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Cauchy Problem in Multi-anisotropic Gevrey Classes for Weakly Hyperbolic Operators.

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Sunto. – *Si dimostra la buona positura del problema di Cauchy per sistemi debolmente iperbolici nell'ambito delle classi Gevrey multi-anisotrope, generalizzanti le classi Gevrey standard. Il risultato è ottenuto sotto le seguenti ipotesi: la parte principale ha coefficienti costanti; i termini di ordine inferiore soddisfano delle condizioni di tipo Levi; infine i coefficienti dei termini di ordine inferiore appartengono a un'opportuna classe Gevrey anisotropa. Nella dimostrazione viene utilizzata la tecnica della quasi-simmetrizzazione di sistemi di tipo Sylvester, adattata alle classi Gevrey multi-anisotrope e tenendo conto dei termini di ordine inferiore.*

Summary. – *We prove the well-posedness of the Cauchy Problem for first order weakly hyperbolic systems in the multi-anisotropic Gevrey classes, that generalize the standard Gevrey spaces. The result is obtained under the following hypotheses: the principal part is weakly hyperbolic with constant coefficients, the lower order terms satisfy some Levi-type conditions; and lastly the coefficients of the lower order terms belong to a suitable anisotropic Gevrey class. In the proof it is used the quasi-symmetrization of Sylvster-type systems, adapted to the case of the multi-anisotropic Gevrey classes and taking into account the lower order terms.*

Introduction.

We consider the Cauchy problem for first order linear Kowalevskian $N \times N$ systems:

$$\begin{cases} \partial_t u(t,x) = A(t,x,D)u(t,x) + B(t,x)K(D)u(t,x) + f(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where $A(t,x,D)$ is a first order partial (or pseudo-) differential operator and $K(D)$ is a pseudo-differential operator of order zero.

We recall that for weakly hyperbolic systems (i.e. the eigenvalues of the symbol of $A(t,x,D)$ are pure imaginary), without supplementary conditions, the Cauchy problem is not well-posed in C^∞ for operators with C^∞ coefficients; an explicit counterexample is constructed for instance by Colombini-Spagnolo

[7] for a second order equation. Only for some classes of hyperbolic systems the C^∞ well-posedness is granted, for instance the symmetric systems (i.e. when the principal symbol is hermitian) or the smoothly symmetrizable systems, including the strictly hyperbolic case (i.e. the eigenvalues of the principal symbol of $A(t,x,D)$ are pure imaginary and distinct). For arbitrary weakly hyperbolic equations, a general result has been obtained in the Gevrey classes, that represent an intermediate case between the C^∞ and the analytic classes in which the well-posedness is given by the Cauchy-Kowalevsky Theorem.

Namely, consider the following global version of the standard Gevrey classes: we say that u belongs to $\gamma^s_{L^2},\ s>1$, if $u \in C^\infty(\mathbb{R}^n)$ and there is a positive constant C such that:

$$\|D^a u(x)\|_{L^2} \leq C^{|a|+1}(a!)^s, \quad \forall a \in \mathbb{N}^n. \tag{2}$$

Therefore, the Cauchy problem is well-posed in the Gevrey classes $\gamma^s_{L^2}$ for $1 \leq s < \frac{M}{M-1}$, where M denotes the maximal multiplicity of the eigenvalues of the matrix-symbol of the system, under the condition of the same Gevrey regularity for the coefficients. The proof of this result can be found in Larsson [20] and Cattabriga [6] (cf. also Hörmander [16] and Rodino [25]) in the case of equations with constant coefficients, Steinberg [26] in the case of equations with smooth characteristic roots, and lastly Bronstein [2] in the general scalar case and Kajitani [19] in the vector case. This result is optimal, unless we assume some additional conditions on the principal part and the lower order terms (see for instance Leray [21] and Leray-Ohya [22]).

In this paper we obtain a well-posedness theorem which generalizes in some cases the above mentioned results, enlarging the class of admissible initial data. Namely, let us begin by defining in short the multi-anisotropic Gevrey classes, previously studied by Friberg [10], Gindikin-Volevich [13], Corli [8], Hakobyan-Markaryan [14], [15] and Zanghirati [27].

We start with a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0\}$, assuming that the origin belongs to $\mathcal{P}$, all the other vertices have integer coordinates and the normal vector $\nu(x)$ at each point $x \in \partial\mathcal{P}$ (outside vertices and coordinate hyperplanes) has strictly positive components. Then, we set:

$$\frac{1}{\mu} = \min_{1 \leq j \leq n,\, x \in \partial\mathcal{P}} \nu_j(x)$$

and

$$k(a,\mathcal{P}) = \inf\{t > 0 : t^{-1}a \in \mathcal{P}\}, \quad \forall a \in \mathbb{R}^n_+.$$

In Section 1 we present in detail the properties of the polyhedra satisfying these conditions and define the following function associated to $\mathcal{P}$:

$$|\xi|_{\mathcal{P}} = \left(\sum_{v \in \mathcal{V}(\mathcal{P})} \xi^{2v} \right)^{\frac{1}{2\mu}}, \quad \forall \xi \in \mathbb{R}^n, \tag{3}$$

where $\mathcal{V}(\mathcal{P})$ is the set of the vertices of $\mathcal{P}$.

Therefore, in Section 2 we define the multi-anisotropic Gevrey classes $\gamma_{L^2}^{s,\mathcal{P}}$, $s > 1$, associated to the complete polyhedron $\mathcal{P}$: namely, we say that u belongs to $\gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n)$ if $u \in C^\infty(\mathbb{R}^n)$ and there is a constant $C > 0$ such that:

$$\|D^a u\|_{L^2} \leq C^{|a|+1} (\mu k(a, \mathcal{P}))^{s\mu k(a,\mathcal{P})}, \quad \forall a \in \mathbb{N}^n. \tag{4}$$

In particular, for every $\mathcal{P}$ we have the inclusion:

$$\gamma_{L^2}^{s}(\mathbb{R}^n) \subset \gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n).$$

In Section 2 we also present the properties of the multi-anisotropic Gevrey classes. In particular, we prove the equivalence with a new definition of $\gamma_{L^2}^{s,\mathcal{P}}$ by means of the Fourier transform (cf. Theorem 2.1); namely, we will prove that u belongs to $\gamma_{L^2}^{s,\mathcal{P}}$ if and only if its Fourier transform $\hat{u}$ satisfies for suitable positive constants C, ε the condition:

$$\left\| \hat{u}(\xi) \exp\left(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}} \right) \right\|_{L^2} \leq C.$$

Then we discuss the inclusions among the classes associated to different polyhedra and with different Gevrey orders, and finally describe their topology and algebraic properties. Note that the classes $\gamma_{L^2}^{s,\mathcal{P}}$ are not closed under multiplication, as already observed by Hakobyan-Markaryan [14]; we study the largest class $\gamma_{L^2}^{s,\mathcal{P}^*}$ such that $\gamma_{L^2}^{s,\mathcal{P}^*} \cdot \gamma_{L^2}^{s,\mathcal{P}} \subset \gamma_{L^2}^{s,\mathcal{P}}$ (see Definition 1.3 and Proposition 2.3).

In this functional frame we study the Cauchy problem (1) for first order systems. In particular, we deal with N-block Sylvester systems, that will be described in formulas (22) and (23).

Let us announce our main result, cf. Theorem 3.3.

We fix a polyhedron $\mathcal{P} \subset \mathbb{R}^n$ and ask in (1) *the following conditions:*

1. *the principal part $A(t, x, D)$ is a Sylvester block matrix, has constant coefficients (we shall then simply write $A(D)$), is weakly hyperbolic and its eigenvalues have maximal multiplicity equal to M;*

2. *the coefficients of the lower order terms $B(t, x)$ are in $C^\infty([0, T], \gamma_{L^2}^{s,\mathcal{P}^*}(\mathbb{R}^n))$, where $\mathcal{P}^*$ is associated to $\mathcal{P}$ as before, cf. Definition* 1.3;

3. *$f(t, x)$ belongs to $C^\infty([0, T], \gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n))$;*

4. *the lower order terms satisfy the following Levi-type condition for a positive constant $k < M$:*

$$|K(\xi)| \leq C \frac{|\xi|_{\mathcal{P}}^k}{\langle \xi \rangle^{M-1}}, \quad \forall \xi \in \mathbb{R}^n. \tag{5}$$

Then the Cauchy problem (1) *admits a unique solution in* $C^\infty([0,T],\gamma^{s,\mathcal{P}}_{L^2}(\mathbb{R}^n))$ *for any data in* $\gamma^{s,\mathcal{P}}_{L^2}(\mathbb{R}^n)$ *if:*

$$1 < s < \frac{M}{k}.$$

Similarly, we may also treat weakly hyperbolic partial differential operators of higher order, which can be reduced to first order systems in the standard way. Let us give the following elementary example of a fourth order operator in $\mathbb{R}_t \times \mathbb{R}^2_{x,y}$ and test on it our result, compared with Bronstein [2].

We fix first the polyhedron $\mathcal{P} \subset \mathbb{R}^2$ of vertices:

$$\mathcal{V}(\mathcal{P}) = \{(0,0),(0,3),(1,0)\}.$$

The corresponding multi-anisotropic Gevrey classes $\gamma^{s,\mathcal{P}}_{L^2}$ are then defined by the condition:

$$\|D^i_x D^j_y u\| \le C^{i+j+1}(3i+j)^{s(3i+j)}, \quad \forall (i,j) \in \mathbb{N}^2.$$

Such $\gamma^{s,\mathcal{P}}_{L^2}$ represent an example of the simpler anisotropic Gevrey classes, relevant subset of the multi-anisotropic ones, cf. Section 1; for them $\mathcal{P} = \mathcal{P}^*$ and therefore $\gamma^{s,\mathcal{P}^*}_{L^2}$ coincides with $\gamma^{s,\mathcal{P}}_{L^2}$.

For the 4−th order operator $P = P(t,x,y,D_x,D_y,D_t)$, the conditions of Theorem 3.3 with $M = 4, k = 3$ turn out to be equivalent to the following:

$$P = P_4(D_t,D_x,D_y) + P_3(t,x,y,D_t,D_x,D_y) + \\ P_2(t,x,y,D_t,D_x,D_y) + P_1(t,x,y,D_t,D_x,D_y) + c_6(t,x,y),$$

where $P_4(D_t,D_x,D_y)$ is any weakly hyperbolic operator with constant coefficients, and the lower order terms are of the form:

$$\begin{aligned} P_3 &= c_1(t,x,y)D_y^3 \\ P_2 &= c_2(t,x,y)D_y^2 \\ P_1 &= c_3(t,x,y)D_x + c_4(t,x,y)D_y + c_5(t,x,y)D_t \end{aligned}$$

with coefficients $c_i(t,x,y) \in \gamma^{s,\mathcal{P}}_{L^2}(\mathbb{R}^2)$.

So Theorem 3.3 ensures the well-posedness in $\gamma^{s,\mathcal{P}}_{L^2}$, for $1 < s < \frac{4}{3}$. Observe that $\gamma^{\frac{4}{3},\mathcal{P}}_{L^2}(\mathbb{R}^2)$ strictly includes $\gamma^{\frac{4}{3}}_{L^2}(\mathbb{R}^2)$ given by the result of Bronstein [2].

For more examples involving the multi-anisotropic Gevrey classes, we address to Section 3.

The techniques used in the proofs are Fourier analysis, the quasi-symmetrization of Sylvester matrices (cf. D'Ancona-Spagnolo [9], Jannelli [17]), approximated energy estimates and the algebraic properties of the multi-anisotropic Gevrey classes proved in Section 2.

