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Transverse Homology Groups

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Sunto. – In questa nota viene fornita una trattazione geometrica della teoria dell’omologia di intersezione.

Summary. – We give, here, a geometric treatment of intersection homology theory.

Introduction.

In [2] M. Goreski and R. MacPherson developed a theory of intersection of homology cycles on a pseudomanifold $X$, generalization of the Poincaré-Lefschetz theory. In their paper the authors attached to an oriented stratified pseudomanifold $X$ a collection of groups $\{IH^p_n(X)\}_{n \geq 0}$, called intersection homology groups, by using cycles and homologies, as in classical simplicial theory, such that their supports meet the strata of $X$ according to the perversity $\bar{p}$.

In this note we give a different geometrical approach to this theory by showing that the objects may be defined starting from singular geometric cycles.

More in detail, given a stratified pseudomanifold $X$, we construct (sections 2. and 3.) a collection of groups $\{H^p_n(X)\}_{n \geq 0}$, called $p$-transverse homology groups of $X$, and we prove (section 4.) that for each $n \geq 0$ the group $H^p_n(X)$ coincides, up to an isomorphism, with the homology intersection group $IH^p_n(X)$.

We think that our different approach could make easier to investigate some important questions and to outcome new results.

For example, the groups $H^p_n(X)$ have functorial properties, if one defines appropriate maps between stratified pseudomanifolds, and they result invariant with respect to a suitable definition of homotopy. The above investigation and their relative results will be object of a later paper.

For the reader’s convenience we include in this paper a section that provides the definitions of stratified pseudomanifold and homology intersection groups.

1. – Preliminaries.

A stratified pseudomanifold $X$ of dimension $m$ is a geometric m-cycle (i.e. the closure of the union of the m-simplices in any triangulation of $X$, and each (m-1)-
simplex is a face of exactly two \( m \)-simplices), with a filtration by subpolyhedra

\[
X = X_m \supset X_{m-1} = X_{m-2} \supset \ldots \supset X_1 \supset X_0
\]

such that for each point \( x \in X_i - X_{i-1} \) there is a filtered space

\[
Y = Y_m \supset Y_{m-1} \supset \ldots \supset Y_1 \supset Y_0 = \text{a point}
\]

and a map \( Y \times D^i \to X \) which for each \( j \), takes \( Y_j \times D^i \) PL-homeomorphically to a neighborhood of \( x \) in \( X_j \). (Here, \( D^i \) is the PL-disk of dimension \( i \)).

If \( X_i - X_{i-1} \) is not empty, it is a PL manifold of dimension \( i \), called the \( i \)-dimensional stratum of the stratification.

A perversity \( \tilde{p} \) is a sequence of integers \( (p_j)_{j \geq 0} \) such that \( p_2 = 0 \) and \( p_{j+1} = p_j \) or \( p_{j+1} = p_j + 1 \).

A compact polyhedron \( P \subseteq X \) of dimension \( \leq h \) is said to be \( (\tilde{p}, h) \)-allowable to \( X \) if \( \dim (P \cap X_i) \leq i + h - m + p_{m-i} \) for each \( i \leq m - 2 \).

Let \( IC^\tilde{p}(X) \) be the chain complex of simplicial chains whose support is \( (\tilde{p}, .) \)-allowable to \( X \), for each \( n \geq 0 \) the \( n \)th homology group of \( IC^\tilde{p}(X) \), denoted by \( IH^\tilde{p}_n(X) \), is the homology intersection group introduced by Goreski and MacPherson in [2].

### 2. – Transverse cycles and transverse homology.

Let \( X \) be a stratified pseudomanifold of dimension \( m \), and let \( \tilde{p} \) a perversity.

A \( \tilde{p} \)-transverse \( n \)-cycle without boundary of \( X \) is a pair \( (C, f) \), where \( C \) is an oriented geometric \( n \)-cycle without boundary and \( f : C \to X \) is a simplicial map such that the polyhedron \( f(C) \) is \( (\tilde{p}, n) \)-allowable to \( X \), that is

\[
\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i}
\]

for each polyhedron \( X_i \) of the filtration of \( X \) which meets \( f(C) \).

