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A simple Necessary and Sufficient Condition for Well-Posedness of 2×2 Differential Systems with Time-dependent Coefficients.

LORENZO MENCHERINI

Sunto. – *Dato il Problema di Cauchy*

$$\partial_t u(x,t) + A(t)\partial_x u(x,t) = 0 \quad u(0,x) = u_\circ(x)$$

Nishitani [N], *dopo aver effettuato, mediante una matrice di cambiamento di base costante, la trasformazione della matrice*

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \qquad t \in [0,T]$$

reale, analitica e iperbolica, nella matrice complessa

$$A^\sharp(t) = \begin{bmatrix} c^\sharp(t) & a^\sharp(t) \\ \overline{a^\sharp}(t) & -c^\sharp(t) \end{bmatrix} = \begin{bmatrix} i\frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -i\frac{a-b}{2} \end{bmatrix},$$

ha dimostrato che il Problema di Cauchy considerato è ben posto in C^∞ in un intorno di zero se e solo se vale la condizione

$$h|a^\sharp|^2 \geq Ct^2|D^\sharp|^2,$$

dove

$$D^\sharp = \dot{a}^\sharp c^\sharp - \dot{c}^\sharp a^\sharp \quad \text{e} \quad h = -\det A = |a^\sharp|^2 - |c^\sharp|^2.$$

In questo breve lavoro invece diamo una semplicissima condizione equivalente a quella di Nishitani (e quindi necessaria e sufficiente per la buona positura), in cui compaiono solamente gli elementi di $A(t)$ e non le loro derivate.

Summary. – *Given the Cauchy Problem*

$$\partial_t u(x,t) + A(t)\partial_x u(x,t) = 0 \quad u(0,x) = u_\circ(x) \quad x \in \mathbf{R}$$

Nishitani [N], *by making use of a change of basis by a constant matrix, transformed the real, analytic, hyperbolic matrix*

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \qquad t \in [0,T]$$

into the complex matrix

$$A^\sharp(t) = \begin{bmatrix} c^\sharp(t) & a^\sharp(t) \\ \overline{a^\sharp}(t) & -c^\sharp(t) \end{bmatrix} = \begin{bmatrix} i\frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -i\frac{a-b}{2} \end{bmatrix},$$

and showed that the given Cauchy Problem is well posed in $\mathcal{C}^\infty$ in a neighborhood of zero if and only if (see also [MS]) the following condition

$$h|a^\sharp|^2 \geq Ct^2|D^\sharp|^2$$

is satisfied, where

$$D^\sharp = \dot{a}^\sharp c^\sharp - \dot{c}^\sharp a^\sharp \quad \text{and} \quad h = -\det A = |a^\sharp|^2 - |c^\sharp|^2.$$

In this short note, we give a very simple condition, which is equivalent to that of Nishitani (and then a necessary and sufficient for the Well-Posedness), but where only the elements of $A(t)$, appear and not their derivatives.

1. – The main result.

Since $A(t)$ is analytic, there exists an integer $\upsilon \geq 0$ so that $A(t) = t^\upsilon A_1(t)$ with $A_1(t) \neq 0$. Let us set

$$A_1(0) = \begin{bmatrix} d_\circ & a_\circ \\ b_\circ & -d_\circ \end{bmatrix}.$$

By the formula of Nishitani, we can see easily that the Cauchy Problem for the matrix $A(t)$ is well posed in $\mathcal{C}^\infty$ if and only if it is for the Cauchy Problem with the matrix $A_1(t)$. For the condition of Well-Posedness, one is automatically reduced to that of the matrix $A_1(t)$, so it is natural to denote the latter matrix again by $A(t)$. On this matrix the following result holds

THEOREM 1. – *If*

$$A(0) = \begin{bmatrix} d_\circ & a_\circ \\ b_\circ & -d_\circ \end{bmatrix} \neq 0,$$

then the following two conditions are equivalent

$$h \geq C\, t^2\, |D^\sharp|^2 \tag{1}$$

$$h \geq C|d_\circ(a+b) - \big(a_\circ + b_\circ\big)d|^2. \tag{2}$$

REMARK 1. – *If the determinant of $A(t)$ does not vanish at the origin, both the conditions are satisfied and then the constraint of determinant equal to zero at zero time does not constitute any loss of generality.*

