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On simple and stable homogeneous bundles


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On Simple and Stable Homogeneous Bundles.

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Sunto. – Nell’ articolo abbiamo voluto analizzare il rapporto tra i concetti di stabilità e semplicità per un fibrato vettoriale omogeneo su una varietà proiettiva.
Il teorema principale mostra come un fibrato omogeneo non sia destabilizzato dai suoi sottofibrati omogenei se e solo se esso è il prodotto tensoriale fra un fibrato omogeneo stabile ed una rappresentazione irriducibile.
Daremos quindi un esempio di un fibrato omogeneo, che risulta semplice, ma non stabile.

Summary. – In this work we will analyze the relation between the stability and the simplicity of a homogeneous vector bundle on a projective variety.
Our main theorem shows that a homogeneous bundle is not destabilized by its homogeneous subbundles if and only if it is the tensor product of a stable homogeneous bundle and an irreducible representation.
Then we give an example of a homogeneous bundle, which is simple, but not stable.

1. – Introduction.

In this article we want to examine the relation between the concepts of stability and simplicity for a homogeneous vector bundle.

We start considering a homogeneous rational variety $X := G/P$, with $G$ complex simple Lie group and $P$ a parabolic subgroup; a homogeneous vector bundle $E$ on $G/P$ will be then, as we will see later, given by a representation $\rho$ of $P$; thus we’ll can write $E = E_\rho$ in the just specified sense.

We will define the notions of simplicity and $H$-stability in section 2; now we show the results we obtained.

The first result for homogeneous bundles on $X = G/P$ is the Ramanan theorem: if $\rho$ is an irreducible representation of $P$, then $E_\rho$ is a stable bundle.

From this, we have that symmetric powers of $TP^m$, or symmetric powers of universal and quotient bundle on grassmannians are stable, because for a representation $\rho$ of $P$, we have $E_{S^m}\rho \simeq S^mE_\rho$, for all $m \geq 0$.

More generally, in [9] Rohmfeld establishes the following semistability-criterion for homogeneous bundles:
Rohmfeld criterion for semistability: (1) $E_\rho$ is H-semistable $\iff \mu_H(F) \leq \mu_H(E_\rho)$ for every homogeneous subbundle $F$ induced by a subrepresentation of $\rho$;  
(2) If $E_\rho$ is indecomposable and $\mu_H(F) < \mu_H(E_\rho)$ for every homogeneous subbundle $F$ of $E_\rho$ induced by a subrepresentation of $\rho$, then $E_\rho$ is stable.

Actually Rohmfeld states his theorem in a slightly different form, which is not suitably written: indeed, the Euler sequence provides a counterexample to the last statement of [9].

However, the reader can easily check that what really Rohmfeld proved is the just stated criterion.

Our first result in this work is therefore the next theorem, which is a refinement of the preceding criterion:

**Theorem 1.** – (Main theorem) Let $E$ be a homogeneous bundle on the homogeneous rational variety $G/P$ (with the preceding notations); then the two following conditions are equivalent:

(i) For every $F$, subbundle of $E$ induced by a subrepresentation of $\rho$, we have 
$$\mu_H(F) < \mu_H(E);$$

(ii) there exist an irreducible representation $W$ of $G$ and a stable homogeneous subbundle $F_0$ of $E$, such that 
$$E \simeq W \otimes F_0.$$ 

We will see later not only the proof of this theorem, but also an application for the next problem.

It is well known that for every vector bundle, $H$-stability $\Rightarrow$ simplicity, but for rank $\geq 3$ the viceversa is not true: in [5] a counterexample is constructed (the simplest one has rk 3 on $\mathbb{CP}^2$).

Although the notion of homogeneity for vector bundles is a strong hypothesis, in the end of this work we will give an example of a rk 15 homogeneous bundle on $\mathbb{CP}^2$, such that it is simple, but not stable: doing this, we will use the main theorem we told before.

We finally remark that all simple homogeneous bundles on $\mathbb{CP}^2$ of rk $\leq 14$ are stable: this is the content of the conclusive tables.

2. – Notations and preliminaries.

Let $X := G/P$ a homogeneous rational variety, with $G$ complex Lie group and $P$ its parabolic subgroup.

We will give now two definitions:
Definition 1. – Let $E$ be a vector bundle on the homogeneous rational variety $X := G/P$, of $\dim X = d$. Fixed $H \in \text{Pic}(X)$, $H$ ample, $E$ is said to be $H$-stable (respectively semistable) if for all subsheaves $F$ of $E$, $0 \neq F \subset E$, it holds
\[ \mu_H(F) < \mu_H(E) \] (respectively $\leq$),
where
\[ \mu_H(F) := \frac{H^{d-1} \cdot c_1(F)}{rk(F)} \]
is the slope of $F$ with respect to $H$.

Example. – If $G/P = \mathbb{C}P^2$, we will take $H = \mathcal{O}_{\mathbb{C}P^2}(1) = \mathcal{O}(1)$; then, for any vector bundle $E$ on $\mathbb{C}P^2$, by identifying $\mathbb{Z} \cong H^2(\mathbb{P}^2, \mathbb{Z}) \ni c_1(E)$, we have
\[ \mu(E) := \mu_H(E) = \frac{c_1(E)}{rk(E)} \]

Definition 2. – A vector bundle $E$ on a homogeneous rational variety $X = G/P$ is said to be simple, if
\[ h^0(E \otimes E^*) = 1 \]

Remark 1. – $E$ is simple $\iff$ $\text{End}(E) = \{\text{homotheties of } E\}$.