A \( \tilde{p} \)-transverse \( n \)-cycle with boundary of \( X \) is a pair \( (C, f) \), where \( C \) is an oriented geometric \( n \)-cycle with boundary \( \partial C \), and \( f : C \to X \) is a simplicial map such that \( f(C) \) is \( (\tilde{p}, n) \)-allowable to \( X \) and \( f(\partial C) \) is \( (\tilde{p}, n - 1) \)-allowable to \( X \), that is

\[
\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i}
\]

for each polyhedron \( X_i \) of the filtration of \( X \) which meets \( f(C) \), and

\[
\dim (f(\partial C) \cap X_i) \leq i + n - m - 1 + p_{m-i}
\]

for each polyhedron \( X_i \) of the filtration of \( X \) which meets \( f(\partial C) \).

Clearly, given a \( \tilde{p} \)-transverse \( (n+1) \)-cycle with boundary \( (C, f) \) of \( X \), the singular \( n \)-cycle \( \partial(C, f) = (\partial C, f/\partial C) \) is a \( \tilde{p} \)-transverse \( n \)-cycle without boundary of \( X \), called boundary of \( (C, f) \).
Remark 2.1. – If \( \bar{p} = (0, 1, 2, \ldots ; j, j + 1, \ldots) \) every singular n-cycle is a \( \bar{p} \)-transverse n-cycle.

Remark 2.2. – Let \((C, f)\) be a \( \bar{p} \)-transverse n-cycle of \( X \), then if \( h \geq n \) each h-cycle \((C', f')\) is a \( \bar{p} \)-transverse h-cycle provided that \( f'(C') \subseteq f(C) \).

Two \( \bar{p} \)-transverse n-cycles without boundary of \( X \) \((C, f)\) and \((C', f')\) are said \( \bar{p} \)-transverse homologous if there exists a \( \bar{p} \)-transverse (n+1)-cycle with boundary \((W, F)\) such that

\[
\begin{align*}
1) \quad \partial W &= C \cup -C' \\
2) \quad F/_{C} &= f, \quad F/_{C'} = f'
\end{align*}
\]

where \(-C'\) is the cycle obtained from \( C' \) by reversing the orientation.

\((W, F)\) is said a \( \bar{p} \)-transverse homology between \((C, f)\) and \((C', f')\).

Let \((C, f)\) be a \( \bar{p} \)-transverse n-cycle of \( X \), consider the map \( F : C \times I \to X \) defined by \( F(y, t) = f(y) \) for each \( t \in I \). By Remark 2.2 it follows that \((C \times I, F)\) is a \( \bar{p} \)-transverse (n+1)-cycle of \( X \).

The gluing property of the geometric cycles with boundary can be extended to \( \bar{p} \)-transverse cycles with boundary as follows.

Let \((C', f')\) and \((C'', f'')\) be two \( \bar{p} \)-transverse (n+1)-cycles with boundary, and suppose that \( C \) is an oriented geometric n-cycle contained, up to a PL homeomorphism, in \( \partial C' \) and in \( \partial C'' \), and such that \( f''/_{C} = f''/_{C'} \). Consider the oriented geometric (n+1)-cycle \( W \) obtained by gluing \( C' \) and \( C'' \) by a PL map \( C \to C \) orientation reversing, and \( f : W \to X \) the unique map which extends \( f' \) and \( f'' \) to \( W \) (usually denoted by \( f' \cup f'' \)). We have that \( f(W) = f'(C') \cup f''(C'') \), and hence \( \dim f(W) = \max \dim (f'(C'), f''(C'')) \). So \((W, f)\) is a \( \bar{p} \)-transverse (n+1)-cycle of \( X \).

As immediate consequence of the above we have that the \( \bar{p} \)-transverse homology relation is an equivalence relation.

Proposition 2.3. – Let \( x \) be a point of a stratified pseudomanifold \( X \) of dimension \( m \). If \( x \notin X_{\nu} \), for each subpolyhedron \( X_{i} \) of the filtration of \( X \) such that \( i < m - n - p_{m-i} \), then an n-cycle \((C, c_{x})\), where \( c_{x} \) is the constant map to \( x \), is a \( \bar{p} \)-transverse n-cycle. Furthermore two \( \bar{p} \)-transverse n-cycles \((C, c_{x})\) and \((C', c_{x'})\) are \( \bar{p} \)-transverse homologous.

