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A simple Necessary and Sufficient Condition for Well-Posedness of $2 \times 2$ Differential Systems with Time-dependent Coefficients.

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**Sunto.** – Dato il Problema di Cauchy

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_0(x)$$

Nishitani [N], dopo aver effettuato, mediante una matrice di cambiamento di base costante, la trasformazione della matrice

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \quad t \in [0, T]$$

reale, analitica e iperbolica, nella matrice complessa

$$A^2(t) = \begin{bmatrix} \frac{c^5(t)}{a^5(t)} & a^4(t) \\ \frac{d^5(t)}{a^5(t)} & -c^4(t) \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -\frac{a-b}{2} \end{bmatrix},$$

ha dimostrato che il Problema di Cauchy considerato è ben posto in $C^\infty$ in un intorno di zero se e solo se vale la condizione

$$h |a^5|^2 \geq C t^2 |D^5|^2,$$

dove

$$D^5 = \dot{a}\dot{c}^5 - \ddot{c} a^5 \quad e \quad h = - \det A = |a^5|^2 - |c^5|^2.$$

In questo breve lavoro invece diamo una semplicissima condizione equivalente a quella di Nishitani (e quindi necessaria e sufficiente per la buona positura), in cui compaiono solamente gli elementi di $A(t)$ e non le loro derivate.

**Summary.** – Given the Cauchy Problem

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_0(x) \quad x \in \mathbb{R}$$

Nishitani [N], by making use of a change of basis by a constant matrix, transformed the real, analytic, hyperbolic matrix

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \quad t \in [0, T]$$

into the complex matrix

$$A^2(t) = \begin{bmatrix} \frac{c^5(t)}{a^5(t)} & a^4(t) \\ \frac{d^5(t)}{a^5(t)} & -c^4(t) \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -\frac{a-b}{2} \end{bmatrix},$$
and showed that the given Cauchy Problem is well posed in $C^\infty$ in a neighborhood of zero if and only if (see also [MSJ]) the following condition

$$h|a^2|^2 \geq C \varepsilon^2 |D^2|$$

is satisfied, where

$$D^2 = \dot{a}^2 c^2 - \dot{c}^2 a^2$$

and

$$h = -\det A = |a^2|^2 - |c^2|^2.$$
although we’ll keep the symbols $a^i(t)$ and $c^i(t)$ in cases where these would be more significant. The three coordinate functions $x(t), y(t), z(t)$ characterize completely the matrix $A(t)$. Indeed

$$
\begin{bmatrix}
  x(t) \\
  y(t) \\
  z(t)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  1 & 1 & 0 \\
  0 & 0 & 2 \\
  1 & -1 & 0
\end{bmatrix} \begin{bmatrix}
  a(t) \\
  b(t) \\
  d(t)
\end{bmatrix}
$$

and on the right hand side the $3 \times 3$ matrix is invertible. In this vectorial form, we’ll indicate the matrix $A^2(t)$ by $P(t)$. Besides, in order to avoid confusion between this triplet of functions and the generic point $(x, y, z)$ of the euclidean reference space, we shall denote the latter by $(X, Y, Z)$.

The proof is split in three steps.

**First Step:** we show that the relation (1) is rotation-invariant for time-independent angles around the Z-axis in the real 3-dimensional space whose orthogonal axes are $X, Y, Z$. To this end it is sufficient to show that the quantities $h(t) \text{ and } |D^2(t)|$ are rotation-invariant for constant angles about the Z-axis, in the sense of

$$h(T_\vartheta P) = h(P) \quad \text{and} \quad |D^2(T_\vartheta P)| = |D^2(P)|,$$

where we have denoted by $T_\vartheta$ the rotation operator relative to an angle $\vartheta$ around the Z-axis. The invariance of $h$ follows by the fact that $h = x^2 + y^2 - z^2$, whereas the respective invariance of $x^2 + y^2$ and $z^2$ is trivial. As to $|D^2|$, we have

$$D^2(T_\vartheta P) = (a^2 e^{i\vartheta}) c^2 - c^2(a^2 e^{i\vartheta}) = e^{i\vartheta} D^2(P)$$

whence

$$|D^2(T_\vartheta P)| = |D^2(P)|.$$

**Second Step:** We recall that (see the example 5 in [N]) the

$$h \geq C d^2$$

imply (1). Conversely here we’ll show that if a certain matrix satisfies the conditions

$$a_\alpha \neq 0 \quad b_\alpha = 0 = d_\alpha,$$

at time zero, then also the opposite implication holds and then (3) and (1) are equivalent. We observe preliminarily that

$$|D^2|^2 = \frac{1}{4} [d (a - b) - d (\dot{a} - \dot{b})]^2 + \frac{1}{4} (\dot{a}b - \dot{b}a)^2 \quad h = ab + d^2.$$
Let’s suppose that (1) holds. Then we have
\[
|D^2| \geq C_1|(a - b) \, \dot{d} - (\dot{a} - \dot{b}) \, d| + C_1|\dot{a}b - \dot{ab}|
\]
\[
\geq C_1|||((a - b) \, \dot{d} - |(\dot{a} - \dot{b}) \, d|) + C_1|\dot{a}b - \dot{ab}|
\]
\[
\geq C_1||((a - b) \, \dot{d} - |(\dot{a} - \dot{b}) \, d|).\]
for some constant \(C_1 > 0\).