In this article, we will work essentially with a particular class of vector bundles on $X$: the homogeneous bundles.

To define these, we need to introduce before the following construction.

Definition 3. – Let $\rho : P \longrightarrow \text{GL}(r, \mathbb{C})$ a representation of $P$. We define the vector bundle $E_\rho$ on $G/P$ as the quotient of $G \times \mathbb{C}^r$, with respect to the equivalence relation $\sim$, given by
\[ (g, v) \sim (g', v') \iff \text{there exist } p \in P : g = g'p \text{ and } v = \rho(p^{-1})v' \]

Remark 2.
\[ E_{\rho_1} \oplus E_{\rho_2} \cong E_{\rho_1 \oplus \rho_2}; \]
\[ E_{\wedge^k(\rho)} \cong \wedge^k E_\rho; \]
\[ E_{\rho_1} \otimes E_{\rho_2} \cong E_{\rho_1 \otimes \rho_2}; \]
\[ E_{S^m \rho} \cong S^m E_\rho; \]
\[ E_{\rho^*} \cong (E_\rho)^*. \]
Now, the previous definition allows us to introduce the concept of homogeneity for a vector bundle:

**Definition 4.** – A vector bundle of \( rk = r \) on \( G/P, E \), is homogeneous, if there exists a representation \( \rho : P \rightarrow GL(r, \mathbb{C}) \) s.t. \( E = E_\rho \).

**Remark 3.** – If \( E, F \) are homogeneous bundles on \( G/P \), then \( E \oplus F, E \otimes F \) and \( E^* \) are homogeneous too.

**Example.** – \( O_{\mathbb{C}P^n}(t), T_{\mathbb{C}P^n}(t), S^m(T_{\mathbb{C}P^n}) \) are homogeneous vector bundles, for all \( t \in \mathbb{Z} \), for all \( m \in \mathbb{N} \).

Now we begin by introducing some notations, which we will use in the fourth section: there, we will construct an example of homogenous vector bundle on \( \mathbb{C}P^2 \), which is simple, but not stable.

In that case we will have thus \( G/P = \mathbb{C}P^2 \), i.e. \( G = SL(3, \mathbb{C}) \) and

\[
P := \left\{ \begin{bmatrix} \det A^{-1} & A & y \\ 0 & A \end{bmatrix} \mid A \in GL(2, \mathbb{C}), (x, y) \in \mathbb{C}^2 \right\}
\]

By the definition of the homogeneity of a vector bundle, we are naturally interested in studying the indecomposable (otherwise, the induced vector bundle is decomposable, hence automatically not simple \( \Rightarrow \) not stable) representations of \( P \).

At first, we observe that \( P \simeq GL(2, \mathbb{C}) \times \mathbb{C}^2 \), where the structure of semi-direct product on \( GL(2, \mathbb{C}) \times \mathbb{C}^2 \) is defined by, for \( (A \times a), (B \times \beta) \in GL(2, \mathbb{C}) \times \mathbb{C}^2 \),

\[
(A \times a) \cdot (B \times \beta) := (A \cdot B \times (\hat{B}a) + \beta),
\]

Here “\( A \cdot B \)” indicates the usual row-columns product and \( \hat{B}a := \det B \cdot a \cdot B \).

So we can find the representations of \( P \), by combining representations of \( GL(2, \mathbb{C}) \) and of \( \mathbb{C}^2 \).

Now, the irreducible representations of \( GL(2, \mathbb{C}) \) are, for all \( m \in \mathbb{N} \) and \( l \in \mathbb{Z} \),

\[
\rho^l_m : GL(2, \mathbb{C}) \rightarrow GL(m + 1, \mathbb{C})
\]

\[
A \mapsto (\det A)^l \cdot S^m A
\]

and thus, because of the complete reducibility of \( GL(2, \mathbb{C}) \), if \( \psi : P \rightarrow GL(r, \mathbb{C}) = Aut(V) \) is a representation of \( P \), then

\[
\psi|_{GL(2, \mathbb{C}) \times 0} = \bigoplus_{i=1}^{j} \rho^l_{m_i},
\]
From all these considerations, we can deduce the following theorem (see [10] for more details):

**Theorem 2.** Let $\psi : P \rightarrow Aut(V)$ be a representation, such that

$$\psi|_{GL(2, \mathbb{C}) \ltimes 0} = \bigoplus_{i=1}^{j} \rho_{m_i}^{l_i}.$$ 

Then there exist a $P$-invariant flag

$$0 \subset V_1 \subset V_2 \ldots \subset V_j = V$$

such that

$$\rho_{m_i}^{l_i} \simeq V_i/V_{i-1} =: \text{gr}_i V.$$ 

Moreover, $\psi|_{Id \ltimes \mathbb{C}^2} =: \pi$ induces, for $1 \leq r \leq s \leq j$, operators

$$\pi_{rs} : \mathbb{C}^2 \rightarrow \text{Hom}(\text{gr}_s V, \text{gr}_r V)$$

and, for $1 \leq s < r \leq j$, operators $\pi_{rs} \equiv 0$ (*).