Proof. – \((C, c_{x})\) is a \( \bar{p} \)-transverse n-cycle because

\[ \dim (c_{x}(C) \cap X_{i}) = \dim (x \cap X_{i}) = 0; \quad 0 \leq i + n - m + p_{m-i} \quad \text{if} \quad i \geq m - n - p_{m-i}. \]

Two \( \bar{p} \)-transverse n-cycles are \( \bar{p} \)-transverse homologous because \((C, c_{x})\) and \((-C', c_{x'})\) are the complete boundary of the \( \bar{p} \)-transverse (n+1)-cycle \((W, F)\) where \( W \) is the disjoint union of the cone on \( C \) with the cone on \(-C'\), and \( F \) is the map which carries the first cone to \( x \) and the second cone to \( x' \).
Proposition 2.4. — Let \((C, f)\) be a \(\bar{p}\)-transverse \(n\)-cycle of a stratified pseudomanifold \(X\) of dimension \(m\), and let \(\overline{y}\) be a point of an \(n\)-dimensional simplex \(\Delta\) of \(C\). Then \((C, f)\) is \(\bar{p}\)-transverse homologous to a \(\bar{p}\)-transverse cycle \((C, f')\) where \(f'\) is a map which agrees with \(f\) on \(C - \Delta\) and carries an \(n\)-simplex \(\Delta' \subset \Delta\) (\(\overline{y} \in \Delta'\)) on \(f(\overline{y})\).

Proof. — Let \(H : \Delta \times I \to \Delta\) be a homotopy between the identity and a map \(g : \Delta \to \Delta\) such that \(g / \Delta = \text{id}_\Delta, g(\Delta') = \overline{y}\).

Let \(W = C \times I, F : W \to X\) the map defined by

\[
F(y, t) = \begin{cases} 
  f(y) & \text{if } y \notin \Delta' \\
  f \circ H(y, t) & \text{if } y \in \Delta
\end{cases}
\]

By Remark 2.2 \((W, F)\) is a \(\bar{p}\)-transverse \((n+1)\)-cycle of \(X\) because \(F(W) = f(C)\). So, let \(f' = F / C_{\times \{1\}}\), the \(\bar{p}\)-transverse \(n\)-cycle \((C, f')\) is \(\bar{p}\)-transverse homologous to \((C, f)\) and it satisfies the required conditions.

By Prop. 2.3 the \(\bar{p}\)-transverse \(n\)-cycles \((C, c_x)\) belong to the same \(\bar{p}\)-transverse homology class. Every representative of such a class is called 0-homologous. A fundamental characterization of these \(n\)-cycles is given by the following

Proposition 2.5. — A \(\bar{p}\)-transverse \(n\)-cycle of \(X\) is 0-homologous if and only if it is a boundary.

Proof. — Let \((C, f)\) be a \(\bar{p}\)-transverse \(n\)-cycle 0-homologous and let \((W, F)\) be a \(\bar{p}\)-transverse homology between \((C, f)\) and \((C_0, c_x)\).

Fig. 1.
Consider the cycle \((W', F')\) where \(W'\) is obtained by gluing \(W\) to a cone \(L\) on \(C_0\), and \(F'\) coincides with \(F\) on \(W\) and with the constant map to \(x\) on \(L\). By Remark 2.2 \((W', F')\) is a \(\bar{p}\)-transverse cycle because \(F'(W') = F(W)\). Moreover the boundary of \((W', F')\) is \((C, f)\).

Conversely, assume \((C, f)\) is the boundary of the \(\bar{p}\)-transverse cycle \((V, G)\).

Let \(\Delta\) be a top dimensional simplex of \(C\). Suppose \(f(\Delta) \subseteq X_1\) and \(f(\Delta) \not\subseteq X_{i-1}\). If \(\Delta\) is a face of an \((n+1)\)-simplex \(\Delta'\) of \(V\), then \(G(\Delta') \not\subseteq X_{i-1}\). Let \(\Delta''\) be an \((n+1)\)-simplex of \(V\) contained in \(\Delta'\). (See Fig. 1).

By Prop. 2.4 we can suppose that, up to \(\bar{p}\)-transverse homology, \(G\) is constant on \(\Delta''\), that is \(\bar{G}(\Delta'') = \bar{x} \in X_i\), \(i \geq m - n - p_{m-} \). The cycle \((\partial\Delta'', G/\partial\Delta'')\) is a \(\bar{p}\)-transverse cycle 0-homologous. The pair \((V - \Delta'', G)\) is a \(\bar{p}\)-transverse homology between \((C, f)\) and \((\partial\Delta'', G/\partial\Delta'')\). Then \((C, f)\) is 0-homologous.

\[\]

3. – The transverse homology groups.