Concerning the quasi-symmetrization, the novelty here with respect to the preceding literature is that we use it to deal with the lower order terms; this shows in particular that the standard Gevrey results under Levi conditions (cf. [22]) can be recaptured, at least in part, by means of this technique.

1. – Complete polyhedra.

In this section we recall some definitions and properties concerning complete polyhedra and prepare the study of the multi-anisotropic Gevrey classes treated in the next section.

We recall that a convex polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$. To $\mathcal{P}$ is uniquely associated a finite set $\mathcal{V}(\mathcal{P})$ of linearly independent points, called the set of vertices of $\mathcal{P}$, that is the smallest set whose convex hull is $\mathcal{P}$.

Moreover, if $\mathcal{P}$ has non-empty interior and the origin belongs to $\mathcal{P}$, there exists a finite set of vectors in $\mathbb{R}^n$:

$$\mathcal{N}(\mathcal{P}) = \mathcal{N}_0(\mathcal{P}) \bigcup \mathcal{N}_1(\mathcal{P}),$$

such that $|\nu| = 1, \forall \nu \in \mathcal{N}_0(\mathcal{P})$, and

$$\mathcal{P} = \{z \in \mathbb{R}^n | \nu \cdot z \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P}), \quad \nu \cdot z \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\}.$$

The boundary of $\mathcal{P}$ is made of faces of equation:

$$\begin{aligned} \nu \cdot z = 0 \quad &\text{if} \quad \nu \in \mathcal{N}_0(\mathcal{P}), \\ \nu \cdot z = 1 \quad &\text{if} \quad \nu \in \mathcal{N}_1(\mathcal{P}). \end{aligned}$$

DEFINITION 1.1. – *A complete polyhedron is a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+$ satisfying the following conditions:*

1. $\mathcal{V}(\mathcal{P}) \subset \mathbb{N}^n$ *(i.e. all vertices have integer coordinates);*
2. *the origin* $(0, 0, \ldots, 0)$ *belongs to* $\mathcal{P}$*;*
3. *dim*$(\mathcal{P}) = n$*;*
4. $\mathcal{N}_0(\mathcal{P}) = \{e_1, e_2, \ldots, e_n\}$, *with* $e_j = (0, 0, \ldots, 0, 1_{j-th}, 0, \ldots, 0) \in \mathbb{R}^n$ *for* $j = 1, \ldots, n$*;*
5. *every* $\nu \in \mathcal{N}_1(\mathcal{P})$ *has strictly positive components.*

We note that 5. means that the following inclusion is valid:

$$Q(x) = \{y \in \mathbb{R}^n | 0 \leq y \leq x\} \subset \mathcal{P} \qquad \text{if } x \in \mathcal{P},$$

and if x belongs to a face of $\mathcal{P}$ and $y > x$ then $y \notin \mathcal{P}$.

We can associate to a partial differential operator with constant coefficients a convex polyhedron as in the following:

DEFINITION 1.2. – *Let* $P(D) = \sum_{|a| \leq m} c_a D^a$ $(c_a \in \mathbb{C})$ *be a differential operator in* $\mathbb{R}^n$ *and* $P(\xi) = \sum_{|a| \leq m} c_a \xi^a, \xi \in \mathbb{R}^n$, *its characteristic polynomial. The Newton* *polyhedron, or characteristic polyhedron, associated to* $P(D)$*, or equivalently to* $P(\xi)$*, is the convex hull of the set:*

$$\{0\} \bigcup \{a \in \mathbb{N}^n : c_a \neq 0\}.$$

We observe that the Newton polyhedron of an operator $P(D)$ that is hypoelliptic in the sense of Hörmander [16] is complete, but the converse is not true in general (for the proof see f.i. Friberg [10] or Boggiatto-Buzano-Rodino [1]).

REMARK 1. – *We can consider also polyhedra with rational vertices instead of integer vertices, as in Zanghirati* [27]; *the properties in the following remain valid also in this case.*

There follow some notations related to a complete polyhedron $\mathcal{P}$:
$\mathcal{F}_v(\mathcal{P}) = \{a \in \mathcal{P} : v \cdot a = 1\}, \ \forall v \in \mathcal{N}_1(\mathcal{P}),$ the face of $\mathcal{P}$ with normal vector v;
$\mathcal{F} = \bigcup_{v \in \mathcal{N}_1(\mathcal{P})} \mathcal{F}_v(\mathcal{P})$ the exterior boundary of $\mathcal{P}$;
$\mathcal{V}(\mathcal{P})$ the set of vertices of $\mathcal{P}$;
$\delta\mathcal{P} = \{a \in \mathbb{R}^n_+ : \delta^{-1} a \in \mathcal{P}\}, \ \delta > 0$;
$k(a, \mathcal{P}) = \inf\{t > 0 : t^{-1} a \in \mathcal{P}\} = \max_{v \in \mathcal{N}_1(\mathcal{P})} v \cdot a, \ \forall a \in \mathbb{R}^n_+$;
$m_j = m_j(\mathcal{P}) = \max_{v \in \mathcal{V}(\mathcal{P})} v_j, \ j = 1, \ldots, n$;
$\mu_j = \mu_j(\mathcal{P}) = \max_{v \in \mathcal{N}_1(\mathcal{P})} v_j^{-1}$;
$\mu = \mu(\mathcal{P}) = \max_{j=1,\ldots,n} \mu_j$ the formal order of $\mathcal{P}$;
$\mu^{(0)}(\mathcal{P}) = \min_{\gamma \in \mathcal{V}(\mathcal{P}) \setminus \{0\}} |\gamma|$ the minimum order of $\mathcal{P}$;
$\mu^{(1)}(\mathcal{P}) = \max_{\gamma \in \mathcal{V}(\mathcal{P})} |\gamma|$ the maximum order of $\mathcal{P}$;
$q(\mathcal{P}) = \left(\frac{\mu(\mathcal{P})}{\mu_1(\mathcal{P})}, \ldots, \frac{\mu(\mathcal{P})}{\mu_n(\mathcal{P})}\right)$;
$|\xi|_{\mathcal{P}} = (\sum_{v \in \mathcal{V}(\mathcal{P})} \xi^{2v})^{\frac{1}{2\mu}}, \ \forall \xi \in \mathbb{R}^n$, the weight function associated to the polyhedron $\mathcal{P}$.

In the following, we shall sometimes refer to the asymptotically equivalent expressions $\left(\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v|\right)^{\frac{1}{\mu}}$, or else $\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v|^{\frac{1}{\mu}}$.

REMARK 2. – *The function* $|\xi|_{\mathcal{P}}$ *is a particular case of weight function, as defined in Liess-Rodino* [23]. *Writing* $\langle \xi \rangle = 1 + |\xi|$*, we have for positive constants* c, C*:*

$$c\langle \xi \rangle^{\frac{\mu^{(0)}}{\mu}} \leq |\xi|_{\mathcal{P}} \leq C\langle \xi \rangle^{\frac{\mu^{(1)}}{\mu}}.$$

REMARK 3. – *Any* $a \in \mathbb{R}^n_+$ *belongs to the boundary of* $k(a, \mathcal{P})\mathcal{P}$.
$k(a, \mathcal{P})$ *is bounded as follows:*

$$\frac{|a|}{\mu^{(0)}} \leq k(a, \mathcal{P}) \leq \frac{|a|}{\mu^{(1)}}. \tag{6}$$

PROPOSITION 1.1. – *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ with natural (or rational) vertices $v^l = (v_1^l, \dots, v_n^l)$, $l = 1, \dots, n(\mathcal{P})$, then:*

1. *for every $j = 1, \dots, n$, there is a vertex v^{l_j} of $\mathcal{P}$ such that:*

$$(0, \dots, 0, v_j^{l_j}, 0, \dots, 0) = v_j^{l_j} e_j, \quad v_j^{l_j} = \max_{v \in \mathcal{P}} v_j = m_j.$$

2. *We have:*

$$\mathcal{P} = \bigcap_{\nu \in \mathcal{N}_1(\mathcal{P})} \{a \in \mathbb{R}_+^n : \nu \cdot a \leq 1\};$$

3. *for every $j = 1, \dots, n$ there is at least one $\nu \in \mathcal{N}_1(\mathcal{P})$ such that:*

$$m_j = \nu_j^{-1};$$

4. *if the boundary $\mathcal{F}$ of $\mathcal{P}$ has at least one vertex lying outside the coordinate axes, then the formal order $\mu(\mathcal{P})$ is strictly greater than the maximum order $\mu^{(1)}(\mathcal{P})$.*

5. *if a belongs to $\mathcal{P}$, then:*

$$|\xi^a| \leq \sum_{l=1}^{n(\mathcal{P})} |\xi^{v^l}|, \quad \left(\xi^a = \prod_{j=1}^{n} \xi_j^{a_j}\right).$$

For the proof let us refer for example to Boggiatto-Buzano-Rodino [1] and Hakobyan-Markaryan [14].

Following Hakobyan-Markaryan [14], we define the complementary polyhedron associated to a complete polyhedron $\mathcal{P}$ and give a subadditive property for the related weight functions.

DEFINITION 1.3. – *Let $\mathcal{P}$ be a complete polyhedron and let μ_j, $j = 1, \dots, n$, be as before, then we define the complementary polyhedron $\mathcal{P}^*$ associated to $\mathcal{P}$ as:*

$$\mathcal{P}^* = \{x \in \mathbb{R}_+^n : \ x \cdot \lambda^0 \leq 1\}, \tag{7}$$

where:

$$\lambda^0 = \left(\frac{1}{\mu_1}, \dots, \frac{1}{\mu_n}\right). \tag{8}$$

REMARK 4. – *The polyhedron $\mathcal{P}^*$ has only one face (besides the faces on the coordinate hyperplanes) and $\mathcal{P} \subset \mathcal{P}^*$; the polyhedra $\mathcal{P}$ and $\mathcal{P}^*$ coincide if and only if $\mathcal{P}$ has only one face (the anisotropic case, see the following Example 2).*

REMARK 5. – *The formal orders of $\mathcal{P}$ and $\mathcal{P}^*$ coincide by definition.*

The following proposition shows an important property of the complementary polyhedron, involving the weight functions (for the proof we address to Hakobyan-Markaryan [14]) .

PROPOSITION 1.2. – *Let $\mathcal{P}$ be a complete polyhedron and let $\mathcal{P}^*$ be its complementary polyhedron as in* (7), *then the associated weight functions satisfy for a constant $C > 0$:*

$$|\xi + \eta|_{\mathcal{P}} \le C(|\xi|_{\mathcal{P}} + |\eta|_{\mathcal{P}^*}), \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{9}$$

In the opposite direction, for all polyhedra $\mathcal{P}'$ such that $\mu(\mathcal{P}') = \mu(\mathcal{P})$ and $\mathcal{P}^* \not\subset \mathcal{P}'$, the property (9) *is not valid; namely there are two sequences $\{\xi^{(k)}\}_{k\in\mathbb{N}}, \{\eta^{(k)}\}_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ such that:*

$$\lim_{k\to\infty} \frac{|\xi^{(k)} + \eta^{(k)}|_{\mathcal{P}}}{|\xi^{(k)}|_{\mathcal{P}} + |\eta^{(k)}|_{\mathcal{P}'}} = \infty.$$

REMARK 6. – *When $\mathcal{P}$ has more than one face, the inclusion $\mathcal{P} \subset \mathcal{P}^*$ is strict, so it follows from Proposition* 1.2 *that there are two sequences $\{\xi^{(k)}\}_{k\in\mathbb{N}}, \{\eta^{(k)}\}_{k\in\mathbb{N}}$ such that:*

$$\lim_{k\to\infty} \frac{|\xi^{(k)} + \eta^{(k)}|_{\mathcal{P}}}{|\xi^{(k)}|_{\mathcal{P}} + |\eta^{(k)}|_{\mathcal{P}}} = \infty.$$

For sake of evidence to the reader, let us add here a direct proof of this last assertion. Consider any complete polyhedron $\mathcal{P} \subset \mathbb{R}^n$ having a vertex not lying in the coordinate axes. Arguing by contradiction, if the subadditive property is valid:

$$|\xi + \eta|_{\mathcal{P}} \le C(|\xi|_{\mathcal{P}} + |\eta|_{\mathcal{P}}),$$

for a positive constant C, then by iteration also the generalized subadditive property is true:

$$\left|\sum_{j=1}^{m} \xi^{(j)}\right|_{\mathcal{P}} \le C \sum_{j=1}^{m} |\xi^{(j)}|_{\mathcal{P}} \tag{10}$$

for any $m \in \mathbb{N}$ and for a positive constant C (eventually depending on m).

