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# Hodge Classes and Abelian Varieties of Quaternionic Type.

GIUSEPPE LOMBARDO (*)

**Sunto.** – *In questo articolo viene analizzato lo spazio delle classi di Hodge contenute nella coomologia intermedia di una varietà Abeliana di tipo quaternionico. Vengono costruite $\mathfrak{sl}_2$-rappresentazioni che semplificano lo studio della congettura di Hodge in quanto l'agebricità di una classe implica quella di tutte le altre contenute nelle medesima rappresentazione.*

**Summary.** – *We obtain «coniugacy classes» (with respect to a $\mathfrak{sl}_2$ action) in the space of Hodge cycles in the middle cohomology of an Abelian variety of quaternionic type. The existence of such a class simplifies the study of the Hodge conjecture.*

## 1. – Introduction.

The relationship between abelian varieties of Weil type, which provide an important test for the Hodge conjecture because of the existence of exceptional Hodge classes, and Kuga-Satake varieties was investigated by the present author in [L].

Now, we consider firstly the simple abelian varieties obtained (Poincarè's decomposition) from the Kuga-Satake varieties associated to $K3$-type Hodge structures of dimension 8 and 7 and we analyze in detail the Hodge classes contained in their middle cohomology. The endomorphism ring of these varieties contains a quaternion algebra, so, using tools of representation theory, we construct «conjugacy classes» of cycles with respect to a $sl_2$ action. The algebricity of one element contained in a class implies obviously the algebricity of the elements contained in the same class, and this provides a simplification in the study of the Hodge conjecture for these varieties.

Using the same techniques, in section 5 we study more generally Abelian varieties of quaternionic type in order to obtain «conjugacy classes» of Hodge

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cycles in their middle cohomology. The main result is Theorem 5, in which we obtain a $sl_2$ representation of dimension $2m+1$ contained in $B^m(X)$, where $X$ is a $2m$-dimensional abelian variety of quaternionic type.

The study of abelian varieties whose endomorphism ring is a definite quaternion algebra over $\boldsymbol{Q}$ is very interesting since can provide information about Hodge classes of Weil-type abelian varieties. Indeed, recently van Geemen and Verra studied families of abelian 8-folds of quaternionic type, called of spin(7)-type, and they showed that the Hodge conjecture for infinitely many families of abelian varieties of Weil type follows from the Hodge (2,2)-conjecture for such 8-folds (see [vG-V]).

## 2. – Preliminary notions.

### 2.1. *Hodge structures.*

A rational Hodge structure of weight $k$ is a rational vectorspace $V$ with a decomposition of its complexification $V \otimes \boldsymbol{C} = \oplus_{p+q=k} V^{p,q}$ where $V^{p,q}$ are complex vector subspaces such that $\overline{V^{p,q}} = V^{q,p}$. The type of a Hodge structure of weight $k$ is the $k+1$-tuple $(\dim V^{k,0}, \dim V^{k-1,1}, \dots, \dim V^{0,k})$, and in particular a weight-two Hodge structure is said to be of $K3$-type if it has type $(1, n-2, 1)$.

Equivalently, a rational Hodge structure of weight $k$ can be defined using representation theory as a couple $(V, h)$ with $V$ rational vectorspace and $h : \boldsymbol{C}^* \to GL(V \otimes \boldsymbol{R})$ rational representation such that $h(t) = t^k \cdot Id$ for all $t$ in $\boldsymbol{R}$.

A rational polarized Hodge structure of weight $k$ is a 3-tuple $(V, h, \psi)$ where $\psi$ is a polarization of the weight $k$ Hodge structure $(V, h)$, that is a bilinear map $\psi : V \times V \to \boldsymbol{Q}$ such that

(1) $\psi(h(z)\, v, h(z)\, w) = (z\bar{z})^k \psi(v, w)\ \forall v, w \in V \otimes \boldsymbol{R},\ \forall z \in \boldsymbol{C}^*$
(2) $\psi(v, h(i)\, w) = \psi(w, h(i)\, v)\ \forall v, w \in V \otimes \boldsymbol{R}$
(3) $\psi(v, h(i) v) > 0\ \forall v \in V \otimes \boldsymbol{R} - \{0\}$.