Let \(X\) be a stratified pseudomanifold of dimension \(m\), and let \(\bar{p}\) a perversity.

Denote by \(H^p_n(X) (n \geq 0)\) the set of \(\bar{p}\)-transverse homology classes of \(\bar{p}\)-transverse n-cycles without boundary of \(X\).

Now given two \(\bar{p}\)-transverse n-cycles \((C, f), (C', f')\) of \(X\), it is easy to see that \((C \cup C', f \cup f')\) is a \(\bar{p}\)-transverse n-cycle of \(X\) and that its \(\bar{p}\)-transverse homology class depends only on the \(\bar{p}\)-transverse homology class of \((C, f)\) and \((C', f')\).

Then it makes sense to define:

\[
[(C, f)] + [(C', f')] = [(C \cup C', f \cup f')]
\]

\[\]

**Proposition 3.1.** – \(H^p_n(X), +\) is an abelian group for each \(n \geq 0\).

**Proof.** – The associativity and commutativity property of the addition are evident.

The zero element of \(H^p_n(X), +\) is the class of the n-cycles 0-homologous.

For, the sum of a \(\bar{p}\)-transverse n-cycle \((C, f)\) with a 0-homologous n-cycle \((C_0, c_0)\) is \(\bar{p}\)-transverse homologous to \((C, f)\), and a \(\bar{p}\)-transverse homology between them is given by the cycle \((W, f)\) where \(W\) is the disjoint union of the cylinder \(C \times I\) on \(C\) with the cone \(L\) on \(-C_0\), and \(F : W \rightarrow X\) is the constant map to \(x\) in \(L\), while \(F/_{C \times I}\) is the map defined by \(F(y, t) = f(y)\) for each \(t \in I\).

From Prop. 2.5 it follows that the inverse element of the class of \((C, f)\) is given by the class of \((-C, f)\).

\[\]

4. – The main theorem.

In this section we prove that the \(\bar{p}\)-transverse homology groups \(H^p_n(X)\) are isomorphic to the intersection homology groups \(IH^p_n(X)\) introduced by Goreski and MacPherson in [2].
Let $\xi = \sum m_h \sigma_h$ be a simplicial n-cycle of $X$. Consider the geometric realization of $\xi$, that is the singular geometric cycle $(C_\xi, f_\xi)$ where $C_\xi$ is the geometric cycle obtained by taking, for each $h$, $m_h$ copies $A^n_h$ of standard n-simplex $A^n$ if $m_h > 0$, $-m_h$ copies of $-A^n$ if $m_h < 0$, and gluing two copies $A^n_h$ and $A^n_{-h}$ along an $(n-1)$-face with opposite orientations, and $f_\xi$ is a simplicial map such that $f_\xi(A^n_h) = \sigma_h$. Since $f_\xi(C_\xi)$ coincides with the support $|\xi|$ of the simplicial cycle $\xi$, we have that, if, and only if, $\xi$ lies in $IC_n^0(X)$, then $(C_\xi, f_\xi)$ is a $\bar{p}$-transverse cycle of $X$. Moreover, if $\xi$ is a boundary, then $(C_\xi, f_\xi)$ is also a boundary.

Then, if $n \leq \dim X$, it makes sense to consider the map $\Psi : IH_n^0(X) \to H_n^0(X)$ defined by

$$\Psi([\xi]) = [(C_\xi, f_\xi)]$$

It is easy to see that $\Psi$ is a homomorphism.

In order to prove that $\Psi$ is an isomorphism we need the following lemma:

**Lemma 4.1.** Let $(C, f)$ be a $\bar{p}$-transverse n-cycle of $X$. If $\dim f(C) < n$, then $(C, f)$ is 0-homologous.

**Proof.** Since $\dim f(C) < n$, we have that $H_n(f(C)) = 0$. So $(C, f)$ is the boundary of a singular geometric cycle $(C', F)$ of $f(C)$. Being $(C, f)$ a $\bar{p}$-transverse cycle and $F(C') \subseteq f(C)$, by Remark 2.2 $(C', F)$ is a $\bar{p}$-transverse cycle of $X$. The assertion follows by Prop. 2.5.