REMARK 2. – *To avoid too much heavy formalism, from now on we shall use the following notation*

$$x(t) = \mathcal{R}e\big(a^\sharp(t)\big), \quad y(t) = \mathcal{I}m\big(a^\sharp(t)\big), \quad z(t) = \mathcal{I}m\big(c^\sharp(t)\big),$$

although we'll keep the symbols $a^\sharp(t)$ e $c^\sharp(t)$ in cases where these would be more significant. The three coordinate functions $x(t), y(t), z(t)$ characterize completely the matrix $A(t)$. Indeed

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \\ d(t) \end{bmatrix}$$

and on the right hand side the 3×3 matrix is invertible. In this vectorial form, we'll indicate the matrix $A^\sharp(t)$ by $P(t)$. Besides, in order to avoid confusion between this triplet of functions and the generic point (x, y, z) of the euclidean reference space, we shall denote the latter by (X, Y, Z).

The proof is split in three steps.

First Step: we show that the relation (1) is rotation-invariant for time-independent angles around the Z-axis in the real 3-dimensional space whose orthogonal axes are X, Y, Z. To this end it is sufficient to show that the quantities $h(t)$ e $|D^\sharp(t)|$ are rotation-invariant for constant angles about the Z-axis, in the sense of

$$h(T_\vartheta P) = h(P) \quad \text{and} \quad \left|D^\sharp(T_\vartheta P)\right| = \left|D^\sharp(P)\right|,$$

where we have denoted by T_ϑ the rotation operator relative to an angle ϑ around to the Z-axis. The invariance of h follows by the fact that $h = x^2 + y^2 - z^2$, whereas the respective invariance of $x^2 + y^2$ and z^2 is trivial. As to $|D^\sharp|$, we have

$$D^\sharp(T_\vartheta P) = (a^\sharp e^{i\vartheta})^\cdot c^\sharp - \dot{c}^\sharp(a^\sharp e^{i\vartheta}) = e^{i\vartheta} D^\sharp(P)$$

whence

$$\left|D^\sharp(T_\vartheta P)\right| = \left|D^\sharp(P)\right|.$$

Second Step: We recall that (see the example 5 in [N]) the

(3) $$h \geq C\, d^2$$

imply (1). Conversely here we'll show that if a certain matrix satisfies the conditions

(4) $$a_\circ \neq 0 \qquad b_\circ = 0 = d_\circ,$$

at time zero, then also the opposite implication holds and then (3) and (1) are equivalent. We observe preliminarily that

$$|D^\sharp|^2 = \frac{1}{4}\, [\dot{d}\,(a - b) - d\,(\dot{a} - \dot{b})]^2 + \frac{1}{4}\,(a\dot{b} - \dot{a}b)^2 \qquad h = ab + d^2.$$

Let's suppose that (1) holds. Then we have

$$\begin{aligned}|D^\sharp| &\geq C_1|(a-b)\,\dot{d}-(\dot{a}-\dot{b})\,d|+C_1|\dot{a}b-a\dot{b}|\\ &\geq C_1\big||(a-b)\,\dot{d}|-|(\dot{a}-\dot{b})\,d|\big|+C_1|\dot{a}b-a\dot{b}|\\ &\geq C_1\big||(a-b)\,\dot{d}|-|(\dot{a}-\dot{b})\,d|\big|.\end{aligned}$$

for some constant $C_1 > 0$.

Notice then that

$$a_\circ \neq 0 \Rightarrow \quad \exists\, C_2 > 0 : |a(t)-b(t)| \geq C_2 \Rightarrow |(a-b)\,\dot{d}| \geq C_2|\dot{d}(t)|$$

and that, by the analyticity of $d(t)$, we have

$$d_\circ = 0 \Rightarrow \exists\, C_3 > 0 : |d(t)| \leq C_3\, t\, \dot{d}(t) \Rightarrow |(\dot{a}-\dot{b})\,d| \leq C_3\, t\, |\dot{d}(t)|,$$

so that

$$|D^\sharp| \geq C_1\, \big||(a-b)\,\dot{d}|-|(\dot{a}-\dot{b})\,d\,|\big| \geq C_2\, |\dot{d}(t)|.$$