### 2.2. *Hodge classes.*

Let $V$ be a rational Hodge structure of even weight $2m$, the space of its Hodge classes is $B(V) := V \cap V^{m,m}$. In case of smooth projective varieties $X$, we define the space of its (codimension $p$) Hodge cycles

$$B^p(X) := H^{2p}(X, \boldsymbol{Q}) \cap H^{p,p}(X) \subset H^{2p}(X, \boldsymbol{C})$$

and the direct summand $Hdg(X) = \oplus_p B^p(X)$ is called the Hodge ring of $X$.

2.3. *Hodge conjecture.*

Let $X$ be a smooth projective variety, there are defined cycle class maps

$$\psi_p\colon CH^p(X)_{\boldsymbol{Q}} \to H^{2p}(X, \boldsymbol{Q})$$
$$\sum a_i Z_i \mapsto \sum a_i [Z_i]$$

from the Chow group (with rational coefficients) $CH^p(X)_{\boldsymbol{Q}}$ which associates to a subvariety $Z_i \subset X$ of codimension $p$ its cohomology class $[Z_i]$. It is possible to show that the image of $\psi_p$ is contained in $B^p(X)$, and we have the

HODGE $(p, p)$ CONJECTURE. – The map $\psi_p\colon CH^p(X)_{\boldsymbol{Q}} \to B^p(X)$ is surjective.

2.4. *Kuga-Satake varieties.*

Let $(V, h, \psi)$ be a weight 2 polarized Hodge structure of $K3$-type and let $\{g_1, \ldots, g_n\}$ be a basis of $V$ in which the symmetric bilinear form $Q = -\psi$ is given by $Q = d_1 X_1^2 + d_2 X_2^2 - d_3 X_3^2 - \ldots - d_n X_n^2$ $(d_i \in \boldsymbol{Q}_{>0})$. We consider the Clifford algebra $C_n = \dfrac{T^{\otimes} V}{I(Q)}$ quotient of the tensor algebra $T^{\otimes} V$ by the the two-sided ideal $I(Q) = \langle v \otimes v - Q(v) \rangle$, and in the following we write simply $a_1 \ldots a_k$ istead of $\overline{a_1 \otimes \ldots \otimes a_k}$. Let $C_n^+$ be the even Clifford subalgebra of $C_n$ (i.e. the subalgebra generated by the class of tensor products of elements of $V$ in even number) and let $J = \dfrac{1}{\sqrt{d_1 d_2}} g_1 g_2$. We have obviously $J^2 = -1$ and the left multiplication by $J$ on $C_n^+$ defines a complex structure on $C_{n,\boldsymbol{R}}^+ \stackrel{\text{def}}{=} C_n^+ \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. Let now $C_{n,\boldsymbol{Z}}^+$ be the lattice of linear combinations of elements of the basis of $C_n^+$ with integer coefficients, the quotient

$$KS = \frac{(C_{n,\boldsymbol{R}}^+, J)}{C_{n,\boldsymbol{Z}}^+}$$

is a complex torus. This torus is an abelian variety since it admits the polarization

$$E(v, w) := Tr(\alpha\iota(v)\, w)$$

where $Tr(x)$ is the trace of the map «right multiplication on $C_n^+$ by the element $x \in C_n^+$», $\iota$ is the canonical involution of the Clifford algebra and $\alpha \in C_n^+$ is an element such that we have $\iota(\alpha) = -\alpha$ and $E(v, Jv) > 0$ for all $v$. The abelian variety

$$(KS, E)$$

is called the Kuga-Satake variety associated to the $k3$-type Hodge structure $(V, h, \psi)$.

## 3. – Hodge structure of dimension 8.

We start our analysis with a Lemma on Clifford algebras constructed from a 8-dimensional vector space.

LEMMA 3.1. – *Let $V$ be a 8-dimensional vector space and let $Q_8 = Hyp \oplus Hyp \oplus [a] \oplus [b] \oplus [c] \oplus [d]$ $(a, b, c, d \in \boldsymbol{Q}_{>0})$ be a quadratic form on $V$, we have*

$$C^+(Q_8) \cong gl_4(\boldsymbol{H}(\sqrt{abcd}))$$

*where $\boldsymbol{H} = \boldsymbol{Q} \oplus \boldsymbol{Q}i \oplus \boldsymbol{Q}j \oplus \boldsymbol{Q}k$ is a quaternion algebra with $i^2 = -ab$, $j^2 = -ac$, $k^2 = -a^2bc$, $ij = k$, $jk = i$, $ki = j$ and $\boldsymbol{H}(\sqrt{abcd}) = \boldsymbol{H} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(\sqrt{abcd})$.*

