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Inverse Results for Generalized Favard-Kantorovich and Favard-Durrmeyer Operators in Weighted Function Spaces.

Grzegorz Nowak

Sunto. – *Consideriamo le modificazioni di tipo Kantorovich e Durrmeyer degli operatori generalizzati di Favard e proviamo i teoremi inversi di approssimazione per funzioni f tali che* $w_{2m}f \in L^p(R)$, *dove* $1 \le p \le \infty$ *e* $w_{2m}(x) = (1+x^{2m})^{-1}$, $m \in N_0$.

Summary. – *We consider the Kantorovich and the Durrmeyer type modifications of the generalized Favard operators and we prove inverse approximation theorems for functions f such that* $w_{2m}f \in L^p(R)$, *where* $1 \le p \le \infty$ *and* $w_{2m}(x) = (1+x^{2m})^{-1}$, $m \in N_0$.

1. – Preliminaries.

Let

$$L^p_{2m}(R) = \{f\colon \|w_{2m}f\|_p < \infty\} \qquad \text{for } 1 \le p \le \infty$$

be the weighted function space, where

$$\|g\|_p = \left(\int_{-\infty}^{\infty} |g(x)|^p dx\right)^{1/p} \qquad \text{if } 1 \le p < \infty,$$

$$\|g\|_\infty = \operatorname*{essup}_{x \in R} |g(x)|,$$

and $w_{2m}(x) = (1+x^{2m})^{-1}$, $m \in N_0 = N \cup \{0\}$. Let us introduce, formally, the generalized Favard operators F_n for functions $f\colon R \to R$:

$$F_n f(x) = \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x;\gamma) \qquad (x \in R, n \in N),$$

where

$$p_{n,k}(x;\gamma) = \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n} - x\right)^2\right)$$

and $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to zero (see [7]). In the case where $\gamma_n^2 = \theta/(2n)$ with a positive constant θ, F_n become the known Favard operators introduced by J. Favard [6] as discrete analogs of the singular Weierstrass integral. Some approximation properties of the classical Favard operators for continuous functions f on R are presented in [1], [2], and for the generalized operators F_n are given e.g. in [7], [11]. Denote by F_n^* the Kantorovich-type modification of operators F_n, defined by

$$F_n^* f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \qquad (x \in R, n \in N),$$

and by $\widetilde{F}_n$ the generalized Durrmeyer-type modification of operators F_n:

$$\widetilde{F}_n f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)dt \qquad (x \in R, n \in N),$$

where $f \in L_{2m}^p(R)$. Some estimates concerning the rate of pointwise convergence of $F_n^* f$ and $\widetilde{F}_n f$ can be found in [9], [10].

Let r be a positive integer. Define the r-th weighted modulus of smoothness of a function $f \in L_{2m}^p(R)$, $m \in N_0$, $1 \le p \le \infty$, as

$$\omega_r(f;\delta)_{2m,p} = \sup_{|h|\le\delta} \|w_{2m} \Delta_h^r f\|_p,$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f(x + h(r/2 - i))$$

and let us introduce the Lipschitz classes

$$\mathrm{Lip}_r(a;2m,p) = \{f \in L_{2m}^p(R)\colon \omega_r(f;\delta)_{2m,p} = O(\delta^a) \quad \text{as} \;\; \delta \to 0+\},$$

where $0 < a \le r$. In [8] we presented the direct theorems for the Favard-Kantorovich and Favard-Durrmeyer operators on $L_{2m}^p(R)$. Namely, under the assumption $n\gamma_n^2 \ge c$ for all $n \in N$, with a positive absolute constant c, we proved that

$$\|w_{2m}(F_n^* f - f)\|_p \le K(m,c)(\omega_2(f;\gamma_n)_{2m,p} + \gamma_n^2 \|w_{2m} f\|_p)$$

and

$$\|w_{2m}(\widetilde{F}_n f - f)\|_p \le K(m,c)(\omega_2(f;\gamma_n)_{2m,p} + \gamma_n^2\|w_{2m}f\|_p),$$

where $K(m,c)$ denotes some positive constant depending only on m and c. In this paper we present inverse theorems for these operators.

Throughout the paper, the symbols $K(r,m,c)$, $K_j(r,m,c)$ $(j=1,2,\dots)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. – Preliminary results.

Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence and let $n\gamma_n^2 \ge c$ for all $n \in N$, where c is a positive absolute constant. As it is known [11], for $\upsilon \in N_0, n \in N, x \in R$ we have

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{\upsilon} p_{n,k}(x;\gamma) \le 15A_c \left(\frac{2}{e}\right)^{\upsilon/2} \sqrt{(2\upsilon)!}\,\gamma_n^{\upsilon}, \tag{1}$$

where $A_c = \max\{1, (2c\pi^2)^{-1}\}$. A simple calculation and the known Schwarz inequality lead to

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{\upsilon} p_{n,k}(t;\gamma)dt \le \sqrt{(2\upsilon)!!}\,\frac{\gamma_n^{\upsilon}}{n} \qquad (k \in Z). \tag{2}$$

Let us choose $n \in N, j \in N_0$ and let us write

$$G_{n,j}^{*} f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^{j} \int_{k/n}^{(k+1)/n} f(t)\,dt \tag{3}$$

and

$$\widetilde{G}_{n,j} f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^{j} \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt, \tag{4}$$

where $f \in L_{2m}^p(R)$, $1 \le p \le \infty$. Obviously $G_{n,0}^{*} f(x) = F_n^{*} f(x)$ and $\widetilde{G}_{n,0} f(x) = \widetilde{F}_n f(x)$.

LEMMA 1. – *Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to 0 and let $n\gamma_n^2 \ge c$ for all $n \in N$, where c is a positive absolute constant. Then for*

$j \in N_0, f \in L_{2m,p}(R), m \in N_0, 1 \le p \le \infty$, *we have*

$$\|w_{2m} G^*_{n,j} f\|_p \le K(m,c)\sqrt{(2j+4m)!}\,\gamma_n^j \|w_{2m} f\|_p, \tag{5}$$

$$\|w_{2m} \widetilde{G}_{n,j} f\|_p \le K(m,c)\sqrt{(2j+4m)!}\,\gamma_n^j \|w_{2m} f\|_p. \tag{6}$$

PROOF. – In view of the definition (3),

$$\begin{aligned}
&(1+x^{2m})^{-1}|G^*_{n,j} f(x)| \\
&\le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\big(1+x^{2m}\big)^{-1}\left|\frac{k}{n}-x\right|^j \int_{k/n}^{(k+1)/n} (1+t^{2m})(1+t^{2m})^{-1}|f(t)|dt \\
&\le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\big(1+x^{2m}\big)^{-1}\left|\frac{k}{n}-x\right|^j \left(1+\left(\frac{|k|+1}{n}\right)^{2m}\right) \\
&\qquad \cdot \int_{k/n}^{(k+1)/n} (1+t^{2m})^{-1}|f(t)|dt \\
&\le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left((1+3^{2m})\left|\frac{k}{n}-x\right|^j + 3^{2m}\left|\frac{k}{n}-x\right|^{j+2m}\right) \\
&\qquad \cdot \int_{k/n}^{(k+1)/n} (1+t^{2m})^{-1}|f(t)|dt.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
\|w_{2m} G^*_{n,j} f\|_1 \le& \left((1+3^{2m})\sqrt{(2j)!!}\,\gamma_n^j + 3^{2m}\sqrt{(2j+4m)!!}\,\gamma_n^{j+2m}\right) \\
&\cdot \sum_{k=-\infty}^{\infty} \int_{k/n}^{(k+1)/n} (1+t^{2m})^{-1}|f(t)|dt \\
\le& ((1+3^{2m}) + 3^{2m}\gamma_n^{2m})\sqrt{(2j+4m)!!}\,\gamma_n^j \|w_{2m} f\|_1.
\end{aligned}$$

However for $p=\infty$, from (1) it follows that

$$\begin{aligned}
&\|w_{2m} G^*_{n,j} f\|_\infty \\
&\le \|w_{2m} f\|_\infty \sup_{x\in R}\left(\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left((1+3^{2m})\left|\frac{k}{n}-x\right|^j + 3^{2m}\left|\frac{k}{n}-x\right|^{j+2m}\right)\right) \\
&\le 15 \cdot \sqrt{(2j+4m)!}\,((1+3^{2m}) + 3^{2m}\gamma_n^{2m})A_c \gamma_n^j \|w_{2m} f\|_\infty.
\end{aligned}$$

Finally by the Riesz-Thorin theorem and the boundedness of the sequence (γ_n) we have (5).

In view of the definition (4)

$$
\begin{aligned}
&(1+x^{2m})^{-1}|\widetilde{G}_{n,j}f(x)| \\
&\le n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\big(1+x^{2m}\big)^{-1}\left|\frac{k}{n}-x\right|^{j} \\
&\quad\cdot\int_{-\infty}^{\infty} p_{n,k}(t;\gamma)\left(1+\left(t-\frac{k}{n}+\frac{k}{n}-x+x\right)^{2m}\right)(1+t^{2m})^{-1}|f(t)|dt \\
&\le \max\{1;3^{2m-1}\}n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left|\frac{k}{n}-x\right|^{j} \\
&\quad\cdot\int_{-\infty}^{\infty} p_{n,k}(t;\gamma)\left(1+\left|\frac{k}{n}-t\right|^{2m}+\left|\frac{k}{n}-x\right|^{2m}\right)(1+t^{2m})^{-1}|f(t)|dt.
\end{aligned}
$$