Finally, $\psi$ is completely reducible $\iff \pi \equiv 0$.

**Definition 5.** (fundamental) Let $\psi$ be as in the preceding theorem. Then $(\rho_{m_1}^{l_1}, \ldots, \rho_{m_j}^{l_j})$ is the type of $\psi$, and $j$ is the index.

**Theorem 3.** Define $Q = T^{P^2}(-1)$. If $\rho_m^j$ is an irreducible representation of $P$, then

$$\text{(1)} E_{\rho_m^j} \simeq S^m T^{P^2}(l-m) \simeq S^m Q(l)$$

$$\text{(2)} (\rho_m^j)^{\ast} \simeq \rho_m^{-l-m}.$$

**Remark 4.** The type of a representation of $P$ doesn’t determine uniquely the $P$-invariant flag of theorem 2, in general: this depends on the fact that, when we have three or more irreducible components of $\psi|_{GL(2, \mathbb{C}) \ltimes 0}$ ($\iff j \geq 3$), $\Rightarrow$ we can arrange on the matrix associated to $\psi$ the correspondent diagonal blocks in many different ways (provided we still have a representation, i.e. $\pi_{rs} \equiv 0$ for all $1 \leq s < r \leq j$, as we said before in (*)).

Now, we will display some results, which will help us to write the matrix-form for a representation of $P$; the first is the

**Theorem 4.** (see [10]) Let $\psi : P \rightarrow Aut(V)$ be a representation of $P$, of type
\((\rho_{m_1}^1, \rho_{m_2}^2)\); if \(\psi\) is indecomposable, then necessarily we have

\[
|m_2 - m_1| = 1 \text{ and } \begin{cases} 
  l_2 = l_1 + 1, & \text{if } m_1 < m_2; \\
  l_2 = l_1 + 2, & \text{if } m_2 < m_1.
\end{cases}
\]

Moreover, the operator \(\pi_{12}\) is uniquely determined, up to a scalar factor \(\lambda \in \mathbb{C}^\star\), as follows: for \((x, y) \in \mathbb{C}^2\):

(i) if \(m_1 < m_2\), we call \(m = m_1\) and

\[
\Rightarrow \pi_{12}(x, y) = \begin{bmatrix}
(m + 1) \cdot x & y & 2y & 0 \\
x & 2y & 0 & x \\
y & x & x & (m + 1) \cdot y
\end{bmatrix} =: D^{m+1}
\]

is a \((m + 1) \times (m + 2)\) matrix;

(ii) if \(m_2 < m_1\), we call now \(m = m_2\)

\[
\Rightarrow \pi_{12}(x, y) = \begin{bmatrix}
y & y & 0 \\
x & y & 0 & x \\
-x & -x & y & -x
\end{bmatrix} =: I^{m+1}
\]

is a \((m + 1) \times (m + 2)\) matrix.

As a consequence of this theorem, one can verify that in both previous cases it holds

\[
\mu(E_{\rho_{m_2}^\psi}) = \mu(E_{\rho_{m_1}^\psi}) + \frac{3}{2}
\]

**Definition 6.** – In theorem 4 (where \(\psi\) is indecomposable!), \(\pi_{12}\) is said a connection-operator.

We reported this theorem, because it is the point of departure to prove the following more general results:

**Proposition 1.** – (see [10]) Let \(\psi\) be an indecomposable representation of \(P\), of type \((\rho_{m_1}^1, \rho_{m_2}^2, \ldots, \rho_{m_t}^t)\); we call \(n := \text{min}\{m_i \mid i = 1, \ldots, t\}\) and \(N := \text{max}\{m_i \mid i = 1, \ldots, t\}\). Then

(i) for each irreducible component \(\rho_m^1\) of \(\psi\), with \(m \neq n, N\), there exist \(h, k \in \{1, \ldots, t\}\) s.t.

\[
m_h = m - 1 \quad \text{and} \quad m_k = m + 1;
\]
(ii) for \( m = n, N \), there exist \( h, k \in \{ 1, \ldots, t \} \) s.t.

\[
m_h = n + 1 \quad \text{and} \quad m_k = N - 1.
\]

In conclusion, the proposition says that the set \( \{ m_i \mid i = 1, ..., t \} \) is connected.

**Theorem 5.** – (see [10]) Let \( \psi \) be an indecomposable representation of \( P \), of index \( t \). Then there exists an uniquely determined filtration

\[
0 \subset V_1 \subset V_2 \subset ... \subset V_k
\]

with the following properties:

1. \( H_i := V_i/V_{i-1} = \bigoplus_{j \in M_i} \rho_{m_j}^j \), where \( M_i \subset \{ 1, ..., t \} \) is such that

\[
\mu(H_i) := \mu(\rho_{m_j}^j) = \text{constant (with respect to } j)\]

for all \( j \in M_i \);

2. \( \mu(H_i) = \mu(H_{i-1}) + \frac{3}{2} \) for all \( i \in \{ 2, ..., k \} \);

3. \( \psi|_{S^2} \) induces, for each \( i \in \{ 1, ..., k - 1 \} \), a homogeneous not-trivial operator of degree 1

\[
\Theta_i : \mathbb{C}^2 \rightarrow \text{Hom} \left( H_{i+1}, H_i \right).
\]

**Definition 7.** – In the same hypothesis of theorem 5, we call \( \Psi \upharpoonright \cdots (H_1, H_2, ..., H_k) \) the \( \mu \)-filtration of the representation \( \psi \).