PROOF. – Let $\{e_1, \ldots, e_8\}$ be the diagonal basis for $Q_8$ such that $Q_8 = \mathrm{diag}(2, -2, 2, -2, a, b, c, d)$; the center of $C_8^+$ is generated by the element $z = e_1 e_2 \ldots e_8$ which satisfies

$$z^2 = (-1)^{7+6+5+4+3+2+1}(2)(-2)(2)(-2)\, abcd = 16abcd\,.$$

Using the properties of Clifford algebras, we can show that

$$C_8^+ \cong (C_6^+ \oplus e_7 C_6^-) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(\sqrt{z})\,. \qquad (*)$$

Moreover, we have

$$C_6^+ = C_4^+ \oplus e_5 C_4^- \oplus e_6 C_4^- \oplus e_5 e_6 C_4^+$$

$$C_6^- = C_4^- \oplus e_5 C_4^+ \oplus e_6 C_4^+ \oplus e_5 e_6 C_4^-\,.$$

We observe that

$$e_7 C_4^- = e_5 e_6 (e_5 e_6 e_7 C_4^-)\,, \quad e_6 C_4^- = e_5 e_7 (e_5 e_6 e_7 C_4^-)\,, \quad e_5 C_4^- = e_6 e_7 (e_5 e_6 e_7 C_4^-)$$

(multiplication by constants does not change linear spaces), therefore

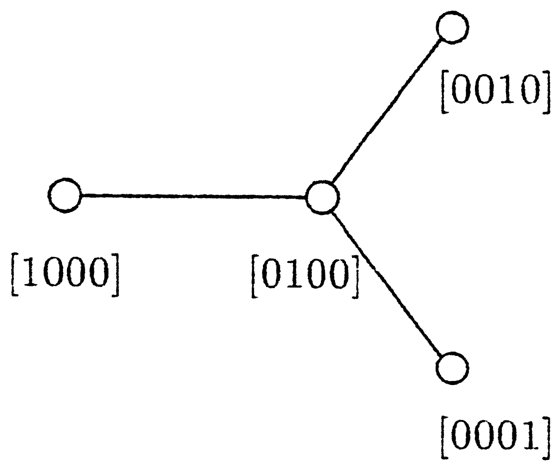
$$\begin{aligned} C_8^+ &= C_4^+ \oplus e_5 C_4^- \oplus e_6 C_4^- \oplus e_5 e_6 C_4^+ \oplus e_7 (C_4^- \oplus e_5 C_4^+ \oplus e_6 C_4^+ \oplus e_5 e_6 C_4^-) \\ &= (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \oplus e_5 e_6 (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \oplus e_5 e_7 (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \\ &\quad \oplus (e_6 e_7)(C_4^+ \oplus e_5 e_6 e_7 C_4^-) \\ &= (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \oplus e_5 e_6 (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \oplus e_5 e_7 (C_4^+ \oplus e_5 e_6 e_7 C_4^-) \\ &\quad \oplus (-a e_6 e_7)(C_4^+ \oplus e_5 e_6 e_7 C_4^-)\,. \end{aligned}$$

Let $B = C_4^+ \oplus e_5 e_6 e_7 C_4^-$, from ( $*$ ) one has

$$C_8^+ \cong (B \oplus e_5 e_6 B \oplus e_5 e_7 B \oplus -a e_6 e_7 B) \otimes \boldsymbol{Q}(\sqrt{abcd}).$$

Moreover, $B \cong gl_4(\boldsymbol{Q})$ (see [L, Thm 6.2] ). The symbols $i = e_6 e_5$, $j = e_7 e_5$ and $k = -a e_6 e_7$ satisfy the rules of a quaternion algebra, therefore we have $C^+(Q_8) \cong gl_4(\boldsymbol{H}(\sqrt{abcd}))$ as required. ■

We consider now the Lie algebra $\mathfrak{so}_8(\boldsymbol{C})$, the Dynkin diagram associated to such an algebra is $D_4$




Dynkin diagram $D_4$

So, we can denote by [0010] and [0001] be the two half-spin representations od $\mathfrak{so}_8(\boldsymbol{C})$ (see [F-H] for details) and we have