Applying (1) and (2), we get

$$\|w_{2m}\widetilde{G}_{n,j}f\|_1 \le 15A_c\max\{1;3^{2m-1}\}\sqrt{(2j+4m)!!}\,\gamma_n^{j}(1+2\gamma_n^{2m})\|w_{2m}f\|_1$$

and

$$\|w_{2m}\widetilde{G}_{n,j}f\|_\infty \le 15A_c\max\{1;3^{2m-1}\}\sqrt{(2j+4m)!}\,\gamma_n^{j}(1+2\gamma_n^{2m})\|w_{2m}f\|_\infty.$$

Finally by the Riesz-Thorin theorem and the boundedness of the sequence (γ_n) we have (6).

LEMMA 2. – *Let* $r\in N, 0<a<r,\quad 0<h\le 1,\quad 0<\delta\le 1,\quad f\in L_{2m,p}(R)$, $m\in N_0,\ 1\le p\le\infty$. *If the inequality*

$$\omega_r(f;h)_{2m,p}\le K(\delta^a+h^r+(h/\delta)^r\omega_r(f;\delta)_{2m,p})$$

is true for certain constant K, then $f\in \mathrm{Lip}_r\,(a;2m,p)$.

The proof of this Lemma is similar to that of Lemma in [3] (p. 696).

LEMMA 3. – *Suppose that* $\gamma=(\gamma_n)_{n=1}^{\infty}$ *is a positive sequence convergent to* 0 *and that* $n\gamma_n^{r/2+1}\ge cK(r)$, *where* $r\in N, r\ge 2, K(r)=\max_{n\in N}\{\gamma_n^{r/2-1}\}$ *and c is positive absolute constant. Let* $a_r=1$ *for even r and* $a_r=2$ *for odd r. Then for* $f\in L_{2m,p}(R), m\in N_0, 1\le p\le\infty$, *we have*

$$\|w_{2m}((F_n^*f)^{(r)}-(n/a_r)^rF_n^*\Delta_{a_r/n}^{r}f)\|_p\le K(r,m,c)\|w_{2m}f\|_p \tag{7}$$

for all $n \in N$ *such that* $n\gamma_n > 2(a_r)^2 r^2$, *and*

$$\|w_{2m}((\widetilde{F}_n f)^{(r)} - (n/a_r)^r \widetilde{F}_n \Delta^r_{a_r/n} f)\|_p \le K(r,m,c)\|w_{2m} f\|_p \tag{8}$$

for all $n \in N$ *such that* $n\gamma_n > r^2/4$.

PROOF. – We consider an even r. Let $r = 2r_1$ with $r_1 \in N$. Then

$$\begin{aligned}
&n^r F_n^*(\Delta^r_{1/n} f(x)) \\
&= n^{2r_1+1} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \sum_{i=0}^{2r_1} \binom{2r_1}{i} (-1)^i \int_{(k+r_1-i)/n}^{(k+r_1-i+1)/n} f(t)dt \\
&= n^{2r_1+1} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i \sum_{k=-\infty}^{k=\infty} p_{n,k-r_1+i}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \\
&\quad + n^{2r_1+1} \sum_{i=0}^{r_1-1} \binom{2r_1}{2r_1-i} (-1)^{2r_1-i} \sum_{k=-\infty}^{k=\infty} p_{n,k-r_1+2r_1-i}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \\
&\quad + n^{2r_1+1} \binom{2r_1}{r_1} (-1)^{r_1} \sum_{k=-\infty}^{k=\infty} p_{n,k-r_1+i}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \\
&= n^{2r_1+1} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i \sum_{k=-\infty}^{\infty} \left(p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma)\right) \int_{k/n}^{(k+1)/n} f(t)dt \\
&\quad + n^{2r_1+1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{r_1} (-1)^{r_1} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma) \\
&= p_{n,k}(x;\gamma)\left\{\exp\left(\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n}-x\right) - \frac{(r_1-i)^2}{2n^2\gamma_n^2}\right) + \exp\left(-\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n}-x\right) - \frac{(r_1-i)^2}{2n^2\gamma_n^2}\right)\right\} \\
&= p_{n,k}(x;\gamma) \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{-(r_1-i)^2}{2n^2\gamma_n^2}\right)^l \left\{\left(-\frac{2n}{r_1-i}\left(\frac{k}{n}-x\right)+1\right)^l + \left(\frac{2n}{r_1-i}\left(\frac{k}{n}-x\right)+1\right)^l\right\} \\
&= p_{n,k}(x;\gamma) \sum_{l=0}^{\infty} \frac{(-1)^l (r_1-i)^{2l}}{l!} 2^{-l} n^{-2l} \gamma_n^{-2l} \sum_{j=0}^{[l/2]} \binom{l}{2j} 2 \cdot \frac{2^{2j} n^{2j}}{(r_1-i)^{2j}} \left(\frac{k}{n}-x\right)^{2j} = \\
&= p_{n,k}(x;\gamma) \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{j=0}^{[l/2]} \binom{l}{2j} 2^{2j+1-l} \left(\frac{k}{n}-x\right)^{2j} n^{2j-2l} \gamma_n^{-2l} (r_1-i)^{2l-2j} + 2p_{n,k}(x;\gamma).
\end{aligned}$$