From the above lemma it follows:

**Remark 4.2.** $H_n^0(X) = 0$ for each $n > \dim X$.

**Lemma 4.3.** Let $(C, f)$ be a $\bar{p}$-transverse n-cycle of $X$. If $\dim f(C) = n$, then there exists a $\bar{p}$-transverse n-cycle $(C', f')$ $\bar{p}$-transverse homologous to $(C, f)$ such that $f'$ is injective on each top dimensional simplex of $C'$.

**Proof.** Let $P'$ be the subpolyhedron of $C$ consisting of n-simplices on which $f$ is one to one, with their faces, and let $P''$ be the subpolyhedron consisting of the remaining n-simplices and their faces. Let $Q = P' \cap P''$, we observe that each (n-1)-simplex of $P' - Q$ (or $P'' - Q$) is a face of two n-simplices of $P'$ (or $P''$) whereas each (n-1)-simplex of $Q$ is a face of an n-simplex of $P'$ and a face of an n-simplex of $P''$. The polyhedron $P'$ (or $P''$) is not in general a cycle just because $Q$ is not in general a cycle.

Briefly, the idea of the proof is that of obtaining, starting from $C = P' \cup P''$, by appropriate identifications on $Q$, two $\bar{p}$-transverse cycles without boundary $(C', f')$ and $(C'', f'')$ such that $(C'', f'')$ is 0-homologous and $(C', f')$ is $\bar{p}$-transverse homologous to $(C, f)$.
Step 1  Construction of \((C', f')\) and \((C'', f'')\).

Let \(\sigma = (V_1, ..., V_n)\) be an \((n-1)\)-simplex of \(Q\) and let \(\tau = (V_0, V_1, ..., V_n)\) the \(n\)-simplex of \(P''\) such that \(\sigma \subset \tau\). There exists an unique \((n-1)\)-face \(\sigma_1 \neq \sigma\) of \(\tau\) such that \(f(\sigma_1) = f(\sigma)\). If \(\sigma_1 = (V_0, ..., V_{i-1}, V_{i+1}, ..., V_n)\) is contained in \(Q\) we identify \(\sigma\) to \(\sigma_1\) by the orientation-reversing simplicial isomorphism \(\varphi\) defined by putting
\[
\varphi(V_r) = V_s \Leftrightarrow f(V_r) = f(V_s).
\]
If \(\sigma_1\) is not contained in \(Q\), it is a face of another simplex \(\tau_1 \neq \tau\) of \(P''\) and there exists an unique \((n-1)\)-face \(\sigma_2 \neq \sigma_1\) of \(\tau_1\) such that \(f(\sigma_1) = f(\sigma_2)\). If \(\sigma_2\) is contained in \(Q\) we identify \(\sigma_2\) to \(\sigma_1\) as before, otherwise we can iterate the proceeding and we obtain a sequence of simplices \(\sigma \subset \tau \supset \sigma_1 \subset \tau_1 \supset \sigma_2 \supset ... \supset \sigma_h \subset \tau_h\). We show that the simplices of the previous sequence are distinct.

Suppose by induction that the assert is true for the sequence \(\sigma \subset \tau \supset \sigma_1 \subset \tau_1 \supset \sigma_2 \supset ... \supset \tau_{l-1} \supset \sigma_l\), then we have

1) \(\tau_l \neq \tau_{l-1}\) by construction;
2) \(\tau_l \neq \tau_i\) if \(i < l - 1\) because otherwise \(\sigma_l, \sigma_i, \sigma_{i+1}\) should be distinct faces of \(\tau_l = \tau_i\) with \(f(\sigma_l) = f(\sigma_i) = f(\sigma_{i+1})\);
3) \(\sigma_{l+1} \neq \sigma_i\) by construction;
4) \(\sigma_{l+1} \neq \sigma_i\) if \(i < l\) because otherwise \(\sigma_i = \sigma_{l+1}\) should be a face of distinct simplices \(\tau_{l-1}, \tau_i, \tau_l\).

From the above and from the compactness of \(C\) it follows that there exists an \((n-1)\)-simplex \(\sigma_r \subset Q\) such that \(f(\sigma_r) = f(\sigma)\). We identify as before \(\sigma\) to \(\sigma_r\).