LEMMA 3.2. – *Let $(V, h, \psi)$ be a general weight-2 rational Hodge structure of dimension 8 and K3-type with polarization*

$$\psi \cong Hyp \oplus Hyp \oplus [a] \oplus [b] \oplus [c] \oplus [d] \qquad (a, b, c, d \in \boldsymbol{Q}_{>0}).$$

*If $abcd = 1$ the associated Kuga-Satake variety is isogenous to $(A_+ \times A_-)^4$ where $A_\pm$ are not-isogenous abelian varieties of dimension 8 with $H^1(A_\pm, \boldsymbol{C})$ isomorphic respectively to $[0010]^2$ or to $[0001]^2$.*

PROOF. – Let $KS$ be the Kuga-Satake variety associated to $(V, h, \psi)$ and let $q = abcd$. It has dimension $dim_{\boldsymbol{C}} KS = \dfrac{2^{8-1}}{2} = 64$ and, since $\mathrm{End}(KS) \cong$

$C_8^+ \cong gl_4(\boldsymbol{H}(\sqrt{q}))$, if $q=1$ we obtain $C_8^+ \cong gl_4(\boldsymbol{H} \oplus \boldsymbol{H})$. Hence, by Poincaré's theorem we have $KS \sim X^4$ with $X \sim A_+ \times A_-$, where $A_+$ and $A_-$ are not isogenous abelian varieties of dimension $dim_{\boldsymbol{C}} A_{\pm} = \dfrac{2^6}{2^2 2} = 8$. Moreover, $H^1(KS, \boldsymbol{C}) \cong C_8^+(\psi_{\boldsymbol{C}})$ and using the isomorphism described in [F-H, p. 305] we have $H^1(KS, \boldsymbol{C}) \cong gl([0010]) \oplus gl([0001])$ (we recall that $dim[0010] = dim[0001] = 8$). Therefore $H^1(KS, \boldsymbol{C}) \cong [0010]^8 \oplus [0001]^8$ and, from Poincaré's decomposition, we have that $H^1(A_+ \times A_-, \boldsymbol{C}) \cong [0010]^2 \oplus [0001]^2$ so $H^1(A_+, \boldsymbol{C}) \cong [0010]^2$ and $H^1(A_-, \boldsymbol{C}) \cong [0001]^2$. Indeed, we cannot have $H^1(A_+, \boldsymbol{C}) \cong H^1(A_-, \boldsymbol{C}) \cong [0010] \oplus [0001]$ since it implies $A_+ \sim A_-$ and these varieties are not isogenous from Poincaré's theorem. ■

We can now prove the following

THEOREM 3.3. – *Let $A_+$ and $A_-$ be the simple Abelian varieties occurring in the decomposition of the Kuga-Satake variety associated to a general 8-dimensional K3 Hodge structure $(V, h, \psi)$. For each $A_{\pm}$ there is a 9-dimensional $\mathfrak{sl}_2$ representation of Hodge cycles contained in $H^8(A_{\pm}, \boldsymbol{Q}) \cap H^{4,4}(A_{\pm})$.*

PROOF. – We consider the variety $A_+$ (the situation is obviously analoguous for $A_-$). Writing $H^1(A_+, \boldsymbol{C}) \cong [0010]_a \oplus [0010]_b$ in order to distinguish the isomorphic copies, we have a $SL_2$-action (or, equivalently, a $\mathfrak{sl}_2$-action) on $H^1(A_+, \boldsymbol{C}) \cong [0010]_a \oplus [0010]_b$. The element $H = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL_2$ acts as $t$ on $[0010]_a$ and as $t^{-1}$ on $[0010]_b$. We can extend this action to $\wedge^n H^1(A_1, \boldsymbol{C})$ in the obvious way and we consider the one-dimensional space $\wedge^8 [0010]_a \cong [0000]$, the element $H$ acts on this space as $t^8$. Using repeatedly the element $Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ on $\wedge^8[0010]_a$, since the weights decrease by two under the action of $Y$ we obtain a $\mathfrak{sl}_2$-representation of dimension 9. This representation is unique (it is maximal), therefore it must be defined over $\boldsymbol{Q}$ (that is, there exists a 9-dimensional subspace contained in $H^8(X, \boldsymbol{Q})$ such that the representation is the $\boldsymbol{C}$-tensorization of this subspace). Since $(V, h, \psi)$ is of type K3, on $V_{\boldsymbol{C}}$ the matrix of $h : \boldsymbol{C}^* \to GL(V_{\boldsymbol{R}})$ can be diagonalized as $h(z) = \mathrm{diag}(z^2, z\bar{z}, z\bar{z}, z\bar{z}, \bar{z}^2, z\bar{z}, z\bar{z}, z\bar{z})$ and writing $z = \varrho e^{i\theta}$ we have $h(z) = \varrho^2 h(e^{i\theta})$. So if we choose $z \in S^1$ we obtain $h(z) = \mathrm{diag}(z^2, 1, 1, 1, z^{-2}, 1, 1, 1) \in SO_8(\boldsymbol{C})$.