**Corollary 1.** – Let \( \psi \) be an indecomposable representation of \( P \), with \( \mu \)-filtration \((H_1, H_2, ..., H_k)\) and respective operators \( \{\Theta_i\}_{i \in \{1, ..., k - 1\}} \). Then

\[
\exp \left( \begin{bmatrix}
0 & \Theta_1 & 0 & 0 \\
0 & \Theta_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \Theta_{k-1} \\
\end{bmatrix} \right) = \psi|_{S^2} \}
\]

Here and in the following we will intend \( V \) as in \( \mathbb{C}T^2 = \mathbb{P}(V) \); now let \( I^{p,q}V \) be the irreducible representation of \( SL(V) \) corresponding to the Young diagram with \( p \) boxes in the first row, and \( q \) boxes in the second one.

This is all what we need to know to compute the cohomology groups we said; in fact, we have the following well known results:

**Proposition 2.** – Let \( \rho_m^j \) be the irreducible representation of \( P \) defining
$S^m Q(l)$; then, if we identify $\rho^l_m$ with the homogeneous bundle it induces,

\[
(1) \quad c_1(\rho^l_m) = \left( \frac{1}{2} m + l \right) \cdot (m + 1);
\]

\[
(\nabla)
\]

\[
(2) \quad \mu(\rho^l_m) = \left( \frac{1}{2} m + l \right).
\]

**Theorem 6.** The first cohomology-groups of $E_{\rho^l_m}$ are

\[
H^0(E_{\rho^l_m}) \cong \Gamma^{m+l,1} V;
\]

\[
H^1(E_{\rho^l_m}) \cong \Gamma^{m-1,l+m+1} V;
\]

\[
H^2(E_{\rho^l_m}) \cong \Gamma^{l-3,1-l,m-3} V.
\]


The goal of this section is the proof of the main theorem (see Introduction): however, we need before the Ramanan construction of «CS-subbundle» ([9]). It allows us to verify the failure of H-stability of a homogeneous vector bundle on its homogeneous subbundles, instead of on all its subsheaves (thus we call «CS-subbundles» those ones which are contradicting stability).

We’re going now to introduce some results, which the reader could find in [9] in a more detailed way:

**Definition 8.** Let $E$ be a not-stable homogeneous vector bundle. A coherent subsheaf $0 \neq F \subset E$ is said to be SCS (i.e., «strong contradicting stability») in $E$, if the two following conditions are fulfilled:

(i) $F$ is H-stable and $E/F$ is torsion-free;

(ii) for all coherent subsheaf $Q$, with $0 \neq Q \subset E/F$, we have

\[
\mu_H(Q) \leq \mu_H(F).
\]

**Lemma 1.** Let $0 \neq U_1, U_2 \subset E$ be two coherent subsheaves of $E$, with $E/U_1$ and $E/U_2$ torsion-free. If $U_1$ is H-stable and $U_2$ satisfies condition (ii) of the preceding definition, then

\[
U_1 \cap U_2 \neq 0 \quad \text{and} \quad U_1 \subsetneq U_2 \quad \Rightarrow \quad \mu_H(U_1) < \mu_H(U_2)
\]
Lemma 2. – Let $E$ be a vector bundle on $G/P$ and
\[ M := \{ c_1(F) \mid 0 \neq F \subset E, F \text{ coherent subsheaf of } E \} \]
Then $\text{sup } M < +\infty$ and $\text{sup } M = \max M$.

Proposition 3. – (Existence of SCS-subsheaves) Let $E$ be a $H$-not stable vector bundle; then $E$ contains a SCS-subsheaf $F$.

Theorem 7. – (Uniqueness of SCS-sheaves) Let $E$ be a $H$-not stable vector bundle on $G/P$, and let $0 \neq U_1, U_2 \subset E$ be two SCS-subsheaves of $E$, s.t. $U_1 \cap U_2 \neq 0$. Then it is $U_1 = U_2$.

With the assumption that $E$ is not $H$-stable, we now define the not-empty (by proposition 3) set
\[ M_0 := \{ \text{SCS - subsheaves of } E, \text{ with minimal rank } =: r_0 \text{ and maximal slope } \mu_0 \} \]
Proposition 3 and theorem 7 say us that $M_0$ contains SCS-subsheaves of $E$, for which every two intersect only trivially ($*$).

Now we call
\[ \bar{F} := \bigoplus_{i \in I} F_i, \]
where $I$ is the maximal set of indices for elements of $M_0$, which form a direct sum; since $\text{rk}E$ is finite and by ($*$), then $I$ is finite too.

Now, if $M \in M_0 - \{ F_i \mid i \in I \}$, then it is $M \cap \bar{F} \neq 0$; in fact, if $M \cap \bar{F} = 0$, then we have to add a new element for $M$ to $I$, because of the maximality of $I$ with respect to this property ($\oplus$).

Moreover, the sheaf $\bar{F}$ has maximal slope $\mu_H(\bar{F}) = \mu_0$; therefore $\bar{F}$ satisfies condition (ii) of the definition of SCS-subsheaf. By lemma 1, we have $M \subset \bar{F}$ and thus the subsheaf $\bar{F}$ is uniquely determined.

