\int_Q \partial_t U^n \cdot U^n \times \phi \, dx \, dt + vA \int_Q \nabla U^n \cdot U^n \times \nabla \phi \, dx \, dt = v \int_Q \nabla \varphi^n \cdot U^n \times \phi \, dx \, dt \tag{41}$$

we pass to the limit by using the convergences stated in lemma 9 to get

$$\int_Q \partial_t M \cdot M \times \phi \, dx \, dt + vA \int_Q \nabla M \cdot M \times \nabla \phi \, dx \, dt = v \int_Q \nabla \varphi \cdot M \times \phi \, dx \, dt \tag{42}$$

which shows that M satisfies in the sense of distributions the equation

$$M \times \partial_t M = v(\nabla \cdot (M \times A \nabla M) + M \times \nabla \varphi)$$

and the Neumann boundary condition

$$M \times (A \nabla \cdot n) M = 0.$$

Moreover M satisfies the saturation condition $|M(t,x)|^2 = 1$ a.e and we have $M \in L^\infty(0,T; \mathbb{H}^1(\Omega))$, $\nabla \varphi \in L^\infty(0,T; \mathbb{L}^2(\mathbb{R}^3))$ and $\partial_t M \in L^2(0,T; \mathbb{L}^2(\Omega))$. The initial condition is obtained as follows. We have

$$|U^n(0) - M(0)|^2_{\mathbb{L}^2(\Omega)} \leq 2 \int_0^t |U^n(s) - M(s)|_{\mathbb{L}^2(\Omega)} |\partial_t U^n(s) - \partial_t M(s)|_{\mathbb{L}^2(\Omega)} \, ds + |U^n(t) - M(t)|^2_{\mathbb{L}^2(\Omega)} \tag{43}$$

which implies the estimate

$$|U^n(0) - M(0)|^2_{\mathbb{L}^2(\Omega)} \leq 2TC |U^n - M|_{L^2(0,T;\mathbb{L}^2(\Omega))} + |U^n - M|^2_{L^2(0,T;\mathbb{L}^2(\Omega))}.$$

Using the strong convergence of U^n to M in $L^2(0,T; \mathbb{L}^2(\Omega))$ and $U^n(0) = M_o$ we deduce that M satisfies the initial condition $M(0) = M_o$ in $\mathbb{L}^2(\Omega)$. Hence the theorem is proved for regular initial data. Now let M_o satisfying the hypothesis of the theorem. Let M_o^p be a regular sequence such that $M_o^p \to M_o$ in $\mathbb{H}^1(\Omega)$. For

ε fixed, let U^p be the solution of the problem (37) satisfying the energy inequality (39). Since operator R^ε is Lipchitz then $(U^p)_p$ is a Cauchy sequence in the space defined by the energy inequality (39). Setting U^ε the strong limit of U^p in that space we deduce that U^ε is a global solution of the approximated problem associated to the initial data M_o satisfying the energy inequality (39). The remainder of the proof follows.

Now we discuss the global existence of weak solutions to GLL equations (31) in a particular case; see Moser [11] for the LL equation (1) for example. For weak solutions satisfying the saturation condition we have $\partial_t M = -M \times (M \times \partial_t M)$ and the equations may be written as $M \times (M \times \partial_t M - \mathcal{H}(M)) = 0$. This equation is degenerated with respect to $\partial_t M$. We shall discuss only the simplest case where the exchange constant $A = 0$ given by

$$\begin{cases} \partial_t M = -M \times \nabla\varphi \text{ in } \mathbb{R}^+ \times \Omega \\ M(0) = M_o \text{ in } \Omega \\ \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \end{cases} \tag{44}$$

where $M_o \in \mathbb{L}^2(\Omega)$ with $|M_o(x)|^2 = 1$ a.e. For a discussion on the case $A \neq 0$ without any coupling with the stray field equation, we refer to [15]. The map $M \in \mathbb{L}^2(\Omega) \to \nabla\varphi \in \mathbb{L}^2(\Omega)$ is linear and continuous and satisfies the estimate $|\nabla\varphi|_{\mathbb{L}^2(\mathbb{R}^3)} \leq |M|_{\mathbb{L}^2(\Omega)}$.