Let $L_i$ be the functionals defined by $L_i(\mathrm{diag}(a_{jj})) = a_{ii}$ (see [F-H] p. 163), we have

$$L_i(h(z)) = \begin{cases} z^2 & i = 1 \\ 1 & i = 2, 3, 4 \end{cases}$$

In order to understand if the obtained $\mathfrak{sl}_2$ representation is contained in $B^4(A_1)$ we have to determine if its elements are contained in $H^{4,4}(A_1)$. We observe that the weights of the half-spin representation [0010] are $\frac{1}{2}(\pm L_1 \pm L_2 \pm L_3 \pm L_4)$ with an even number of minus. We can compute the matrix of the representation $\tilde{h}: S^1 \to GL([0010])$ using $h$ and these weights

$$\begin{aligned} \frac{1}{2}(L_1+L_2+L_3+L_4)(h(z)) &= \sqrt{z^2 \cdot 1 \cdot 1 \cdot 1} = z \\ &\vdots \\ \frac{1}{2}(-L_1-L_2-L_3-L_4)(h(z)) &= \sqrt{z^{-2} \cdot 1 \cdot 1 \cdot 1} = z^{-1} = \bar{z} \end{aligned}$$

and we obtain $\tilde{h}(z) \cong (z, z, z, z, \bar{z}, \bar{z}, \bar{z}, \bar{z})$. Therefore, $\wedge^8[0010]$ has type $(4, 4)$ and the whole representation is contained (by conjugation) in $B^4(A_1) = H^8(A_1, \boldsymbol{Q}) \cap H^{4,4}(A_1)$. Looking at the $SL_2$-action, it is easy to find the spaces in which the cycles are contained and the situation can be summarized as follows

| weights | spaces | number of cycles |
|---|---|---|
| $t^8$ | $\wedge^8[0010]_a$ | 1 |
| $t^6$ | $\wedge^7[0010]_a \otimes [0010]_b$ | 1 |
| $t^4$ | $\wedge^6[0010]_a \otimes \wedge^2[0010]_b$ | 1 |
| $t^2$ | $\wedge^5[0010]_a \otimes \wedge^3[0010]_b$ | 1 |
| $t^0$ | $\wedge^4[0010]_a \otimes \wedge^4[0010]_b$ | 1 |
| $t^{-2}$ | $\wedge^3[0010]_a \otimes \wedge^5[0010]_b$ | 1 |
| $t^{-4}$ | $\wedge^2[0010]_a \otimes \wedge^6[0010]_b$ | 1 |
| $t^{-6}$ | $[0010]_a \otimes \wedge^7[0010]_b$ | 1 |
| $t^{-8}$ | $\wedge^8[0010]_b$ | 1. |

Moreover, we observe that the cycles are the [0000] representations (since are invariants) and that (by explicit computation) there are 10 [0000] representations contained in $H^8(A_1, \boldsymbol{Q}) \cap H^{4,4}(A_1)$. The last cycle necessarily must be contained in $\wedge^4[0010]_a \otimes \wedge^4[0010]_b$ otherwise, using the $SL_2$-action, the number of cycles necessarily increases. ■

## 4. – Hodge structure of dimension 7.

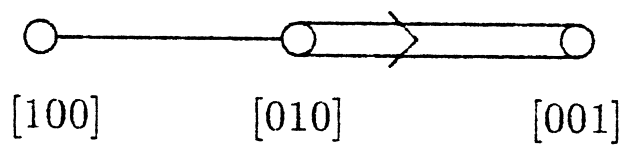
Now we consider Kuga-Satake varieties constructed from a 7-dimensional K3-type Hodge sructure. We show that these varieties can be decomposed in Abelian 8-folds of quaternionic type and, repeting the same argument of the previous section, we compute the number of $\mathfrak{sl}_2$ representations contained in the space of their codimension 4 Hodge cycles. We obtain the following