Notice then that
\[
a_o \neq 0 \Rightarrow \exists C_2 > 0 : |a(t) - b(t)| \geq C_2 \Rightarrow |(a - b) \, \dot{d}| \geq C_2|\dot{d}(t)|
\]
and that, by the analyticity of \(\dot{d}(t)\), we have
\[
d_o = 0 \Rightarrow \exists C_3 > 0 : |d(t)| \leq C_3 \, t \, \dot{d}(t) \Rightarrow |(\dot{a} - \dot{b}) \, d| \leq C_3 \, t \, |\dot{d}(t)|,
\]
so that
\[
|D^2| \geq C_1 \, ||((a - b) \, \dot{d}) - |(\dot{a} - \dot{b}) \, d|| \geq C_2 \, |\dot{d}(t)|.
\]
If we now apply the condition of Nishitani, we have
\[
\sqrt{h} \geq C \, t \, |D^2| \geq C_2 \, t \, |\dot{d}| \geq C_4 \, |d| \quad \text{for some constant} \quad C_4 > 0
\]
that is,
\[
h \geq C_6 \, d^2, \quad \text{where} \quad C_6 = C_4^2,
\]
which we can write as
\[
h \geq C_6 \, Im^2a^2.
\]

Third Step. Suppose that for \(t = 0\) we have \(P(0) = P_o = (X_o, Y_o, Z_o)\) with \(h(0) = X_o^2 + Y_o^2 + Z_o^2 = 0\). We now consider the rotation \(T_0\) of the whole space \(XYZ\), around to the \(Z\)-axis, which transforms the generator \(OP_o\) of the cone \(h = 0\) in the generator \(OP_0\) whose equations are \(\tilde{X} = \tilde{Z}\) and \(\tilde{Y} = 0\). In this way, in the new variables \((\tilde{X}, \tilde{Y}, \tilde{Z}) = T_0(X, Y, Z)\), the conditions (4) are satisfied, where we know that (1) and (3) are equivalent. By this rotation, the triple of functions \((x(t), y(t), z(t))\) goes to the correspondent triple \((\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))\), so that the condition (1) of well-posedness (unchanged for the first step) for the second step is equivalent to the
\[
h(t) \geq C \, \tilde{y}^2(t),
\]
which becomes therefore a necessary and sufficient condition for well-posedness in the new variables. Thus, we have simplified considerably the condition (1) of Nishitani. Now we want to express (5) as a function of the elements of the starting matrix. Therefore we apply \(T_0^{-1} = T_{-0}\) to (5). We have already observed that \(h(t)\) is rotation-invariant for a time-independent angle. On the right hand
side of (5) we have \( \tilde{y}^2(t) \): note that \( |\tilde{Y}| \) is the distance of \((\tilde{X}, \tilde{Y}, \tilde{Z})\) from the plane \( \pi : \tilde{Y} = 0 \), determined by the point \( P_0 = (\tilde{X}_0, 0, \tilde{Z}_0) \) and by the axis \( \tilde{X} = 0 = \tilde{Y} \) of the cone \( \tilde{X}^2 + \tilde{Y}^2 - \tilde{Z}^2 = 0 \). As the rotations are isometries, this distance is unchanged under \( T_{\varphi} \). Under this transformation the point \((\tilde{X}, \tilde{Y}, \tilde{Z})\) is changed into \((X, Y, Z)\) and the plane \( \pi : \tilde{Y} = 0 \) is transformed into the plane

\[
Y_0X - X_0Y = 0
\]

because it is the plane through the axis \( X = 0 = Y \) and the initial point \( P_0 = (X_0, Y_0, Z_0) \). Finally we have the assertion simply by using the formula of the distance point from the plane, which in this particular case becomes:

\[
dist(P, \pi) = \frac{|Y_0X - X_0Y|}{\sqrt{X_0^2 + Y_0^2}}.
\]

We observe that the denominator of the previous expression does not vanish because \( A(0) \neq 0 \) and \( h(0) = 0 \). Going back from the pointwise notation to the functional notation we have

\[
h(t) \geq C |Y_0x(t) - X_0y(t)|^2
\]

which in the variables of the original matrix \( A(t) \) becomes

\[
h(t) \geq C |d_0(a(t) + b(t)) - (a_0 + b_0)d(t)|^2. \quad \square
\]

**Corollary 1. - The Cauchy Problem**

\[
\partial_t u(x, t) + A(t) \partial_x u(x, t) = 0 \quad u(0, x) = u_0(x) \quad x \in \mathbb{R}
\]

where

\[
A(t) = t^\nu \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix}, \quad t \in [0, T]
\]

is a real, analytic, hyperbolic matrix, and \( \nu \in \mathbb{N} \) is such that \( ||t^{-\nu}A(0)|| \neq 0 \), is Well-Posed in \( C^\infty \) in a neighborhood of zero if and only if the condition

\[
a(t) b(t) + d^2(t) \geq C \left| d(0) [a(t) + b(t)] - [a(0) + b(0)] d(t) \right|^2
\]

is satisfied.

**REFERENCES**


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