So if ν is the normal vector to the hyperplane containing the vertices $\{m_j e_j,\ j = 1, \dots n\}$ of $\mathcal{P}$ lying on the coordinate axes, then the other vertices v of $\mathcal{P}$ satisfy:

$$v \cdot \nu > 1.$$

So, taking $m = n$ and $\xi^{(j)} = t^{\nu_j} e_j,\ j = 1, \dots, n$, we have:

$$\left|\sum_{1\le j\le n} \xi^{(j)}\right|_{\mathcal{P}} = \left(\sum_{v\in\mathcal{V}(\mathcal{P})} |t^{\nu\cdot v}|\right)^{\frac{1}{\mu}} > C \sum_{1\le j\le n} |t^{\nu_j m_j}|^{\frac{1}{\mu}} = C \sum_{1\le j\le n} |\xi^{(j)}|_{\mathcal{P}},\ \forall C > 0$$

for t large, and so (10) *can't be satisfied for all $t > 0$ by any $C > 0$.*

In fact, for all $j = 1, \dots, n$, $v_j m_j = 1$, whereas in the second sum there exists $v \in \mathcal{V}(\mathcal{P})$ such that $v \cdot v > 1$.

Now we give some examples of complete polyhedra:

1) If $P(D)$ is an elliptic operator of order m, then its Newton polyhedron is complete and is the polyhedron of vertices:

$$\mathcal{V}(\mathcal{P}) = \{0, me_j,\ j = 1, \dots, n\},$$

therefore $\mathcal{P}$ is so defined:

$$\mathcal{P} = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \dots, n,\ \sum_{j=1}^{n} x_j \leq m\}.$$

The set $\mathcal{N}_1(\mathcal{P})$ is reduced to a vector:

$$v = m^{-1} \sum_{j=1}^{n} e_j = (m^{-1}, \dots, m^{-1})$$

and:

$$m_j(\mathcal{P}) = \mu_j(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \mu(\mathcal{P}) = m, \quad j = 1, \dots, n;$$
$$k(a, \mathcal{P}) = m^{-1}|a| = m^{-1} \textstyle\sum_{j=1}^{n} a_j, \quad a \in \mathbb{R}^n_+.$$

The weight associated to $\mathcal{P}$ is:

$$|\xi|_{\mathcal{P}} = \langle \xi \rangle = 1 + |\xi|.$$

2) If $P(\xi)$ is a quasi-elliptic polynomial of order m (cf. for example Hörmander [16], Rodino [25] and Zanghirati [28]), then its characteristic polyhedron $\mathcal{P}$ is complete and has vertices:

$$\mathcal{V}(\mathcal{P}) = \{0, m_j e_j,\ j = 1, \dots, n\},$$

where m_j are fixed strictly positive integers.

The set $\mathcal{N}_1(\mathcal{P})$ is again reduced to a vector:

$$v = \sum_{j=1}^{n} m_j^{-1} e_j.$$

The polyhedron $\mathcal{P}$ is given by:

$$\mathcal{P} = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 0, \dots, n,\ \sum_{j=1}^{n} m_j^{-1} x_j \leq 1\}$$

and:

$$\mu_j(\mathcal{P}) = m_j, \quad j = 1, \dots, n;$$

$$\mu^{(0)}(\mathcal{P}) = \min_{j=1,\dots,n} m_j;$$

$$\mu(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \max_{j=1,\dots,n} m_j = m;$$

$$k(a, \mathcal{P}) = m^{-1} q \cdot a, \quad a \in \mathbb{R}^n_+.$$

The weight associated to $\mathcal{P}$ is:

$$|\xi|_{\mathcal{P}} = (1 + |\xi_1^{m_1}| + \ldots + |\xi_n^{m_n}|)^{\frac{1}{m}}.$$

3) Take as $\mathcal{P} \subset \mathbb{R}^2$ the polyhedron of vertices:

$$\mathcal{V}(\mathcal{P}) = \{(0,0), (0,3), (1,2), (2,0)\},$$

then $\mathcal{P}$ is complete and:

$$\mathcal{N}_1(\mathcal{P}) = \left\{ v_1 = \left(\frac{1}{3}, \frac{1}{3}\right),\ v_2 = \left(\frac{1}{2}, \frac{1}{4}\right) \right\}.$$

So $\mathcal{P}$ is defined by:

$$\mathcal{P} = \left\{ (x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0,\ \frac{1}{3}x + \frac{1}{3}y \leq 1,\ \frac{1}{2}x + \frac{1}{4}y \leq 1 \right\}.$$

The quantities related to $\mathcal{P}$ are the following:

$$m_1(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = 2,$$
$$m_2(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = m(\mathcal{P}) = 3,$$
$$\mu(\mathcal{P}) = 4.$$

We observe that in this case the formal order $\mu(\mathcal{P})$ is strictly bigger than the maximum order $m(\mathcal{P})$, as $\mathcal{P}$ has a vertex lying outside the coordinate axes. We have:

$$k(a,\mathcal{P}) = \begin{cases} a \cdot v_1 = \frac{1}{3}a_1 + \frac{1}{3}a_2 & \text{if } a_2 \geq 2a_1 \\ a \cdot v_2 = \frac{1}{2}a_1 + \frac{1}{4}a_2 & \text{if } a_2 \leq 2a_1 \end{cases}.$$

The weight associated to $\mathcal{P}$ is:

$$|\xi|_{\mathcal{P}} = (1 + |\xi_1^2| + |\xi_2^3| + |\xi_1\xi_2^2|)^{\frac{1}{4}}.$$

2. – The multi-anisotropic Gevrey classes.

We now define the multi-anisotropic Gevrey classes associated to a complete polyhedron.

We give two equivalent definitions, one in terms of the estimate of the derivatives, the other in terms of the Fourier transform. Aiming at a self-contained presentation, we also study the topology, the inclusion of multi-anisotropic Gevrey classes of different order and associated to different polyhedra, and the related algebraic properties.

DEFINITION 2.1. – *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and $s \in \mathbb{R},\ s > 1$. We* *denote by* $\gamma_{L^2}^{s,\mathcal{P}}$ *the multi-anisotropic Gevrey class of order* s *associated to* $\mathcal{P}$*, i.e. the set of all* $u \in C^\infty(\mathbb{R}^n)$ *such that:*

$$\exists C > 0 : \ \|D^a u(x)\|_{L^2} \leq C^{|a|+1}(\mu k(a,\mathcal{P}))^{s\mu k(a,\mathcal{P})}, \quad \forall a \in \mathbb{N}^n, \tag{11}$$

with the notations of the previous section.

REMARK 7. – *The following conditions are equivalent:*

1. *u belongs to* $\gamma_{L^2}^{s,\mathcal{P}}$;
2. *there is a constant* $C > 0$ *such that:*

$$\|D^a u(x)\|_{L^2} \leq C^{j+1} j^{js}$$

for any $a \in \mathbb{N}^n$*, where* $j = j(a) = \min\{i \in \mathbb{N} : \ k(a,\mathcal{P}) \leq i\}$.

This follows immediately from the inequality:

$$j - 1 \leq k(a,\mathcal{P}) \leq j.$$

REMARK 8. – *It follows from* (11) *that if u belongs to* $\gamma_{L^2}^{s,\mathcal{P}}$*, then u satisfies for a new constant* $C > 0$*:*

$$\|D^a u(x)\|_{L^\infty} \leq C^{|a|+1}(\mu k(a,\mathcal{P}))^{s\mu k(a,\mathcal{P})}, \quad \forall a \in \mathbb{N}^n,$$

in view of the Sobolev embedding theorem.

From now on $\mathcal{P}$ will denote a complete polyhedron and we shall suppose $s > 1$ for the Gevrey order.

The space $\gamma_{L^2}^{s,\mathcal{P}}$ can be endowed with a natural topology. Namely, we denote by $\gamma_{L^2,C}^{s,\mathcal{P}}$, $C > 0$, the space of the functions $u \in C^\infty(\mathbb{R}^n)$ such that:

$$\|u\|_{s,\mathcal{P},C} = \sup_{a \in \mathbb{N}^n} C^{-|a|}(\mu k(a,\mathcal{P}))^{-s\mu k(a,\mathcal{P})}\|D^a u(x)\|_{L^2} < \infty. \tag{12}$$

With such a norm, $\gamma_{L^2,C}^{s,\mathcal{P}}$ is a Banach space. Then:

$$\gamma_{L^2}^{s,\mathcal{P}} = \bigcup_{C>0} \gamma_{L^2,C}^{s,\mathcal{P}},$$

endowed with the topology of inductive limit.

The multi-anisotropic Gevrey functions can be defined also by means of the Fourier transform, using the weight $|\xi|_{\mathcal{P}}$, as in the following theorem.

THEOREM 2.1. – *The following two conditions are equivalent:*

1. *u belongs to* $\gamma_{L^2}^{s,\mathcal{P}}$;
2. *there exist two constants* $C, \varepsilon > 0$ *such that the Fourier transform of u satisfies:*

$$\left\| \hat{u}(\xi) \exp\left(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}\right) \right\|_{L^2} \leq C. \tag{13}$$

We use the following two lemmas. The proofs are analogous to the ones in Corli [8] and Calvo [3] for the local Gevrey classes $G^{s,\mathcal{P}}$; but here we don't need a cut-off function and can proceed in an easier way, thanks to the properties of the Fourier transform in the space L^2. In order to be self-contained, we give full details of the argument.

LEMMA 2.1. – *If u belongs to $\gamma_{L^2}^{s,\mathcal{P}}$, then there is a constant $C > 0$ such that:*

$$\left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}} + N^s}{CN^s} \right)^{\mu N} \right\|_{L^2} \leq C, \quad N = 1, 2, \ldots$$

PROOF. – By means of the properties of the Fourier transform we have:

$$\|\xi^a \hat{u}(\xi)\|_{L^2} = \|\widehat{D^a u}(\xi)\|_{L^2} = (2\pi)^{\frac{n}{2}} \|D^a u(x)\|_{L^2} \leq C(CN^s)^{\mu k(a,\mathcal{P})}$$

if $a \cdot v \leq N$, for all $v \in \mathcal{N}_1(\mathcal{P})$, i.e. $k(a, \mathcal{P}) \leq N$.