THEOREM 4.1. – *Let $(V, h, \psi)$ be a general weight 2 polarized Hodge structure of $\dim V = 7$ and $\dim V^{2,0} = 1$ with polarization $\psi \cong \mathrm{Hyp} \oplus \mathrm{Hyp} \oplus [a] \oplus [b] \oplus [c]$ $(a, b, c \in \boldsymbol{Q}_{>0})$, the associated Kuga-Satake variety is isogenous to four copies of an Abelian variety A of dimension 8. The space of codimension 4 Hodge cycles of A contains one 9-dimensional, one 5-dimensional and two 1-dimensionals $\mathfrak{sl}_2$-representations.*

PROOF. – The matrix of the representation $h : S^1 \to GL(V_{\boldsymbol{R}})$ can be diagonalized on $V_{\boldsymbol{C}}$ as $h(z) = \mathrm{diag}(z^2, 1, 1, z^{-2}, 1, 1, 1) \in SO_7(\boldsymbol{C})$ (see Lemma 3.2), therefore

$$L_i(h(z)) = \begin{cases} z^2 & i = 1 \\ 1 & i = 2, 3 \end{cases}$$

(see 3.2). From the isomorphism $C^+ \cong gl_4(\boldsymbol{H})$ (analogous to Lemma 3.1, see also [vG]), we have the decomposition $KS \sim A^4$ where $A$ is a simple Abelian variety of dimension $\dim_{\boldsymbol{C}} A = 8$ with $\mathrm{End}(A) = \boldsymbol{H}$. In similar way to Lemma 3.2, $C^+(\psi_{\boldsymbol{C}}) \cong gl(S)$ where $S$ is the spin representation of $\mathfrak{so}_7(\boldsymbol{C})$ which has dimension 8 and weights $\frac{1}{2}(\pm L_1 \pm L_2 \pm L_3)$. The Dynkin diagram associated to $\mathfrak{so}_7(\boldsymbol{C})$ is $B_3$ (see [F-H])




Dynkin diagram $B_3$

therefore we can write $S = [001]$. Now, we repeat the argument of 3.3; the Hodge structure on this space is given by $\tilde{h}(z) = \mathrm{diag}(z, z, z, z, \bar{z}, \bar{z}, \bar{z}, \bar{z})$, $H^1(A, \boldsymbol{C}) \cong [001]_a \oplus [001]_b$ and we find a $SL_2$-representation of dimension 9 in $H^8(A, \boldsymbol{Q}) \cap H^{4,4}(A)$ starting from $\wedge^8 [001] = [000]$ which has type $(4, 4)$.

We can also compute (with the aid of the computer program «Lie») the number of the $[000]$ representations in each component of

$\wedge^8([001]_a \oplus [001]_b)$ and we obtain

| weights | spaces | number of cycles |
|---|---|---|
| $t^8$ | $\wedge^8[001]_a$ | 1 |
| $t^6$ | $\wedge^7[001]_a \otimes [001]_b$ | 1 |
| $t^4$ | $\wedge^6[001]_a \otimes \wedge^2[001]_b$ | 2 |
| $t^2$ | $\wedge^5[001]_a \otimes \wedge^3[001]_b$ | 2 |
| $t^0$ | $\wedge^4[001]_a \otimes \wedge^4[001]_b$ | 4 |
| $t^{-2}$ | $\wedge^3[001]_a \otimes \wedge^5[001]_b$ | 2 |
| $t^{-4}$ | $\wedge^2[001]_a \otimes \wedge^6[001]_b$ | 2 |
| $t^{-6}$ | $[001]_a \otimes \wedge^7[001]_b$ | 1 |
| $t^{-8}$ | $\wedge^8[001]_b$ | 1. |

Obviously, we have the following situation

(1) one 9-dimensional representation generated by $\wedge^8[001]$,

(2) one 5-dimensional representation generated by the cycle contained in $\wedge^6[001]_a \otimes \wedge^2[001]_b$ and not contained in the previous $\mathfrak{sl}_2$ representation