By proceeding in this manner for each \((n-1)\)-simplex of \(Q\) we obtain, by appropriate subdivision, two geometric \(n\)-cycles \(C'\) and \(C''\) without boundary. Finally we denote by \(f' : C' \to X\) and \(f'' : C'' \to X\) the PL maps naturally induced by \(f\), and so we obtain the singular cycles \((C', f')\), \((C'', f'')\). These cycles are \(\bar{p}\)-transverse cycles because \(f'(C') \subseteq f(C), f''(C'') \subseteq f(C)\) and \((C, f)\) is a \(\bar{p}\)-transverse cycle. Observe that \(f'\) is one to one on each \(n\)-simplex of \(C'\), while the map \(f''\) is not one to one on each \(n\)-simplex of \(C''\), so \(\dim f''(C'') < n\) and hence \((C'', f'')\) is 0-homologous by Lemma 4.1.

Step 2  \((C', f')\) is \(\bar{p}\)-transverse homologous to \((C, f)\).

Let \((D, g)\) be a \(\bar{p}\)-transverse \((n+1)\)-cycle whose boundary is \((C'', f'')\) and let \(M_p\) be the simplicial mapping cylinder of the quotient map \(p : C \to C' \cup C''\). By gluing \(M_p\) to the geometric cycle \(D\) along \(C''\) we obtain the geometric cycle \(W = M_p \cup D\) such that \(\partial W = C \cup -C'\).

Let \(F : W \to X\) the PL map defined by
\[
F(y, t) = f(y) \text{ if } t \neq 0 \\
F(p(y), 0) = f(y) \\
F(y) = g(y) \text{ if } y \in D
\]
Being $F(W) \subset f(C) \cup g(D)$, the singular $(n+1)$-cycle $(W, F)$ is a $\bar{p}$-transverse $(n+1)$-cycle of $X$. Furthermore we have $F/C = f$, $F/C' = f'$ and hence $(C', f')$ is $\bar{p}$-transverse homologous to $(C, f)$.

**Theorem 4.4.** $\Psi : IH^p_n(X) \to H^p_n(X)$ is an isomorphism.

**Proof.**

$\Psi$ is onto.

Let $(C, f)$ be a $\bar{p}$-transverse n-cycle of $X$.

If $\dim f(C) < n$, by Lemma 4.1 $(C, f)$ is 0-homologous, and hence $[(C, f)]$ is the image of the zero element of $IH^p_n(X)$.

Suppose $\dim f(C) = n$. From Lemma 4.3 $(C, f)$ is $\bar{p}$-transverse homologous to an n-cycle $(C', f')$ where $f'$ is injective on each top dimensional simplex $\tau_n$ of $C'$. So $(C', f')$ is the geometric realization of the simplicial cycle $\sum f'(\tau_n)$.

$\Psi$ is 1-1.

Let $[\zeta]$ an element of $IH^p_n(X)$ such that $\Psi([\zeta]) = [(C_\zeta, f_\zeta)] = 0$ and let $(W, F)$ be a $\bar{p}$-transverse $(n+1)$-cycle of $X$ whose boundary is $(C_\zeta, f_\zeta)$. We denote by $S$ the set of $(n+1)$-simplices of $W$ on which $F$ is one to one. We will prove that $\zeta = \partial \sum_{\tau_{n+1} \in S} F(\tau_{n+1})$.

Suppose $\zeta = \sum m_h \sigma_h$, and let $\Delta^n_h$ be an n-symplex of $C_\zeta$ such that $f_\zeta(\Delta^n_h) = \sigma_h$. We need to show that, if $m_h \neq 0$, there exists an $(n+1)$-symplex $\bar{\tau} \in S$ such that $\sigma_h = F(\Delta^n_h) \prec F(\bar{\tau})$.

Let $\Delta^n_h \prec \tau$. 
If $\tau \in S$, it is the required $\bar{\tau}$.

If not, there exists an unique face $\sigma'$ of $\tau$ such that $F(\sigma') = -\sigma_h$. $\sigma'$ does not lie in $C_\xi = \partial W$ (because otherwise $m_h = 0$), and hence there exists an unique $(n+1)$-simplex $\tau' \neq \tau$ of $W$ such that $\sigma' \prec \tau'$. If $\tau' \in S$, it is the required $\bar{\tau}$. If not, we repeat the previous procedure starting from $\sigma' \prec \tau'$.

Arguing as in proof of Lemma 4.1, the above procedure has a finite number of steps, and it determines $\bar{\tau} \in S$.

Observe that, if $S = \emptyset$, then $\xi = 0$.

\section*{References}


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