If we now apply the condition of Nishitani, we have

$$\sqrt{h} \geq C\, t\, |D^\sharp| \geq C\, C_2\, t\, \big|\dot{d}\big| \geq C_4\, |d| \quad \text{for some constant} \quad C_4 > 0$$

that is,

$$h \geq C_6\, d^{\,2}, \quad \text{where} \quad C_6 = C_4^2,$$

which we can write as

$$h \geq C_6\, \mathcal{I}m^2 a^\sharp.$$

Third Step. Suppose that for $t = 0$ we have $P(0) = P_\circ = (X_\circ, Y_\circ, Z_\circ)$ with $h(0) = X_\circ^2 + Y_\circ^2 - Z_\circ^2 = 0$. We now consider the rotation T_ϑ of the whole space XYZ, around to the Z-axis, which trasforms the generator $OP_\circ$ of the cone $h = 0$ in the generator $\widetilde{OP}_\circ$ whose equations are $\widetilde{X} = \widetilde{Z}$ and $\widetilde{Y} = 0$. In this way, in the new variables $(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = T_\vartheta(X, Y, Z)$, the conditions (4) are satisfied, where we know that (1) and (3) are equivalent. By this rotation, the triple of functions $(x(t), y(t), z(t))$ goes to the corrispondent triple $(\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$, so that the condition (1) of well-posedness (unchanged for the first step) for the second step is equivalent to the

$$h(t) \geq C\, \widetilde{y}^{\,2}(t), \tag{5}$$

which becomes therefore a necessary and sufficient condition for well-posedness in the new variables. Thus, we have simplified considerably the condition (1) of Nishitani. Now we want to express (5) as a function of the elements of the starting matrix. Therefore we apply $T_\vartheta^{-1} = T_{-\vartheta}$ to (5). We have already observed that $h(t)$ is rotation-invariant for a time-independent angle. On the right hand side of (5) we have $\widetilde{y}^2(t)$: note that $|\widetilde{Y}|$ is the distance of $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ from the plane $\pi : \widetilde{Y} = 0$, determined by the point $\widetilde{P}_\circ = (\widetilde{X}_\circ, 0, \widetilde{X}_\circ)$ and by the axis $\widetilde{X} = 0 = \widetilde{Y}$ of the cone $\widetilde{X}^2 + \widetilde{Y}^2 - \widetilde{Z}^2 = 0$. As the rotations are isometries, this distance is unchanged under $T_{-\vartheta}$. Under this transformation the point $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ is changed into (X, Y, Z) and the plane $\pi : \widetilde{Y} = 0$ is transformed into the plane

$$Y_\circ X - X_\circ Y = 0$$

because it is the plane through the axis $X = 0 = Y$ and the initial point $P_\circ = (X_\circ, Y_\circ, Z_\circ)$. Finally we have the assertion simply by using the formula of the distance point from the plane, which in this particular case becomes:

$$dist\ (P, \pi) = \frac{|Y_\circ X - X_\circ Y|}{\sqrt{X_\circ^2 + Y_\circ^2}}.$$

We observe that the denominator of the previous expression does not vanish because $A(0) \neq 0$ e $h(0) = 0$. Going back from the pointwise notation to the functional notation we have

$$h(t) \geq C|Y_\circ x(t) - X_\circ y(t)|^2$$

which in the variables of the original matrix $A(t)$ becomes

$$h(t) \geq C\Big|d_\circ\big(a(t) + b(t)\big) - (a_\circ + b_\circ)d(t)\Big|^2. \qquad \square$$

COROLLARY 1. – *The Cauchy Problem*

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_\circ(x) \quad x \in \mathbf{R}$$

where

$$A(t) = t^\upsilon \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix}, \qquad t \in [0, T]$$

is a real, analytic, hyperbolic matrix, and $\upsilon \in \mathbf{N}$ is such that $\|t^{-\upsilon}A(0)\| \neq 0$, is Well-Posed in C^∞ in a neighborhood of zero if and only if the condition

$$a(t)\, b(t) + d^2(t) \geq C \left| d(0)\, [a(t) + b(t)] - [a(0) + b(0)]\, d(t) \right|^2$$

is satisfied.

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