(3) two 1-dimensional representations contained in $\wedge^4[001]_a \otimes \wedge^4[001]_b$. ■

REMARK 4.2. – We observe that, starting from a 6-dimensional K3-type Hodge structure, we obtain (see [L]) that $KS \sim A^4$ where $A$ is an Abelian fourfold of Weil type. The endomorphism ring of $A$ is an imaginary quadratic field, and $H^1(A, \mathbf{C}) \cong [001] \oplus [010]$ where $[001]$ and $[010]$ are the half-spin representations of $\mathfrak{so}_6(\mathbf{C})$. These representations are not isomorphic, so in this case we don't have a $\mathfrak{sl}_2$-action and we cannot find conjugacy classes. A direct computation of the $[000]$-representations shows that we have 3 cycles, $\wedge^4[010]$, $\wedge^4[001]$ and a third cycle contained in $\wedge^2[010] \otimes \wedge^2[001]$ (this cycle is $[E^2]$ where $E$ denotes the polarization of the variety).

## 5. – Abelian Varieties of quaternionic type.

Let now $X$ be an Abelian variety of dimension $2m$ with $\mathrm{End}_{\mathbf{Q}}(X)$ a totally definite quaternion algebra $\mathbf{H}$. From [Ab1] we have that the Mumford-Tate group of $X$ is isomorphic to $SO(2m, \mathbf{C})$ and $H^1(X, \mathbf{C}) = W \oplus W$ where $W$ is the standard representation of $SO(2m, \mathbf{C})$. Hence, as is 3.3, we have a $\mathfrak{sl}_2$-representation of Hodge cycles corresponding to the action of $\mathbf{H}$. Now, we study this situation more in detail and we prove the following

THEOREM 5.1. – *Let $X$ be an Abelian variety of type III of dimension $2m$. The space of codimension $m$ Hodge cycles contains a $\mathfrak{sl}_2$-representation of dimension $2m+1$.*

PROOF. – Let $x \in \boldsymbol{H}$ be a generical element of $\boldsymbol{H}$ with non trivial imaginary part, $\boldsymbol{Q}(x)$ is a imaginary quadratic field therefore there are $a, b \in \boldsymbol{Q}$ such that $x^2 + ax + b = 0$ and $A, B \in \boldsymbol{Q}$ such that $x^{4m} + Ax^{2m} + B = 0$. Let $M_x$ be the action of $x$ on $H^{2m}(X, \boldsymbol{Q})$, on $H^{2m}(X, \boldsymbol{C})$ $M_x$ has eigenvalues $x^{2m}, x^{2m-1}\bar{x}, \ldots, \bar{x}^{2m}$. Now, we consider $\mathrm{Ker}(M_x^2 + AM_x + B)$; if $v \in (H^{2m}(X, \boldsymbol{C}))^{p,q}$ we have that $(M_x^2 + AM_x + B)v = (x^{2p}\bar{x}^{2q} + Ax^p\bar{x}^q + B)v$. Observing that the equation $y^2 + Ay + B = 0$ has solutions $y = x^{2m}$ and $y = \bar{x}^{2m}$ we obtain that

$$V = \mathrm{Ker}(M_x^2 + AM_x + B) = (H^{2m}(X, \boldsymbol{C}))^{2m,0} \oplus (H^{2m}(X, \boldsymbol{C}))^{0,2m}.$$

From [Ab1], the element $x \in \boldsymbol{Q}(x)$ acts as $x$ on the first copy of $W$ and as $\bar{x}$ on the second, and each copy contains $m$ $(1, 0)$ forms and m $(0, 1)$ forms. Summarizing, we have the decomposition $H^1(X, \boldsymbol{C}) = H_+^{1,0} \oplus H_-^{1,0} \oplus H_+^{0,1} \oplus H_-^{0,1}$ (where $+$ and $-$ mean the action as $x$ or as $\bar{x}$ respectively). From this decomposition, we have that the subspace $V_{\boldsymbol{Q}}$ such that $V_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \boldsymbol{C} = V$ is contained in $H^{2m}(X, \boldsymbol{Q}) \cap H^{m,m}(X)$. Hence, using the $\boldsymbol{H}$-action on $V$ we obtain a $2m+1$ dimensional representation contained in $H^{2m}(X, \boldsymbol{Q}) \cap H^{m,m}(X)$. ■

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