To solve this equation we introduce the following iterative scheme. Let $M^0 \in W^{1,\infty}(0,T;L^2(\Omega))$ such that $|M^0(t,x)|^2 = 1$ a.e and consider the sequence $(M^n)_n$ defined for $n \geq 0$ by

$$\begin{cases} \partial_t M^{n+1} = -M^{n+1} \times \nabla\varphi^n \text{ in } \mathbb{R}^+ \times \Omega \\ M^{n+1}(0) = M_o \text{ in } \Omega \\ \nabla \cdot (\nabla\varphi^n + \chi(\Omega)M^n) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3. \end{cases} \tag{45}$$

For given $M^n \in W^{1,\infty}(0,T;L^2(\Omega))$ M^{n+1} satisfies a linear system of differential equations. Hence M^{n+1} exists in $W^{1,\infty}(0,T;L^2(\Omega))$ and satisfies the saturation condition $|M^{n+1}(t,x)|^2 = 1$ for a.e. Moreover we have

$$\partial_t(M^{n+1} - M^n) = -(M^{n+1} - M^n) \times \nabla\varphi^n - M^n \times (\nabla\varphi^n - \nabla\varphi^{n-1})$$

which implies the estimate

$$\frac{1}{2}\frac{d}{dt}|M^{n+1}(t) - M^n(t)|^2_{\mathbb{L}^2(\Omega)} \leq |M^n(t) - M^{n-1}(t)|_{\mathbb{L}^2(\Omega)}|M^{n+1}(t) - M^n(t)|_{\mathbb{L}^2(\Omega)} \tag{46}$$

since we have $|M^n(t,x)|^2 = 1$ a.e and $|\nabla\varphi^n(t) - \nabla\varphi^{n-1}(t)|_{\mathbb{L}^2(\Omega)} \leq |M^n(t) - M^{n-1}(t)|_{\mathbb{L}^2(\Omega)}$. Setting $V^n(t) = \exp(-t)|M^n(t) - M^{n-1}(t)|_{\mathbb{L}^2(\Omega)}$ we get $V^{n+1}(t) \leq$

$\int_0^t V^n(s)\, ds$ and finally, $V^{n+1}(t) \leq \frac{t^n}{n!}|V^1|_{L^\infty(0,T)}$. Using this estimate we deduce that for all n and p we have

$$|V^{n+p}(t) - V^n(t)| \leq |V^1|_{L^\infty(0,T)} \frac{T^n}{n!} \tag{47}$$

where we used $\frac{n!}{(n+p-1)!} \leq \frac{1}{(n+1)^p}$ if $p \geq 2$ and $\sum_{k=1}^{p-1} \frac{T^k}{(n+1)^k} \leq \frac{1}{1-\frac{T}{n+1}} \leq 1$ for n large. Finally we obtain that $(V^n)_n$ is a Cauchy sequence in $L^2(0,T)$ and then $M^n \to M$ strongly in $L^2(0,T;\mathbb{L}^2(\Omega))$. We proved the result

THEOREM 4. – *Let $M_o \in \mathbb{L}^2(\Omega)$ be such that $|M_o(x)|^2 = 1$ a.e. Then for all $T > 0$ fixed there exists a global weak solution $M \in W^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$ of the problem* (44) *satisfying the saturation condition.*

REMARK 4. – *The existence proof of weak solutions to the general GLL equation* (31) *is not proved in this work.*

5. – Concluding remarks.

Understanding magnetization damping in magnetic films is of paramount importance, both from a fundamental scientific and from a practical point of view. As has been presented in this paper we have taken a significant step towards understanding magnetization damping and have shown that the dynamic character of the ferromagnetic structure is greatly affected by the material damping factor. This means also that the switching strongly depends on the damping parameter. For lower values of a, the rotation of magnetic moment is almost entirely precessional. Note that at high damping the magnetization rotates more or less directly towards the effective field direction. Study of a in bulk metallic ferromagnets has drawn a significant interest but the damping mechanism in bulk ferromagnets is not yet fully understood. Recent interest on the basic physics community in this topic is motivated by the following fact. Viewing the magnetization damping as a function of the thickness of samples reveals that for thick samples the damping parameter remains constant, equal to the “bulk” value. However, as the film thickness decreases the damping parameter increases rapidly [9]. It would be very interesting to take into account the dependence of a on film thickness. The results of this paper will stimulate further interest in the magnetic dynamics of thin film magnetics especially in connection with the aspect above.

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