Finally, we have only to show the properties which characterize our $\bar{F}$: in particular, in the second of the following theorems we’ll see that $\bar{F}$ is a homogeneous $H$-semistable subbundle of $E$. Once again, here we quote some results from [9]:

Proposition 4. – If $\bar{F}$ is defined as above and $A \in M_0$, then there exists $i \in I$ s.t. $A \simeq F_i$.

Theorem 8. – If $\bar{F}$ is as above, then $\bar{F}$ is a homogeneous $H$-semistable vector subbundle of $E$, with $\mu_H(\bar{F}) \geq \mu_H(E)$.

Definition 9. – Let $\bar{F}$ be as above; we call it the CS-subbundle of $E$. 
From this construction of $\bar{F}$, it follows the next

**Corollary 2.** Let $H$ be an ample fixed element in $\text{Pic}(G/P)$. If $E_\rho$ is a $H$-not stable homogeneous vector bundle on $G/P$, then $E_\rho$ contains a homogeneous CS-subbundle; i.e., there exists a homogeneous subbundle $\bar{F} = \bigoplus_{i \in I} F_i$ induced by a subrepresentation of $\rho$, s.t.

$$\mu_H(\bar{F}) \geq \mu_H(E_\rho),$$

where the $F_i$’s are homogeneous subbundles of $E_\rho$, $H$-stable and with the same slope and rank.

**Corollary 3.** (see [6]) $\mu_H(\bar{F}) \geq \mu_H(E_\rho)$ for every $F$, homogeneous subbundle induced by a subrepresentation of $\rho \Leftrightarrow E_\rho$ is $H$-semistable.

We are now ready to prove the next

**Theorem 9.** (Main theorem) Let $E = E_\rho$ be a homogeneous vector bundle on $G/P$; the following conditions are equivalent:

(i) For every homogeneous subbundle $F$ given by a subrepresentation of $\rho$, we have $\mu_H(F) < \mu_H(E)$;

(ii) There exist an irreducible representation $W$ of $G$ and a homogeneous $H$-stable ($\Rightarrow$ simple) bundle (not necessarily a homogeneous subbundle) $F_0$ of $E$, s.t.

$$E = W \otimes F_0.$$

**Proof:**

((i) $\Rightarrow$ (ii)) We only have the two following possibilities:

(a) $E$ is $H$-stable, and we have already finished, because $E = E \otimes C$;

(b) Otherwise $E$ is $H$-not stable, and therefore, by hypothesis (i) and corollary 2, it is necessarily $\bar{F} = E$, where $\bar{F}$ is as in the same corollary.

Now, $\bar{F} = \bigoplus_{i \in I} F_i$ and in this direct sum we can group the $F_i$’s which result isomorphic; thus we get

$$\bar{F} = \bigoplus_i W_i \otimes F_i,$$

with $W_i$ vector spaces and $F_i$ pairwise not isomorphic. Now, by using the $H$-semistability of $\bar{F}$, we will prove that there is only one summand in the above direct sum.

In fact, if there were at least two distinct $F_i \otimes W_i$ in $\bar{F}$, then each of these would be a homogeneous subbundle of $E$ (using for this the same Rohmfeld’s argument, with the $\rho$-invariance and the Krull-Schmidt theorem’s application which is in [2]), of rank $< \text{rk}(E)$, but with $\mu_H(F_i \otimes W_i) = \mu_H(F_i) = \mu_H(E)$, in opposition to the assumption.

Hence $E = \bar{F} = F_0 \otimes W$, where $F_0$ is one fixed of the $F_i$’s.
The stability of $F_0$ is directly given by corollary 2; therefore we only have to show that $W$ is an irreducible representation of $G$.

- To prove that $W$ is a representation, we need at first to define an action of $G$ on $W$: but we can do this in a natural way, after we observed that

$$\text{Hom}(F_0, \bar{F}) = \text{Hom}(F_0, W \otimes F_0) = W \otimes \text{Hom}(F_0, F_0) \simeq W$$

because of the simplicity of $F_0$.

The first term of this chain is $\text{Hom}(F_0, F_0) \simeq H^0(F_0^* \otimes \bar{F})$, on which there is already a natural action of $G$; therefore $W$ is a representation.

- Now, by contradiction, if $W$ wasn't irreducible, then it would be decomposable, that is we would have $W = W_1 \oplus W_2$, with $W_i$ not-trivial subbundles.

But so it were also $E = F_0 \otimes W = F_0 \otimes W_1 \oplus F_0 \otimes W_2$, where the $F_0 \otimes W_i$'s are both homogeneous (again by [2]) proper subbundles of $E$, with $\mu_H(F_0 \otimes W_i) = \mu_H(E)$.

Hence we found a contradiction to the hypothesis and therefore (i) $\Rightarrow$ (ii) also in the $H$-not stable case.

$((i) \Leftarrow (iii)$ We can suppose $E$ not $H$-stable, because otherwise the thesis is obvious. Hence, let $E$ be not-stable.

By (ii), we have immediately the $H$-semistability of $E$; so, it suffices to show that the only homogeneous subbundle of $E$, given by a subrepresentation of $\rho$, with slope $\mu_H(E)$ is $E$ itself.