Let now $a = vN$ for any $v \in \mathcal{V}(\mathcal{P})$, then summing up the previous inequalities for $a = 0,\ a = vN$, for all $v \in \mathcal{V}(\mathcal{P})$, we obtain:

$$\|\hat{u}(\xi)\|_{L^2} N^{s\mu N} + \sum_{v \in \mathcal{V}(\mathcal{P})} \|\hat{u}(\xi)\xi^{vN}\|_{L^2} \leq C(CN^s)^{\mu N}.$$

Using the following inequality:

$$n(\mathcal{P})^{\mu N - 1} \sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^{vN}| \leq |\xi|_{\mathcal{P}}^{N\mu} \leq 2^{n(\mathcal{P})(\mu N - 1)} \sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^{vN}|,$$

where $n(\mathcal{P})$ denotes the number of vertices of $\mathcal{P}$ different from the origin, we can conclude that:

$$\begin{aligned} \left\| \hat{u}(\xi) \frac{N^{s\mu N} + \sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^{vN}|}{C(CN^s)^{\mu N}} \right\|_{L^2} &\leq \left\| \hat{u}(\xi) \frac{N^{s\mu N} + \frac{|\xi|_{\mathcal{P}}^{N\mu}}{n(\mathcal{P})^{(\mu N - 1)}}}{C(CN^s)^{\mu N}} \right\|_{L^2} \\ &\leq \left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}} + N^s}{C' N^s} \right)^{\mu N} \right\|_{L^2} \leq C, \end{aligned}$$

for all $N = 1, 2, \ldots$. □

LEMMA 2.2. – *The function u is of class $\gamma_{L^2}^{s,\mathcal{P}}$ if and only if its Fourier transform $\hat{u}$ satisfies:*

$$\left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}}}{CN^s} \right)^{\mu N} \right\|_{L^2} \leq C', \quad N = 1, 2, \ldots \tag{14}$$

PROOF. – The proof of the necessity of (14) is obvious from Lemma 2.1.

Now let us fix $a \in \mathbb{N}^n$ and assume that condition (14) is satisfied.

We know that:

$$|\xi^a| \leq |\xi|_{\mathcal{P}}^{\mu k(a,\mathcal{P})}.$$

In fact, given $a \in \mathbb{N}^n$, then $\frac{a}{k(a,\mathcal{P})}$ belongs to $\mathcal{F}$ and so, by the definition of convex hull, told $v^{l_1}, \ldots, v^{l_r}$ the vertices of the face where $\frac{a}{k(a,\mathcal{P})}$ lies, we have:

$$a = k(a,\mathcal{P}) \sum_{i=1}^{r} \lambda_i v^{l_i}, \quad \sum_{i=1}^{r} \lambda_i = 1, \quad \lambda_i \geq 0 \ ,$$

and using the inequality:

$$x^\beta \leq \sum_{a \in A} c_a x^a,$$

for any linear convex combination $\beta = \sum_{a\in A} C_a a$, where A is a given subset of $\mathbb{R}^n_+$ (cf. [1]), we obtain:

$$\text{(15)} \qquad \begin{aligned} |\xi^a| &= \prod_{j=1}^{n} |\xi_j^{a_j}| \leq \sum_{i=1}^{r} \lambda_i \left(\prod_{j=1}^{n} |\xi_j|^{v_j^{l_j}} \right)^{k(a,\mathcal{P})} \\ &\leq \left(\sum_{v^l \in \mathcal{V}(\mathcal{P})} \prod_{j=1}^{n} |\xi_j|^{2v_j^{l_j}} \right)^{\frac{1}{2}k(a,\mathcal{P})} \leq |\xi|_{\mathcal{P}}^{\mu k(a,\mathcal{P})}. \end{aligned}$$

Now we can estimate:

$$\begin{aligned} \|D^a u(x)\|_{L^2} &= (2\pi)^{-\frac{n}{2}} \|\widehat{D^a u}(\xi)\|_{L^2} = (2\pi)^{-\frac{n}{2}} \|\xi^a \hat{u}(\xi)\|_{L^2} \\ &\leq (2\pi)^{-\frac{n}{2}} \||\xi|_{\mathcal{P}}^{\mu k(a,\mathcal{P})} \hat{u}(\xi)\|_{L^2} \leq (2\pi)^{-\frac{n}{2}} \||\xi|_{\mathcal{P}}^{\mu k(a,\mathcal{P})} |\xi|_{\mathcal{P}}^{-\mu N} |\xi|_{\mathcal{P}}^{\mu N} \hat{u}(\xi)\|_{L^2} \\ &\leq C(CN^s)^{\mu N} \||\xi|_{\mathcal{P}}^{\mu k(a,\mathcal{P}) - \mu N}\|_{L^2} \leq C'(C'N^s)^{\mu N} \end{aligned}$$

if N is big enough to satisfy $\mu k(a,\mathcal{P}) < \mu N - \delta$ with $\||\xi|_{\mathcal{P}}^{-\delta}\|_{L^2} < \infty$, then condition (11) is trivially satisfied by eventually enlarging C. This implies that u belongs to $\gamma_{L^2}^{s,\mathcal{P}}$. □

Now we prove Theorem 2.1.

PROOF. – Let us suppose that u belongs to $\gamma_{L^2}^{s,\mathcal{P}}$, then in view of Lemma 2.2, the Fourier transform of u satisfies:

$$\left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}}^{\frac{1}{s}}}{CN} \right)^{s\mu N} \right\|_{L^2} \leq C, \quad N = 1, 2, \ldots .$$

As $N! \leq N^N$ and $N^N \leq e^N N!$, this condition is equivalent to the following:

$$\left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} C''^{N+1} \right)^{s\mu} \right\|_{L^2} \leq C, \quad N = 1, 2, \ldots .$$

Taking $\varepsilon = 2^{-\frac{1}{s\mu}} C''$, we get:

$$\left\| \hat{u}(\xi) \left(\varepsilon^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \right)^{s\mu} \right\|_{L^2} \leq C \frac{1}{2\varepsilon^{s\mu}} \frac{1}{2^N} .$$

Summing up for $N = 1, 2, \ldots$:

$$\left\| \hat{u}(\xi) \left(\sum_{N=0}^{\infty} \varepsilon^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \right)^{s\mu} \right\|_{L^2} \leq \frac{C}{2\varepsilon^{s\mu}} \sum_{N=0}^{\infty} \frac{1}{2^N},$$

and hence for a new constant $\varepsilon > 0$:

$$\| \hat{u}(\xi) \exp(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}) \|_{L^2} \leq C$$

and so u satisfies (13).

Conversely, if $u \in L^2(\mathbb{R}^n)$ satisfies (13), i.e. $\| \hat{u}(\xi) \exp(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}) \|_{L^2} \leq C$, then, writing the Taylor expansion:

$$\exp(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}) = \left(\exp\left(\frac{\varepsilon}{\mu s} |\xi|_{\mathcal{P}}^{\frac{1}{s}} \right) \right)^{\mu s} = \left(\sum_{N=0}^{\infty} \left(\frac{\varepsilon}{\mu s} \right)^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \right)^{\mu s} \geq \left(\left(\frac{\varepsilon}{\mu s} \right)^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \right)^{\mu s},$$

for all $N = 1, 2, \ldots$; we get:

$$\left\| \hat{u}(\xi) \left(\left(\frac{\varepsilon}{\mu s} \right)^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \right)^{\mu s} \right\|_{L^2} \leq \left\| \hat{u}(\xi) \exp\left(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}} \right) \right\|_{L^2} \leq C,$$

that implies:

$$\left\| \hat{u}(\xi) \left(\frac{|\xi|_{\mathcal{P}}^{\frac{1}{s}}}{C' N} \right)^{s\mu N} \right\|_{L^2} \leq C',$$

so u belongs to $\gamma_{L^2}^{s,\mathcal{P}}$ by Lemma 2.2. □

Theorem 2.1 allows us to define in $\gamma_{L^2}^{s,\mathcal{P}}$ an equivalent topology to (12). Namely, for any $\varepsilon > 0$, we define the Banach space $\gamma_{L^2,\varepsilon}^{s,\mathcal{P}}$ as the set of the functions $u \in \gamma_{L^2}^{s,\mathcal{P}}$ for which the following norm is finite:

$$\|u\|_{L^2,\mathcal{P},\varepsilon} = \| \hat{u} \exp(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}) \|_{L^2}.$$

Then:

$$\gamma_{L^2}^{s,\mathcal{P}} = \bigcup_{\varepsilon>0} \gamma_{L^2,\varepsilon}^{s,\mathcal{P}}$$

endowed by the topology of inductive limit.

Now, using the two equivalent definitions of the multi-anisotropic Gevrey functions in Theorem 2.1, we can analyze some examples:

1) If $\mathcal{P}$ is the Newton polyhedron of an elliptic operator (see Example 1 of Section 1), then $\gamma_{L^2}^{s,\mathcal{P}}$ coincides with $\gamma_{L^2}^{s}$, the set of the standard s-Gevrey functions, i.e. of the functions $u \in C^\infty(\mathbb{R}^n)$ such that for a constant $C > 0$ it is satisfied:

$$\|D^a u(x)\|_{L^2} \le C^{|a|+1}(a!)^s, \quad \forall a \in \mathbb{N}^n,$$

cf. (2), or equivalently there are two constants $C, \varepsilon > 0$ such that:

$$\|\hat{u}(\xi)\exp(\varepsilon\langle\xi\rangle^{\frac{1}{s}})\|_{L^2} \le C.$$

2) If $\mathcal{P}$ is the Newton polyhedron of a quasi-elliptic operator (see Example 2 of Section 1), then:

$$\gamma_{L^2}^{s,\mathcal{P}} = \gamma_{L^2}^{s,q}, \quad \text{where } q = \left(\frac{m}{m_1}, \dots, \frac{m}{m_n}\right),$$

is the set of the anisotropic Gevrey functions (see Hörmander [16], Rodino [25], Zanghirati [28]), i.e. the functions $u \in C^\infty(\mathbb{R}^n)$ satisfying for a $C > 0$:

$$\|D^a u(x)\|_{L^2} \le C^{|a|+1} a^{sa\cdot q} = C^{|a|+1} a_1^{a_1\sigma_1} \dots a_n^{a_n\sigma_n}, \quad \forall a \in \mathbb{N}^n,$$

where $\sigma_i = sq_i,\ i = 1, \dots n$; or equivalently there are two constants $C, \varepsilon > 0$ for which it holds:

$$\|\hat{u}(\xi)\exp(\varepsilon(1 + |\xi_1^{m_1}| + \dots + |\xi_n^{m_n}|)^{\frac{1}{ms}}\|_{L^2} \le C.$$

3) If $\mathcal{P} \subset \mathbb{R}^2$ is the polyhedron of vertices

$$\mathcal{V}(\mathcal{P}) = \{(0,0), (0,3), (1,2), (2,0)\},$$

as in Example 3 of Section 1, then the multi-anisotropic Gevrey class $\gamma_{L^2}^{s,\mathcal{P}}$ is the set of the functions $u \in C^\infty(\mathbb{R}^n)$ satisfying for a constant $C > 0$:

$$\|D^a u(x)\|_{L^2} \le C^{|a|+1}(\mu k(a,\mathcal{P}))^{s\mu k(a,\mathcal{P})}, \quad \forall a \in \mathbb{N}^n,$$

where $k(a,\mathcal{P})$ is calculated in Example 3 of Section 1; or equivalently there are two constants $C, \varepsilon > 0$ such that:

$$\|\hat{u}(\xi)\exp(\varepsilon(1 + |\xi_1^2| + |\xi_2^3| + |\xi_1\xi_2^2|)^{\frac{1}{4s}})\|_{L^2} \le C.$$

In the next Proposition we study the inclusions of the multi-anisotropic Gevrey classes associated to different polyhedra and with different Gevrey orders.