Hence, let $E'$ be another subbundle of $E$, induced by a subrepresentation of $\rho$, with $\mu_H(E) = \mu_H(E')$: we can assume that $\text{rk } E'$ is minimal with respect to the subbundles of $E$ with the same properties.

Thus $E'$ satisfies (i) and so, just by applying the first part of the proof, we know that $E' = W' \otimes F_0'$, with $F_0'$ homogeneous and stable. Now, from the morphism

$$i : W' \otimes F_0' \hookrightarrow F_0 \otimes W$$

(induced by the inclusion $E' \hookrightarrow E$), it follows the existence of a not-zero $\varphi \in \text{Hom}(F_0', F_0)$.

But $\text{Hom}(F_0', F_0)$ is one dimensional, because both $F_0$ and $F_0'$ are stable bundles, with the same slope; thus $\varphi$ itself is an isomorphism, so that

$$\text{Hom}(F_0', F_0) \simeq \text{Hom}(F_0', F_0)^G \simeq \mathbb{C},$$

by the stability ($\Rightarrow$ simplicity) of $F_0$; here by $\text{Hom}(F_0', F_0)^G$ we mean the $G$-invariant subspace.

Finally, coming back to the morphism $i$, we can say that it induces a $G$-invariant map $W' \to W$ and this implies $W' = W$, by using the Schur’s lemma.

Thus we've got

$$E' = F_0 \otimes W' = F_0 \otimes W = E,$$

which is our thesis.
4. – A simple homogeneous bundle, which is not-stable.

In this final section, we will discuss the key-example of a homogeneous, simple, but not-stable vector bundle; we will construct this bundle on $G/P = \mathbb{C}P^2$, so that in this case we will have $G = SL(3, \mathbb{C})$ and

$$P := \left\{ \begin{bmatrix} \det A^{-1} & x & y \\ 0 & A \end{bmatrix} \mid A \in GL(2, \mathbb{C}), (x, y) \in \mathbb{C}^2 \right\}$$

With the notations introduced in section 2, we consider the representation $\psi$ of $P$ of type $(\rho_1^{-2}, \rho_0^0 \oplus \rho_2^{-1} \oplus \rho_4^{-2}, \rho_9^0)$.

We want to write the matrix-form of this $\psi$, determined by its type: by the results we indicated in second section, we obtain a first matrix

$$A := \begin{bmatrix} \rho_1^{-2} & I_1 & D_2 & 0 & 0 \\ 0 & \rho_0^0 & 0 & 0 & 0 \\ 0 & 0 & \rho_2^{-1} & 0 & D_3 \\ 0 & 0 & 0 & \rho_4^{-2} & I_4 \\ 0 & 0 & 0 & 0 & \rho_9^0 \end{bmatrix}$$

Hence the matrix-form of $\psi$ is (by abuse of notation)

$$\psi = \exp A = \begin{bmatrix} \rho_1^{-2} & I_1 & D_2 & 0 & \frac{1}{2}D_2D_3 \\ 0 & \rho_0^0 & 0 & 0 & 0 \\ 0 & 0 & \rho_2^{-1} & 0 & D_3 \\ 0 & 0 & 0 & \rho_4^{-2} & I_4 \\ 0 & 0 & 0 & 0 & \rho_9^0 \end{bmatrix}$$

This matrix is important, because it allows us to investigate the not-stability of $E$: in fact, by corollary 2, if $E$ is not-stable, then it contains the CS-subbundle, which is induced by a subrepresentation of $\psi$. So, examining all the subrepresentation of $\psi$ and calculating the slope of each of these (or better, the slope of each bundle by these induced), we can find a destabilizing subbundle of $E$.

Recalling (see [10]) that two similar matrices associated to representations of $P$ induce the same bundle, we are able to find a destabilizing subbundle of $E$,
considering the following matrix, which is similar to $\psi$

$$
\begin{bmatrix}
\rho_1^{-2} & D_2 & 0 & \frac{1}{2}D_2D_3 & I_1 \\
0 & \rho_2^{-1} & 0 & D_3 & 0 \\
0 & 0 & \rho_4^{-2} & I_4 & 0 \\
0 & 0 & 0 & \rho_3^0 & 0 \\
0 & 0 & 0 & 0 & \rho_0^0
\end{bmatrix}
$$

(3)

and its submatrix

$$
\rho := 
\begin{bmatrix}
\rho_1^{-2} & D_2 & 0 & \frac{1}{2}D_2D_3 \\
0 & \rho_2^{-1} & 0 & D_3 \\
0 & 0 & \rho_4^{-2} & I_4 \\
0 & 0 & 0 & \rho_3^0
\end{bmatrix}
$$

(4)

Now, if $\bar{F} := E_\rho$, then

$$
\mu(\bar{F}) = \frac{3}{14} > \frac{1}{5} = \mu(E)
$$

and we conclude the not-stability of $E$.

Finally, we have to show the simplicity of $E$; we start with the following exact sequence, obtained out of the filtration of $\psi$:

$$
0 \rightarrow F \rightarrow E \rightarrow O \rightarrow 0
$$

(5)

We want to compute $H^0(E)$; hence we need before some information about the first cohomology groups of $\bar{F}$.