Proposition 2.1. – *Let $\mathcal{P}, \mathcal{P}'$ be two complete polyhedra of formal orders μ and μ' respectively. Then, for any $s, s' > 1$, we have the following inclusions of Gevrey classes:*

1. *if $s' < s$, then $\gamma_{L^2}^{s',\mathcal{P}} \subset \gamma_{L^2}^{s,\mathcal{P}}$;*
2. *if $\mathcal{P} \subset \mathcal{P}'$, then $\gamma_{L^2}^{s,\mathcal{P}'} \subset \gamma_{L^2}^{s\frac{\mu'}{\mu},\mathcal{P}}$ if $\frac{\mu'}{\mu} \geq 1$;*
3. *more generally, for any $\mathcal{P}, \mathcal{P}'$, if $H = \max\{h \in \mathbb{Q} : h\mathcal{P} \subset \mathcal{P}'\}$, then $\gamma_{L^2}^{Hs,\mathcal{P}'} \subset \gamma_{L^2}^{s\frac{\mu'}{\mu},\mathcal{P}}$* if $H \geq 1$ and if $\frac{\mu'}{\mu} \geq 1$.

Remark 9. – *If $\frac{\mu'}{\mu} \leq 1$ or $H \leq 1$ we may reset Proposition 2.1, in order to have the Gevrey order bigger than 1. More precisely, we have the other cases:*

if $\mathcal{P} \subset \mathcal{P}'$, then $\gamma_{L^2}^{s\frac{\mu}{\mu'},\mathcal{P}'} \subset \gamma_{L^2}^{s,\mathcal{P}}$ if $\frac{\mu'}{\mu} \leq 1$ in case 2;
$\gamma_{L^2}^{s,\mathcal{P}'} \subset \gamma_{L^2}^{s\frac{1}{H}\frac{\mu'}{\mu},\mathcal{P}}$ if $H \leq 1$ and if $\frac{\mu'}{\mu} \geq 1$;
$\gamma_{L^2}^{Hs\frac{\mu}{\mu'},\mathcal{P}'} \subset \gamma_{L^2}^{s,\mathcal{P}}$ if $H \geq 1$ and if $\frac{\mu'}{\mu} \leq 1$;
$\gamma_{L^2}^{s\frac{\mu}{\mu'},\mathcal{P}'} \subset \gamma_{L^2}^{s\frac{1}{H},\mathcal{P}}$ if $H \leq 1$ and if $\frac{\mu'}{\mu} \leq 1$ in case 3.

Proof.

1. The proof is obvious, in view of (11).
2. Since u belongs to $\gamma_{L^2}^{s,\mathcal{P}'}$, then there is a constant $C > 0$ such that:

$$\|D^a u(x)\|_{L^2} \leq C^{|a|+1}(\mu' k(a,\mathcal{P}'))^{s\mu' k(a,\mathcal{P}')}, \ \forall a \in \mathbb{N}^n$$

and as $\mathcal{P} \subset \mathcal{P}'$ implies that $k(a,\mathcal{P}') \leq k(a,\mathcal{P})$, so u satisfies:

$$\|D^a u(x)\|_{L^2} \leq C'^{|a|+1}(\mu k(a,\mathcal{P}))^{s\mu\frac{\mu'}{\mu}k(a,\mathcal{P})}, \ \forall a \in \mathbb{N}^n$$

for a suitable constant $C' > 0$ depending on C, μ, μ'. This implies that u belongs to $\gamma_{L^2}^{s\frac{\mu'}{\mu},\mathcal{P}}$.

3. Let $\mathcal{P}, \mathcal{P}'$ be two complete polyhedra and let $H = \max\{h \in \mathbb{Q} : h\mathcal{P} \subset \mathcal{P}'\}$, then:

$$k(a,\mathcal{P}') \leq k(a,H\mathcal{P}) = \frac{1}{H}k(a,\mathcal{P}), \quad \forall a \in \mathbb{N}^n,$$

so, if u belongs to $\gamma_{L^2}^{Hs,\mathcal{P}'}$, then:

$$\|D^a u(x)\|_{L^2} \leq C'^{|a|+1}(\mu k(a,\mathcal{P}))^{s\mu\frac{\mu'}{\mu}k(a,\mathcal{P})}, \ \forall a \in \mathbb{N}^n$$

for a suitable constant $C' > 0$ depending on C, μ, μ', H. And so $u \in \gamma_{L^2}^{s\frac{\mu'}{\mu},\mathcal{P}}$. □

Corollary 2.1. – *Given $\mathcal{P}$, let $\mu^{(0)}, \mu^{(1)}$ be defined as in Section 2. Then we have the following inclusions involving the standard Gevrey classes:*

$$\gamma_{L^2}^{s} \subset \gamma_{L^2}^{s\frac{\mu}{\mu^{(1)}}} \subset \gamma_{L^2}^{s,\mathcal{P}} \subset \gamma_{L^2}^{s\frac{\mu}{\mu^{(0)}}}.$$

The proof is obvious, by using Proposition 2.1.

We can observe that the ratio $\frac{\mu}{\mu^{(1)}}$ depends on the slope of the faces of $\mathcal{P}$, so it can be as large as we want by varying $\mathcal{P}$; this implies that for all $k > 0$ and all $s > 1$ there is a complete polyhedron $\mathcal{P}$ such that $\gamma^k_{L^2} \subset \gamma^{s,\mathcal{P}}_{L^2}$.

We now present the algebraic properties of the multi-anisotropic Gevrey classes.

PROPOSITION 2.2.

1. *for any* $u, v \in \gamma^{s,\mathcal{P}}_{L^2}$, *we have* $u + v \in \gamma^{s,\mathcal{P}}_{L^2}$;
2. *for any* $u \in \gamma^{s,\mathcal{P}}_{L^2}$, $k \in \mathbb{C}$, *we have* $ku \in \gamma^{s,\mathcal{P}}_{L^2}$;
3. *for any* $u \in \gamma^{s,\mathcal{P}}_{L^2}$, $a \in \mathbb{N}^n$, *we have* $D^a u \in \gamma^{s,\mathcal{P}}_{L^2}$.

The proof follows easily from Definition 2.1 or from (13).

By multiplying two functions in the same multi-anisotropic Gevrey class, we don't remain in this class, except than in the anisotropic and standard case. The following proposition solves the problem of finding the largest class, whose product with $\gamma^{s,\mathcal{P}}_{L^2}$ is $\gamma^{s,\mathcal{P}}_{L^2}$.

PROPOSITION 2.3. – *Let* $\mathcal{P}$ *be a complete polyhedron and* $\mathcal{P}^*$ *its complementary polyhedron as in Definition* 1.3, *then for any* $s > 1$ *we have:*

$$\gamma^{s,\mathcal{P}}_{L^2} \cdot \gamma^{s,\mathcal{P}^*}_{L^2} \subset \gamma^{s,\mathcal{P}}_{L^2}, \tag{16}$$

whereas for any complete polyhedron $\mathcal{P}'$ *such that* $\mu(\mathcal{P}') = \mu(\mathcal{P})$ *and* $\mathcal{P}^* \not\subset \mathcal{P}'$ *we have:*

$$\gamma^{s,\mathcal{P}}_{L^2} \cdot \gamma^{s,\mathcal{P}'}_{L^2} \not\subset \gamma^{s,\mathcal{P}}_{L^2}. \tag{17}$$

REMARK 10. – *From the inclusion* $\mathcal{P} \subset \mathcal{P}^*$ *and* $\mu(\mathcal{P}) = \mu(\mathcal{P}^*)$, *it follows from Proposition* 2.1 *that* $\gamma^{s,\mathcal{P}^*}_{L^2} \subset \gamma^{s,\mathcal{P}}_{L^2}$ *and the inclusion is strict except than in the anisotropic case.*

Now we prove Proposition 2.3.

PROOF. – We first observe that it is not restrictive to suppose that $\mathcal{P}$ has formal order $\mu = 1$, so also $\mathcal{P}^*$ has formal order equal to 1.

Now let $f \in \gamma^{s,\mathcal{P}}_{L^2}$, $g \in \gamma^{s,\mathcal{P}^*}_{L^2}$. Then, referring to Theorem 2.1, there are two constants $\varepsilon, C > 0$ such that:

$$\| \hat{f}(\xi) \exp(\varepsilon |\xi|^{\frac{1}{s}}_{\mathcal{P}}) \|_{L^2} \le C,$$

$$\| \hat{g}(\xi) \exp(\varepsilon |\xi|^{\frac{1}{s}}_{\mathcal{P}^*}) \|_{L^2} \le C.$$

It is easy to see that the second inequality implies also:

$$\left\| \hat{g}(\xi) \exp\Big(\frac{\varepsilon}{2}|\xi|_{\mathcal{P}^*}^{\frac{1}{s}}\Big) \right\|_{L^1} \leq C.$$

So, for a suitable $\varepsilon' > 0$, that we will determine in the following, we have:

$$\begin{aligned}
\|\widehat{fg}(\xi) \exp(\varepsilon'|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2} &= \|\hat{f}(\xi) * \hat{g}(\xi) \exp(\varepsilon'|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2} \\
&= \left\| \int \hat{f}(\xi-\eta)\hat{g}(\eta) \exp(\varepsilon'|\xi-\eta+\eta|_{\mathcal{P}}^{\frac{1}{s}})\, d\eta \right\|_{L^2} \\
&\leq \left\| \int |\hat{f}(\xi-\eta)\hat{g}(\eta)| \exp(C'(|\xi-\eta|_{\mathcal{P}} + |\eta|_{\mathcal{P}^*})^{\frac{1}{s}})\, d\eta \right\|_{L^2} \\
&\leq C''' \left\| \int \exp(C''\varepsilon'(|\xi-\eta|_{\mathcal{P}}^{\frac{1}{s}})|\hat{f}(\xi-\eta)| \exp(C''\varepsilon'|\eta|_{\mathcal{P}^*}^{\frac{1}{s}})|\hat{g}(\eta)|\, d\eta \right\|_{L^2} \\
&\leq C^{IV} \|\hat{f}(\xi)\exp(C''\varepsilon'(|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2} \|\hat{g}(\xi)\exp(C''\varepsilon'|\xi|_{\mathcal{P}^*}^{\frac{1}{s}})\|_{L^1} \leq C
\end{aligned}$$

by means of Young inequality and Proposition 1.2. Therefore, if $\varepsilon' \leq \frac{\varepsilon}{2C'''}$, we obtain:

$$\|\widehat{fg}(\xi) \exp(\varepsilon'|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2} \leq C.$$

That means that fg belongs to $\gamma_{L^2}^{s,\mathcal{P}}$, as we wanted to prove.

Now if $\mathcal{P}'$ is a complete polyhedron of formal order $\mu = 1$ such that $\mathcal{P}^* \not\subset \mathcal{P}'$, we shall actually prove that the pointwise product is not continuous as a map:

$$\gamma_{L^2}^{s,\mathcal{P}} \times \gamma_{L^2}^{s,\mathcal{P}'} \longrightarrow \gamma_{L^2}^{s,\mathcal{P}}.$$

In fact, arguing by contradiction, we shall prove that if for every $\varepsilon_1, \varepsilon_2 > 0$, there are $\varepsilon > 0$ and $C > 0$ such that for all $f \in \gamma_{L^2,\varepsilon_1}^{s,\mathcal{P}}, g \in \gamma_{L^2,\varepsilon_2}^{s,\mathcal{P}'}$:

$$\|\widehat{fg}(\xi) \exp(\varepsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2} \leq C\|\hat{f}(\xi)\exp(\varepsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{s}})\|_{L^2}\|\hat{g}(\xi)\exp(\varepsilon_2|\xi|_{\mathcal{P}'}^{\frac{1}{s}})\|_{L^2}, \tag{18}$$

then the weight functions must satisfy for a suitable $\delta > 0$:

$$|\theta+\eta|_{\mathcal{P}} \leq \delta(|\theta|_{\mathcal{P}} + |\eta|_{\mathcal{P}'}), \quad \forall \theta, \eta \in \mathbb{R}^n. \tag{19}$$

This contradicts the second part of Proposition 1.2 and gives the conclusion.