Let $F'$ and $F''$ be the homogeneous subbundles of $\bar{F}$, given by subrepresentations of type $(\rho_1^{-2}, \rho_2^{-1})$ and $(\rho_4^{-2}, \rho_3^0)$ respectively ($F' \hookrightarrow (\rho_1^{-2}, \rho_2^{-1})$ and $F'' \hookrightarrow (\rho_4^{-2}, \rho_3^0)$); then we have

$$
0 \rightarrow F' \rightarrow \bar{F} \rightarrow F'' \rightarrow 0
$$

(6)

and now we have to compute $H^0, H^1(F'), H^0, H^1(F'')$.

(a) $F'$:

$$
0 \rightarrow \rho_1^{-2} \rightarrow F' \rightarrow \rho_2^{-1} \rightarrow 0
$$

$$
\Rightarrow 0 \rightarrow 0 \rightarrow H^0(F') \rightarrow 0 \rightarrow C \rightarrow H^1(F') \rightarrow 0 \rightarrow H^2(F') \rightarrow 0
$$

$$
\Rightarrow H^0(F') = 0, \ H^1(F') \simeq C, H^2(F') = 0
$$
(b) $F''$:

\[ 0 \rightarrow \rho_4^{-2} \rightarrow F'' \rightarrow \rho_3^0 \rightarrow 0 \]

(7) \[ 0 \rightarrow 0 \rightarrow H^0(F'') \xrightarrow{\beta} I^{3,3}V \xrightarrow{\gamma} I^{3,3}V \xrightarrow{\alpha} H^1(F'') \rightarrow 0 \]

By Schur’s lemma, $\alpha$ is $\equiv 0$, or it is an isomorphism.

If $\alpha$ is an isomorphism, then we have $\gamma \equiv 0$ and hence $\beta$ is surjective; therefore $\beta$ is an isomorphism, i.e. $H^0(F'') \simeq I^{3,3}V$.

Substituting this in the cohomology sequence associated to (6), we get

\[ 0 \rightarrow 0 \rightarrow H^0(F) \xrightarrow{\delta} I^{3,3}V \xrightarrow{\sigma} C \xrightarrow{\tau} H^1(F) \xrightarrow{\epsilon} I^{3,3}V \rightarrow 0 \]

By the Schur’s lemma $\epsilon$ must be an isomorphism; hence $H^1(F) \simeq I^{3,3}V$ and then $\tau \equiv 0$: this is a contradiction, because $\sigma \equiv 0$, by the same lemma.

Hence $\alpha$ isn’t an isomorphism, but $\alpha \equiv 0$.

This implies $H^1(F'') = 0$, because of the surjectivity of $\alpha$, and thus sequence (7) becomes

\[ 0 \rightarrow 0 \rightarrow H^0(F'') \xrightarrow{\beta} I^{3,3}V \xrightarrow{\gamma} I^{3,3}V \xrightarrow{\alpha} 0 \]

$\Rightarrow \gamma$ is an isomorphism, $\beta \equiv 0$ and $H^0(F'') = 0$.

Coming back now to the cohomology sequence of (6),

\[ 0 \rightarrow H^0(F) \rightarrow 0 \rightarrow C \rightarrow H^1(F) \rightarrow 0, \]

we finally obtain $H^1(F) \simeq C$ and $H^0(F) = 0$.

Hence, with this results the cohomology sequence of (5) is

\[ 0 \rightarrow H^0(E) \xrightarrow{\mu} C \xrightarrow{\theta} C \xrightarrow{\nu} H^1(E) \rightarrow 0 \]

Now, since $\theta \neq 0$, $\theta$ must be an isomorphism. $\Rightarrow \nu \equiv 0, \mu \equiv 0$ and $H^0(E) = 0$.

We will use this information later, to compute $H^0(E \otimes E^*)$.

Now we examine the bundle $F \leftrightarrow (\rho_1^{-2}, \rho_2^{-1}, \rho_3^{-2}, \rho_3^0)$, induced by matrix (4): with the same method exposed before to search for the subrepresentations of $\psi$, we can find all subbundles of $F$ given by sub-representations, and verify that they all have slope $< \mu(F)$:

a) Index 3: (1) $G_1 \leftrightarrow (\rho_1^{-2}, \rho_2^{-1}, \rho_4^{-2})$, which is decomposable.

$\Rightarrow \mu(G_1) = \frac{-3}{10} < \frac{3}{14} = \mu(F)$

b) Index 2: (1) $G_2 \leftrightarrow (\rho_1^{-2}, \rho_2^{-1})$. Then

\[ \mu(G_2) = \frac{-3}{5} < \frac{3}{14} = \mu(F); \]
(2) $G_3 \leftrightarrow (\rho_3^{-2}, \rho_4^{-2})$. Then

$$\mu(G_3) = -\frac{3}{7} < \frac{3}{14} = \mu(\bar{F})$$

\(c\) Index 1: (1) $G_4 = E_{\rho_4^{-2}} = Q(-2)$. Therefore

$$\mu(G_4) = -\frac{3}{2} < \frac{3}{14} = \mu(\bar{F})$$

(2) $G_5 = E_{\rho_4^{-2}} = S^1Q(-2)$. Then

$$\mu(G_5) = 0 < \frac{3}{14} = \mu(\bar{F})$$

This computation tells us by corollary 3 that $\bar{F}$ is semistable. Now, just by using the main theorem we can conclude the stability (⇒ simplicity) of $\bar{F}$: by contradiction, if $\bar{F}$ isn’t stable, then by the main theorem in the expression $\bar{F} = W \otimes F_0$, $F_0$ is a proper homogeneous subbundle of $\bar{F}$, because $F_0$ is stable, while $\bar{F}$ is not by assumption.