To prove the claim, let us observe first that, writing for a given complete polyhedron $\mathcal{Q}$ and $\varepsilon > 0$:

$$\Lambda(\xi) = \exp(\varepsilon|\xi|_{\mathcal{Q}}^{\frac{1}{s}}), \tag{20}$$

we have for suitable $c, C > 0$:

$$C^{-1} \leq \frac{\Lambda(\xi)}{\Lambda(\eta)} \leq C \quad \text{if} \quad |\xi-\eta| < \frac{C}{c}. \tag{21}$$

That follows trivially from the inequality:

$$|\xi|_{\mathcal{Q}} < C|\xi|_{\mathcal{Q}}|\xi-\eta|_{\mathcal{Q}},$$

see for example [12].

Now, inspiring to the proof of Theorem 3.8 in Garello [12], let us take two functions $\phi \in \gamma^{s,\mathcal{P}}_{L^2}, \psi \in \gamma^{s,\mathcal{P}'}_{L^2}$ such that:

$$\phi\psi \not\equiv 0, \quad \hat{\phi}(\xi) \geq 0, \quad \hat{\psi}(\xi) \geq 0,$$

$$supp\,\hat{\phi}, supp\,\hat{\psi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq c\},$$

where the constant $c > 0$ will be taken sufficiently small, to satisfy (21) with the inequalities below. Now, fixed $\theta, \eta \in \mathbb{R}^n$, we define:

$$f(x) = \phi(x)e^{i\eta x},$$

$$g(x) = \psi(x)e^{i\theta x}.$$

Let us observe that obviously $f \in \gamma^{s,\mathcal{P}}_{L^2}, g \in \gamma^{s,\mathcal{P}'}_{L^2}$.

We have:

$$f(x)g(x) = \phi(x)\psi(x)e^{i(\theta+\eta)x}.$$

Taking into account the support of $\hat{\phi}$ and $\hat{\psi}$, we can estimate for suitable constants $C_1, C_2, C_3 > 0$:

$$\left\|\exp\left(\varepsilon_1|\xi|^{\frac{1}{s}}_{\mathcal{P}}\right)\hat{f}(\xi)\right\|^2_{L^2} = \int \exp(2\varepsilon_1|\xi|^{\frac{1}{s}}_{\mathcal{P}})|\hat{f}(\xi)|^2\,d\xi$$

$$= \int_{|\xi-\eta|<c} \exp(2\varepsilon_1|\xi|^{\frac{1}{s}}_{\mathcal{P}})|\hat{\phi}(\xi-\eta)|^2\,d\xi$$

$$\leq C_1\exp(2\varepsilon_1|\eta|^{\frac{1}{s}}_{\mathcal{P}})\|\hat{\phi}(\xi)\|^2_{L^2}.$$

$$\left\|\exp\left(\varepsilon_2|\xi|^{\frac{1}{s}}_{\mathcal{P}'}\right)\hat{g}(\xi)\right\|^2_{L^2} = \int \exp(2\varepsilon_2|\xi|^{\frac{1}{s}}_{\mathcal{P}'})|\hat{g}(\xi)|^2\,d\xi =$$

$$\int_{|\xi-\theta|<c} \exp(2\varepsilon_2|\xi|^{\frac{1}{s}}_{\mathcal{P}'})|\hat{\psi}(\xi-\theta)|^2\,d\xi$$

$$\leq C_2\exp(2\varepsilon_2|\theta|^{\frac{1}{s}}_{\mathcal{P}'})\|\hat{\psi}(\xi)\|^2_{L^2}.$$

$$\left\|\exp\left(\varepsilon|\xi|^{\frac{1}{s}}_{\mathcal{P}}\right)\widehat{fg}(\xi)\right\|^2_{L^2} = \int \exp(\varepsilon|\xi|^{\frac{1}{s}}_{\mathcal{P}})|\widehat{fg}(\xi)|^2\,d\xi =$$

$$\int_{|\xi-(\theta+\eta)|<c} \exp(2\varepsilon|\xi|^{\frac{1}{s}}_{\mathcal{P}})|\widehat{\phi\psi}(\xi-(\theta+\eta))|^2\,d\xi$$

$$\geq C_3\exp(2\varepsilon|\theta+\eta|^{\frac{1}{s}}_{\mathcal{P}})\|\widehat{\phi\psi}\|^2_{L^2}.$$

So, if (18) is satisfied, it would imply (19) for a suitable large $\delta > 0$.

This concludes the proof of Proposition 2.3. □

3. – The Cauchy problem in multi-anisotropic Gevrey classes.

In this section we prove the result announced in the Introduction, cf. Theorem 3.3, that ensures the well-posedness of the Cauchy problem in the multi-anisotropic Gevrey classes under suitable hypotheses. At this aim, we first introduce the notion of Sylvester system and recall the technique of quasi-symmetrization, useful to obtain the well-posedness for systems coming from partial differential equations. Let us consider a $\nu N \times \nu N$ hyperbolic symbol $A(t,x,\xi)$, homogeneous of order one in ξ, of N-block Sylvester type, i.e.

$$A(t,x,\xi) = \begin{pmatrix} B_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & B_0 \end{pmatrix} \langle \xi \rangle, \tag{22}$$

where the ν blocks B_0 are the same $N \times N$ Sylvester matrix:

$$\begin{pmatrix} 0 & 1 & \dots & & 0 \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & 0 & 1 \\ b_1 & b_2 & \dots & & b_N \end{pmatrix}, \tag{23}$$

with symbols $b_j(t,x,\xi)$ homogeneous of order zero in ξ.

REMARK 11. – *We obtain Sylvester matrices from the standard reduction of a scalar equation to a first order system.*

We recall the notion of quasi-symmetrizer for a hyperbolic system:

$$\partial_t u(t,x) = A(t,x,D)u(t,x) + B(t,x)K(D)u(t,x) + f(t,x),$$

where $A(t,x,\xi)$ is a hyperbolic matrix-valued symbol of Sylvester type as in (22).

We denote by S^m the set of the symbols of matrix-valued $m-$th order classical pseudo-differential operators in $\mathbb{R}^n$.

THEOREM 3.1. – *(cf. D'Ancona-Spagnolo* [9]*)*
Assume that $A(t,x,\xi)$ belongs to $C^k([0,T];S^1)$, is homogeneous of order 1 in ξ for large ξ, $k \geq 2$ and that:

1. *$A(t,x,\xi)$ has only pure imaginary eigenvalues;*

2. *$\langle\xi\rangle^{-1}A(t,x,\xi)$ is a uniformly bounded N-block Sylvester matrix.*
Then there exists a quasi-symmetrizer:

$$Q_\varepsilon = Q_\varepsilon(t,x,\xi) \in C^k([0,T];S^0)$$

satisfying the following conditions for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and all $\varepsilon > 0$:

$$\begin{aligned} &\varepsilon^{2(N-1)} \le Q_\varepsilon = Q_\varepsilon^* \le CI, \\ &Q_\varepsilon A + A^* Q_\varepsilon \le C\varepsilon\langle\xi\rangle Q_\varepsilon, \\ &-C\varepsilon^{1-N} Q_\varepsilon \le Q'_\varepsilon \le C\varepsilon^{1-N} Q_\varepsilon. \end{aligned} \tag{24}$$

Here Q'_ε is the time derivative of Q_ε and $C > 0$ is independent of ε, t, x, ξ; if P, Q are two matrices, $P \le Q$ means that $(Pv, v) \le (Qv, v)$, $\forall v \in \mathbb{C}^n$.

The following result takes into account the multiplicity of the roots of the matrix $A(t,x,\xi)$.

THEOREM 3.2. – *(cf. Jannelli* [17]*)*
Under the same hypotheses of Theorem 3.1 *and assuming that the eigenvalues of each Sylvester block of the matrix $A(t,x,\xi)$ have maximal multiplicity equal to M, there exists a quasi-symmetrizer:*

$$Q_\varepsilon = Q_\varepsilon(t,x,\xi) \in C^k([0,T];S^0)$$

satisfying the conditions (24) *with M in place of N, i.e. for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and all $\varepsilon > 0$:*

$$\begin{aligned} &\varepsilon^{2(M-1)} \le Q_\varepsilon = Q_\varepsilon^* \le I, \\ &Q_\varepsilon A + A^* Q_\varepsilon \le C\varepsilon\langle\xi\rangle Q_\varepsilon, \\ &-C\varepsilon^{1-M} Q_\varepsilon \le Q'_\varepsilon \le C\varepsilon^{1-M} Q_\varepsilon, \end{aligned} \tag{25}$$

where $C > 0$ is independent of ε, t, x, ξ.

We shall use Theorem 3.2 to prove our result.
Let us consider the pseudo-differential system in $[0,T]\times\mathbb{R}^n$ (for all $T > 0$):

$$\partial_t w(t,x) = A(D)w(t,x) + B(t,x)K(D)w(t,x) + f(t,x),$$

where:
(i) $A(\xi)$ is a N-block-Sylvester matrix-valued symbol homogeneous of order 1 in ξ for large ξ (with ν Sylvester blocks of order N as in (22));
(ii) $K(\xi)$ is a $\nu N \times \nu N$ matrix-valued symbol of order ≤ 0.
Let us fix a complete polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ and let $s > 1$.

Then we have the following:

THEOREM 3.3. – *Let us assume that:*

(i) $A(\xi)$ *has only pure imaginary eigenvalues and the eigenvalues of each Sylvester block have maximal multiplicity equal to M;*

(ii) *the symbol* $K(\xi)$ *of* $K(D)$ *satisfies the following Levi-type condition for a fixed k,* $0 < k < M$*:*

$$|K(\xi)| \leq C\left(\frac{|\xi|_{\mathcal{P}}^{k}}{\langle\xi\rangle^{M-1}}\right), \quad \text{for a } \; C > 0 \text{ and } \forall \xi \in \mathbb{R}^n; \tag{26}$$

(iii) $B(t,x)$ *belongs to* $C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}^*}(\mathbb{R}^n))$*, where* $\mathcal{P}^*$ *denotes the complementary polyhedron of* $\mathcal{P}$*;*

(iv) $f(t,x)$ *belongs to* $C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n))$.

Then the Cauchy problem

$$\begin{cases} \partial_t w(t,x) = A(D)w(t,x) + B(t,x)K(D)w(t,x) + f(t,x) \\ w(0,x) = w_0(x) \in \gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n) \end{cases} \tag{27}$$

admits a unique solution $w \in C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^n))$ *for:*

$$s < \frac{M}{k}.$$

We have for Theorem 3.3 the following slightly more general version:

THEOREM 3.4. – *Let* (i) *and* (ii) *in Theorem* 3.3 *be satisfied. Fix a complete polyhedron* $\mathcal{P}'$ *and* $s > 1$ *such that* $\gamma_{L^2}^{s,\mathcal{P}'}(\mathbb{R}^n) \subset \gamma_{L^2}^{r,\mathcal{P}}(\mathbb{R}^n)$ *for some r with* $1 < r < \frac{M}{k}$*, cf. Proposition* 2.1. *Substitute the hypotheses* (*iii*) *and* (*iv*) *with the following:*

(iii) $B(t,x)$ *belongs to* $C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}^*}(\mathbb{R}^n))$*, where* $\mathcal{P}^*$ *denotes the complementary polyhedron of* $\mathcal{P}'$*;*

(iv) $f(t,x)$ *belongs to* $C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}'}(\mathbb{R}^n))$.