Therefore $rk(\bar{F}) = 14 = rk W \cdot rk(F_0)$ and we have only three possibilities:

1. $rk(W) = 2$ and $rk(F_0) = 7$; but so

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathcal{Z} \ni c_1(F_0) = \frac{3}{2}$$

Thus this possibility leads to a contradiction;

2. $rk(W) = 7$ and $rk(F_0) = 2$; in this case

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathcal{Z} \ni c_1(F_0) = \frac{3}{7}$$

and, as above, this case isn’t possible;

3. $rk(W) = 14$ and $rk(F_0) = 1$; but

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathcal{Z} \ni c_1(F_0) = \frac{3}{14}$$

But this is another contradiction, and hence $\bar{F}$ is stable.

We are now finally ready to compute $H^0(End(E))$.

Starting from (5), and tensoring it with $E^*$, we get

$$0 \to \bar{F} \otimes E^* \to End(E) \to E^* \to 0, \tag{8}$$

Thus we need to study (i) $\bar{F} \otimes E^*$ and (ii) $E^*$:

(i) If we take the dual of (5) and afterwards we tensor by $\bar{F}$, we obtain

$$0 \to \bar{F} \to E^* \otimes \bar{F} \to End(\bar{F}) \to 0$$

and its cohomology sequence

$$0 \to 0 \to H^0(\bar{F} \otimes E^*) \to \mathbb{C} \to \mathbb{C} \to \ldots \tag{9}$$
where we used the simplicity of $\tilde{F}, H^0(\tilde{F}) = 0$ and $H^1(\tilde{F}) \simeq \mathbb{C}$. But $\tau$ is an isomorphism; $\Rightarrow \sigma \equiv 0$ and, by its injectivity, $H^0(\tilde{F} \otimes E^*) = 0$.

(ii) The dual of (5) is

\[(10) \quad 0 \rightarrow \mathcal{O} \rightarrow E^* \rightarrow F^* \rightarrow 0 \]

Hence, to estimate $H^0, H^1(E^*)$, we need some information about the first cohomology groups of $F^*$.

With the same techniques used for $H^0, H^1(\tilde{F})$, we can compute $H^0(F^*) = 0$. Finally, coming back to the cohomology sequence of (10), we have

\[0 \rightarrow \mathbb{C} \rightarrow H^0(E^*) \rightarrow 0 \]

\[\Rightarrow H^0(E^*) \simeq \mathbb{C}. \]

$\Rightarrow$ from (8) we see that $H^0(End(E)) \simeq \mathbb{C}$, i.e. $E$ is simple.

As conclusion to the article, we report some lists, in which we display the results we obtained in all cases of homogeneous vector bundles on $\mathbb{C}P^2$, of $rk \leq 15$:

**Homogeneous bundles of index 3**

\[
\begin{array}{|c|c|c|}
\hline
\mu\text{-filtration} & \text{Stable} & \text{Simple} \\
\hline
(p_m^l, p_{m+1}^{l+1}, p_{m+2}^{l+2}) \text{ for } m > 0 & \text{yes} & \text{yes} \\
\hline
(p_{m+2}^{l-2}, p_{m+1}^l, p_m^{l+2}) \text{ for } m > 0 & \text{yes} & \text{yes} \\
\hline
(p_m^l \oplus p_{m+2}^{l-1}, p_{m+1}^{l+1}) & \text{no, but it is semistable} \Leftarrow & \text{no} \\
\hline
(p_{m+1}^l, p_m^{l+2} \oplus p_{m+2}^{l+1}) & \text{no, but it is semistable} \Leftarrow & \text{no} \\
\hline
\end{array}
\]

for $l \in \mathbb{Z}$ and $m \in \mathbb{N}$.

**Homogeneous bundles of index 4 and rank 10**

\[
\begin{array}{|c|c|c|}
\hline
\mu\text{-filtration} & \text{Stable} & \text{Simple} \\
\hline
(p_0^{-1}, p_0^0 \oplus p_3^{-1}, p_0^1) & \text{yes} & \text{yes} \\
\hline
(p_0^0 \oplus p_2^{-1}, p_1^1 \oplus p_3^0) & \text{yes} & \text{yes} \\
\hline
(p_1^{-1}, p_0^1 \oplus p_2^0, p_3^1) & \text{no} & \text{no} \\
\hline
\end{array}
\]
Homogeneous bundles of index 5 and rank 15

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<th>(\mu)-filtration</th>
<th>Stable</th>
<th>Simple</th>
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<td>((\rho_1^{-2}, \rho_0^0 \oplus \rho_2^{-1}, \rho_3^0, \rho_4^1))</td>
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<td>no</td>
</tr>
<tr>
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<td>no</td>
<td>yes</td>
</tr>
<tr>
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</tr>
<tr>
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<td>yes</td>
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REFERENCES


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