Then the Cauchy problem

$$\begin{cases} \partial_t w(t,x) = A(D)w(t,x) + B(t,x)K(D)w(t,x) + f(t,x) \\ w(0,x) = w_0(x) \in \gamma_{L^2}^{s,\mathcal{P}'}(\mathbb{R}^n) \end{cases} \tag{28}$$

admits a unique solution $w \in C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}'}(\mathbb{R}^n))$.

REMARK 12. – *The hypothesis* (26) *on the lower order terms implies the Levi condition in the case of the standard Gevrey classes, and it is essential in order to obtain the result of existence; in fact, for an arbitrary hyperbolic operator without Levi conditions, it is not possible to obtain a better result of well-po-*

sedness than in $\gamma^s_{L^2}$, $s < \frac{M}{M-1}$, *where M is the maximal multiplicity of the roots of the principal symbol (cf. Bronstein* [2]*).*

REMARK 13. – *Since* $M \leq N$, *under the assumption:*

$$(29) \qquad |K(\xi)| \leq C\left(\frac{|\xi|^k_{\mathcal{P}}}{\langle\xi\rangle^{N-1}}\right), \quad \text{for } C > 0 \text{ and } \forall \xi \in \mathbb{R}^n,$$

with $k < N$, *Theorem* 3.3 *is valid for* $s < \frac{N}{k}$ *and Theorem* 3.4 *for every* $s > 1$ *with* $\gamma^{s,\mathcal{P}'}_{L^2}(\mathbb{R}^n) \subset \gamma^{\frac{N}{k},\mathcal{P}}_{L^2}(\mathbb{R}^n)$, *basing the proof on the quasi-symmetrization of D'Ancona-Spagnolo (cf. Theorem* 3.1*).*

REMARK 14. – *For Sylvester systems coming after the standard reduction from a scalar Kowalevskian equation:*

$$(30) \qquad D^m_t u + \sum_{|\nu|+j=m, j\neq m} a_{\nu j} D^\nu_x D^j_t u + \sum_{|\nu|+j<m} a_{\nu j}(t,x) D^\nu_x D^j_t u + f(t,x) = 0$$

with hyperbolic principal part of order m and maximal multiplicity of the characteristics equal to M, the Levi-type condition (26) *with* $0 < k < M$ *is equivalent to the following: for all* $(t,x) \in [0,t] \times \mathbb{R}^n$, *there is a constant* $C > 0$ *such that:*

$$(31) \qquad |a_{\nu j}(t,x)\xi^\nu| \leq C|\xi|^k_{\mathcal{P}}\langle\xi\rangle^{m-M-j}, \quad \forall \xi \in \mathbb{R}^n, \ \forall \nu, j : |\nu| + j \leq m-1.$$

It was proved in [4] *that the condition* (31) *implies the well-posedness for* $s < \frac{M}{k}$ *in the case of operators with constant coefficients.*

Instead, the condition (29)*, for systems coming from a differential equation of the form* (30) *with hyperbolic principal part of order m, is equivalent to ask for* $0 < k < m$*: for all* $(t,x) \in [0,t] \times \mathbb{R}^n$, *there is a constant* $C > 0$ *such that:*

$$(32) \qquad |a_{\nu j}(t,x)\xi^\nu| \leq C|\xi|^k_{\mathcal{P}}\langle\xi\rangle^{-j}, \quad \forall \xi \in \mathbb{R}^n, \ \forall \nu, j : |\nu| + j \leq m-1,$$

that implies the well-posedness for $s < \frac{m}{k}$, in the case of operators with constant coefficients (cf. [4]).

REMARK 15. – *The hypothesis* (26) *can be weakened only if we assume some restrictions on the principal part. In the case of scalar equations, if* $P(D) = D^m_t + \sum_{|\nu|+j=m,\, j\neq m} b_{\nu j} D^\nu_x D^j_t + \sum_{|\nu|+j\leq m-1} a_{\nu j} D^\nu_x D^j_t$ *is a differential operator such that its principal part is hyperbolic with characteristics of maximal multiplicity equal to* $M \leq m$, *and its principal symbol satisfies for a constant* $C > 0$ *the condition:*

$$(33) \qquad |b_{\nu j}\xi^\nu| \leq C|\xi|^{m-M-j}_{\mathcal{P}} \qquad \text{for } |\nu| + j = m, \ j \neq m,$$

then, if the lower order terms satisfy for some $k < M$ the condition:

$$|a_{\nu j}\xi^{\nu}| \leq c|\xi|_{\mathcal{P}}^{k-j}\langle\xi\rangle^{m-M} \qquad \text{for } |\nu| + j \leq m-1, \tag{34}$$

then $P(D)$ is $\left(\frac{M}{k}, \mathcal{P}\right)$-hyperbolic. Condition (33) *is satisfied for instance when A is the null matrix.*

REMARK 16. – *If $|\xi|_{\mathcal{P}} = \langle\xi\rangle$, i.e. $\mathcal{P}$ is the Newton polyhedron associated to an elliptic operator, the hypothesis* (32) *with $k = m-1$ is trivially satisfied by any differential operator, and so we recapture for operators with constant principal part the above mentioned result of Bronstein* [2] *of well-posedness in $\gamma^s_{L^2}$, for all $s < \frac{m}{m-1}$, see also D'Ancona-Spagnolo* [9] *in the case when the coefficients of the principal part are x-independent.*

REMARK 17. – *If we let $|\xi|_{\mathcal{P}} = \langle\xi\rangle$ in* (26), *we obtain the well-posedness in $\gamma^s_{L^2}$, for all $s < \frac{M}{k}$, corresponding for our system to the result of Leray-Ohya* [22] *in the case of operators with variable coefficients and constant multiplicity.*

Now we prove Theorem 3.3.

PROOF. – By performing the Fourier transform with respect to the space variables, we get the equivalent formulation of the Cauchy problem (27):

$$\begin{cases} \partial_t\hat{w}(t,\xi) = A(\xi)\hat{w}(t,\xi) + (B(t,x)K(D)w(t,x))\hat{} + \hat{f}(t,\xi) \\ \hat{w}(0,\xi) = \widehat{w_0}(\xi) \end{cases}.$$

Let Q_ε, $\varepsilon > 0$, be the quasi-symmetrizer of the matrix $A(\xi)$ constructed in Theorem 3.2, and define the matrix:

$$Q(\xi) = Q_\varepsilon(\xi), \quad \varepsilon = |\xi|_{\mathcal{P}}^{\frac{k}{M}}\langle\xi\rangle^{-1}, \tag{35}$$

that satisfies the following conditions (in view of Theorem 3.2):

$$\begin{aligned} &i) \quad Q(\xi) \in S^0(\mathbb{R}^n), \\ &ii) \quad |\xi|_{\mathcal{P}}^{2(M-1)\frac{k}{M}}\langle\xi\rangle^{-2(M-1)} I \leq Q \leq I, \\ &iii) \; AQ + QA^* \leq C|\xi|_{\mathcal{P}}^{\frac{k}{M}} Q. \end{aligned} \tag{36}$$

As Q_ε has constant coefficients, then Q'_ε is the null matrix.

We observe that the choice of ε in (35) will allow us to get the best estimates for the terms that will appear in (38) below, as we have to balance between the estimate *ii)* in which it appears a negative power of ε and *iii)* that has a positive power of ε.

We define the radius function:

$$\rho(t) = \rho_0 - \frac{\rho_0}{2T}t\ , \quad t \in [0,T],$$

with ρ_0 depending on the initial data and on $f(t,x)$. Therefore we introduce the following two formal energies for any vector function $w(x)$:

$$\begin{aligned}\mathcal{E}(t,w) &= \int \exp\left(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right)(Q(\xi)\hat{w}(t,\xi),\hat{w}(t,\xi))^{\frac{1}{2}}\,d\xi,\\ \widetilde{\mathcal{E}}(t,w) &= \int \exp\left(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right)(\hat{w}(t,\xi),\hat{w}(t,\xi))^{\frac{1}{2}}\,d\xi,\end{aligned} \tag{37}$$

the integral being extended over all $\mathbb{R}^n$.

We observe that the boundedness of $\mathcal{E}(t,w)$, defined in terms of the matrix $Q(\xi)$, gives indeed a regularity result in view of the estimates (36), ii).

If we used the standard symmetrizer P of A, we could not obtain a regularity result as P is not a positive definite matrix in the case of weakly hyperbolic systems, as pointed out by Jannelli [18].

In fact, for a solution w of the Cauchy problem, we can estimate:

$$\begin{aligned}\frac{d}{dt}(Q\hat{w},\hat{w})^{\frac{1}{2}} &= \frac{1}{2(Q\hat{w},\hat{w})^{\frac{1}{2}}}[(Q\hat{w}',\hat{w})+(Q\hat{w},\hat{w}')]\\ &= \frac{1}{2(Q\hat{w},\hat{w})^{\frac{1}{2}}}[(QA\hat{w},\hat{w})+(Q(B(t,x)Q(D)w)\widehat{\ },\hat{w})+(Q\hat{f},\hat{w})\\ &+(Q\hat{w},A\hat{w})+(Q\hat{w},(B(t,x)Q(D)w)\widehat{\ })+(Q\hat{w},\hat{f})]\\ &= \frac{1}{2(Q\hat{w},\hat{w})^{\frac{1}{2}}}[((QA+A^*Q)\hat{w},\hat{w})+((B(t,x)Q(D)w)\widehat{\ },\hat{w})\\ &+(Q\hat{f},\hat{w})+(Q\hat{w},(B(t,x)Q(D)w)\widehat{\ })+(Q\hat{w},\hat{f})].\end{aligned} \tag{38}$$

We further estimate these terms as follows:

$$((QA+A^*Q)\hat{w},\hat{w}) \le C|\xi|_{\mathcal{P}}^{\frac{k}{M}}(Q\hat{w},\hat{w})$$

from (36), and:

$$\begin{aligned}(Q\hat{f},\hat{w}) &\le (Q\hat{w},\hat{w})^{\frac{1}{2}}(Q\hat{f},\hat{f})^{\frac{1}{2}} \le |\hat{f}|(Q\hat{w},\hat{w})^{\frac{1}{2}},\\ (Q(B(t,x)Q(D)w)\widehat{\ },\hat{w}) &\le (Q\hat{w},\hat{w})^{\frac{1}{2}}(Q(B(t,x)Q(D)w)\widehat{\ },(B(t,x)Q(D)w)\widehat{\ })^{\frac{1}{2}}\\ &\le |(B(t,x)Q(D)w)\widehat{\ }|(Q\hat{w},\hat{w})^{\frac{1}{2}}\end{aligned}$$

by means of (36) and the Schwartz inequality for the scalar product associated to Q. So we obtain:

$$\frac{d}{dt}(Q\hat{w},\hat{w})^{\frac{1}{2}} \le C|\xi|_{\mathcal{P}}^{\frac{k}{M}}(Q\hat{w},\hat{w})^{\frac{1}{2}} + |(B(t,x)K(D)w)\widehat{\ }| + |\hat{f}|.$$

In view of the condition (ii) on $K(\xi)$ and (iii) on $B(t,x)$ and the estimates (36), we obtain:

$$
\begin{aligned}
&\widetilde{\mathcal{E}}(t, B(t,x)K(D)w) \\
&= \int \exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})|(B(t,x)K(D)w)\widehat{}|\; d\xi \\
&= |\exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})\hat{B}(t,\xi) * \widehat{K(D)}w(t,\xi)|_{L^1} \\
&= \left\| \int \exp\,(\rho(t)|\eta + (\xi-\eta)|_{\mathcal{P}}^{\frac{1}{s}})\hat{B}(t,\eta)\widehat{K(D)}w(t,\xi-\eta)\, d\eta \right\|_{L^1} \\
&\le \left\| \int \exp\,(\rho(t)|\eta|_{\mathcal{P}^*}^{\frac{1}{s}})|\hat{B}(t,\eta)| \exp\,(\rho(t)|\xi-\eta|_{\mathcal{P}}^{\frac{1}{s}})|\widehat{K(D)}w(t,\xi-\eta)|\, d\eta \right\|_{L^1} \\
&= \|(\exp\,(\rho(t)|\xi|_{\mathcal{P}^*}^{\frac{1}{s}})|\hat{B}(t,\xi)|) * (\exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})|\widehat{K(D)}w|)\|_{L^1} \\
&\le \|\exp\,(\rho(t)|\xi|_{\mathcal{P}^*}^{\frac{1}{s}})\hat{B}(t,\xi)\|_{L^1} \|\exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})\widehat{K(D)}w\|_{L^1} \\
&\le C' \left\| \exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) \frac{|\xi|_{\mathcal{P}}^{k}}{\langle\xi\rangle^{M-1}}\hat{w} \right\|_{L^1} \\
&\le \int \exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})|\xi|_{\mathcal{P}}^{k-(M-1)\frac{k}{M}}(Q\hat{w},\hat{w})^{\frac{1}{2}}\, d\xi \\
&= \int \exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})|\xi|_{\mathcal{P}}^{\frac{k}{M}}(Q\hat{w},\hat{w})^{\frac{1}{2}}\, d\xi,
\end{aligned}
\tag{39}
$$

where we have also used Proposition 1.2 and the Young inequality.

So we get the following estimate for $\mathcal{E}'(t,w)$:

$$
\mathcal{E}'(t,w) \le \int \exp\,(\rho(t)|\xi|_{\mathcal{P}}^{\frac{1}{s}})|\xi|_{\mathcal{P}}^{\frac{1}{s}}\{\rho'(t) + C|\xi|_{\mathcal{P}}^{-\frac{1}{s}+\frac{k}{M}}\}(Q\hat{w},\hat{w})^{\frac{1}{2}}\, d\xi + \widetilde{\mathcal{E}}(t,f). \tag{40}
$$

If $-\frac{1}{s}+\frac{k}{M} < 0$, i.e. $s < \frac{M}{k}$, we can estimate:

$$
\mathcal{E}'(t,w) \le C\mathcal{E}(t,w) + \widetilde{\mathcal{E}}(t,f)
$$

that, in view of Gronwall's lemma, allows us to conclude that $\mathcal{E}(t,w)$ is bounded for any solution of the Cauchy problem (27).

Till now we have obtained an estimate with the L^1 norm, but by means of the same arguments applied to the first order derivatives $D^\alpha w$ we get the estimate:

$$
\|\exp\,(\rho|\xi|_{\mathcal{P}}^{\frac{1}{s}})\hat{w}(t,\xi)\|_{L^2} \le C,
$$

which gives the regularity of w in $C^\infty([0,T],\gamma_{L^2}^{s,\mathcal{P}})$ for any $s < \frac{M}{k}$ as we wanted to prove.

Finally we prove the uniqueness of the solution of the Cauchy problem (27). If

u, v are two solutions of (27), then their difference $w = u - v$ satisfies the following Cauchy problem:

$$\begin{cases} \partial_t w(t,x) = A(D)w(t,x) + B(t,x)K(D)w(t,x) \\ w(0,x) = 0 \end{cases}.$$

So, proceeding as in the previous steps of the proof, we get

$$\mathcal{E}'(t,w) \leq C\mathcal{E}(t,w),$$

that gives, by means of Gronwall's inequality,

$$\mathcal{E}(t,w) \leq C\mathcal{E}(0,w) = 0.$$

So $w = u - v$ is the null solution, implying that the solution of (27) is unique. □

PROOF OF THEOREM 3.4. – It is analogous to the proof of Theorem 3.3 and is obtained by taking instead of (37) the following energy functions:

$$\mathcal{E}(t,w) = \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)(Q(\xi)\hat{w}(t,\xi), \hat{w}(t,\xi))^{\frac{1}{2}}\, d\xi,$$

$$\widetilde{\mathcal{E}}(t,w) = \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)(\hat{w}(t,\xi), \hat{w}(t,\xi))^{\frac{1}{2}}\, d\xi.$$

The estimate (39) becomes:

$$\begin{aligned}
&\widetilde{\mathcal{E}}(t, B(t,x)K(D)w) \\
&= \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)|(B(t,x)K(D)w\hat{)}|\ \ d\xi \\
&= \|\exp\left(\rho(t)|\xi + (\eta - \xi)|_{\mathcal{P}'}^{\frac{1}{s}}\right)\hat{B}(t,\xi) * \widehat{K(D)w}\|_{L^1} \\
&= \|(\exp\left(\rho(t)|\xi|_{\mathcal{P}^*}^{\frac{1}{s}}\right)|\hat{B}(t,\xi)|) * (\exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)|\widehat{K(D)}w(t,\xi)|)\|_{L^1} \\
&\leq C'\|\exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)\frac{|\xi|_{\mathcal{P}}^{k}}{\langle\xi\rangle^{M-1}}\hat{w}\|_{L^1} \\
&\leq \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)|\xi|_{\mathcal{P}}^{k-(M-1)\frac{k}{M}}(Q\hat{w}, \hat{w})^{\frac{1}{2}}\, d\xi \\
&= \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)|\xi|_{\mathcal{P}}^{\frac{k}{M}}(Q\hat{w}, \hat{w})^{\frac{1}{2}}\, d\xi,
\end{aligned}$$

that allows us to conclude in a similar way as (40):

$$\mathcal{E}'(t,w) \leq \int \exp\left(\rho(t)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\right)|\xi|_{\mathcal{P}'}^{\frac{1}{s}}\{\rho'(t) + C|\xi|_{\mathcal{P}'}^{-\frac{1}{s}}|\xi|_{\mathcal{P}}^{\frac{k}{M}}\}(Q\hat{w}, \hat{w})^{\frac{1}{2}}\, d\xi + \widetilde{\mathcal{E}}(t,f).$$

If $|\xi|_{\mathcal{P}'}^{-\frac{1}{s}} < |\xi|_{\mathcal{P}}^{-r}$ for $r < \frac{k}{M}$, i.e. $\gamma_{L^2}^{s,\mathcal{P}'} \subset \gamma_{L^2}^{r,\mathcal{P}}$, then w is in $C^\infty([0,T], \gamma_{L^2}^{s,\mathcal{P}'})$ as we wanted to prove.

REMARK 18. – *We can prove Theorems* 3.3 *and* 3.4 *also in the case that the matrix* $A = A(t,D)$ *depends on the time variable, under the hypothesis that the quasi-symmetrizer* Q *defined in* (36) *satisfies the condition:*

$$Q' \leq C|\xi|_{\mathcal{P}}^{\frac{k}{M}} Q$$

instead of $Q' \leq C|\xi|_{\mathcal{P}}^{\frac{k}{M}-k}\langle\xi\rangle^{M-1}Q$ *true in the general case. This reflects on some conditions on the matrix* $A(t,D)$*; anyhow, we are not able to express them in terms of the matrix* A*, except in the case when* A *has constant coefficients and therefore* $Q' = 0$*, or* $\mathcal{P}$ *is the Newton polyhedron of an elliptic operator and* $k = M-1$.

Now we give some examples exploiting the result of Theorem 3.3; for simplicity we refer to scalar operators satisfying (31).

1. Let $\mathcal{P}$ be the complete polyhedron in $\mathbb{R}^2$ of vertices

$$\mathcal{V}(\mathcal{P}) = \{(0,0),(0,3),(1,0)\}.$$

Applying condition (31) with $M = 3$ and $k = 2$ to the differential operator of order 4:

$$P = P_4(D_t,D_x,D_y) + P_3(t,x,y,D_t,D_x,D_y) + P_2(t,x,y,D_t,D_x,D_y) + P_1(t,x,y,D_t,D_x,D_y) + c_6(t,x,y),$$

we obtain:

$$\begin{aligned} &P_4 \text{ is hyperbolic with } M = 3,\\ &P_3 = c_1(t,x,y)D_y^3 + c_2(t,x,y)D_xD_y^2,\\ &P_2 = c_3(t,x,y)D_y^2 + c_4(t,x,y)D_tD_y + c_5(t,x,y)D_xD_y,\\ &P_1 \text{ is any operator of order } 1, \end{aligned}$$

so we get the well-posedness in $\gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^2)$, $1 < s < \frac{3}{2}$, under the condition that the coefficients are also in $\gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^2)$ (as $\mathcal{P}^*$ coincides with $\mathcal{P}$).

We observe that the hypothesis on the multiplicity allows us to ask a less restrictive condition on the lower order terms and to obtain a better Gevrey order than in the case $k = 3, M = 4$ mentioned in the Introduction.

2. Let $\mathcal{P}$ be the complete polyhedron in $\mathbb{R}^2$ of vertices

$$\mathcal{V}(\mathcal{P}) = \{(0,0),(4,0),(2,2),(0,3)\},$$

so $\mu = 6$ and the corresponding weight function is

$$|(\xi,\eta)|_{\mathcal{P}} = (1 + |\xi^4| + |\xi^2\eta^2| + |\eta^3|)^{\frac{1}{6}}.$$

The complementary polyhedron $\mathcal{P}^*$ of $\mathcal{P}$ has vertices

$$\mathcal{V}(\mathcal{P}) = \{(0,0),(6,0),(0,4)\}.$$

Applying the condition (31) to the differential operator of order 4:

$$P(t,x,y,D_t,D_x,D_y) = P_4(D_t,D_x,D_y) + P_3(t,x,y,D_t,D_x,D_y) + P_2(t,x,y,D_t,D_x,D_y) + P_1(t,x,y,D_t,D_x,D_y) + c(t,x,y),$$

we ask that:

$$\begin{aligned} &P_4 \text{ is hyperbolic,}\\ &P_3 \equiv 0,\\ &P_2 = c_1(t,x,y)D_x^2 + c_2(t,x,y)D_xD_t + c_3(t,x,y)D_xD_y,\\ &P_1 = c_4(t,x,y)D_x + c_5(t,x,y)D_y + c_6(t,x,y)D_t, \end{aligned}$$

and the coefficients are in $C^\infty([0,T],\gamma_{L^2}^{s,\mathcal{P}^*}(\mathbb{R}^2))$, then the solution belongs to $C^\infty([0,T],\gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^2))$ for any data in $\gamma_{L^2}^{s,\mathcal{P}}(\mathbb{R}^2)$, if $1 < s < \frac{4}{3}$; this extends the classical result as $\gamma_{L^2}^{2}(\mathbb{R}^2) \subset \gamma_{L^2}^{\frac{4}{3},\mathcal{P}}(\mathbb{R}^2)$.

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