Consequently, using the definition (3), we get

$$(9)\quad n^r F_n^*(\Delta_{1/n}^r f(x))$$

$$= \sum_{l=1}^{\infty}\sum_{j=0}^{[l/2]} n^{2(r_1+j-2l)}\gamma_n^{-2l}\frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^* f(x)$$

$$+ 2n^{2r_1+1}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\int_{k/n}^{(k+1)/n} f(t)dt$$

$$+ n^{2r_1+1}\sum_{k=-\infty}^{\infty}\binom{2r_1}{r_1}(-1)^{r_1}p_{n,k}(x;\gamma)\int_{k/n}^{(k+1)/n} f(t)dt$$

$$= \sum_{l=1}^{2r_1}\sum_{j=0}^{[l/2]} n^{2(r_1+j-l)}\gamma_n^{-2l}\frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^* f(x)$$

$$+ \sum_{l=2r_1+1}^{\infty}\sum_{j=0}^{[l/2]} n^{2(r_1+j-l)}\gamma_n^{-2l}\frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^* f(x)$$

$$= S_{n,1}f(x) + S_{n,2}f(x).$$

In view of (5),

$$\|w_{2m}S_{n,2}f\|_p \le K(m,c)\|w_{2m}f\|_p 4^{r_1}n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{r_1^{2l}}{n^{2l}\gamma_n^{2l}2^l}\sum_{j=0}^{[l/2]}\frac{\sqrt{(4j+4m)!}}{(2j)!(l-2j)!}n^{2j}\gamma_n^{2j}4^j r_1^{-2j}.$$

Using the Stirling formula we obtain

$$\|w_{2m}S_{n,2}f\|_p \le K_1(r,m,c)\|w_{2m}f\|_p n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{(r_1^2/2)^l}{(n^2\gamma_n^2)^l}\sum_{j=0}^{[l/2]}(n^2\gamma_n^2)^j 64^j$$

$$\le K_1(r,m,c)\|w_{2m}f\|_p\left\{\frac{(8r_1^2)^{2r_1+1}}{n^2\gamma_n^{2r_1+2}} + n^{2r_1}\sum_{l=2r_1+2}^{\infty}\left(\frac{8r_1^2}{n\gamma_n}\right)^l\right\}.$$

Assuming $(8r_1^2)/(n\gamma_n) < 1$ and using the condition $n\gamma_n^{r_1+1} \ge cK(r)$ we get

$$\|w_{2m}S_{n,2}f\|_p \le K_2(r,m,c)\|w_{2m}f\|_p.$$

From the mean value theorem

$$\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2s} = \begin{cases} 0 & \text{if } 0 < s < r_1,\\ (2r_1)!/2 & \text{if } s = r_1.\end{cases}$$

Therefore

$$S_{n,1}f(x)$$

$$= \sum_{l=r_1}^{2r_1} \sum_{j=0}^{[l/2]} \frac{(-1)^l 2^{2j+1-l}}{l! n^{2l-2j-2r_1} \gamma_n^{2l}} \binom{l}{2j} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i (r_1-i)^{2l-2j} G^*_{n,2j} f(x)$$

$$= \sum_{l=0}^{r_1} \sum_{j=0}^{l-1} \frac{(-1)^{r_1+l} 2^{2j+1-l-r_1}}{(r_1+l)! n^{2l-2j} \gamma_n^{2l+2r_1}} \binom{r_1+l}{2j} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i (r_1-i)^{2r_1+2l-2j} G^*_{n,2j} f(x)$$

$$+ \sum_{l=0}^{r_1} \frac{(-1)^{2r_1-l}}{\gamma_n^{4r_1-2l}} \frac{(2r_1)!}{2^l l! (2r_1-2l)!} G^*_{n,2j} f(x).$$

It is easy to see, by the method of induction, that

$$p^{(\upsilon)}_{n,k}(x;\gamma) = p_{n,k}(x;\gamma) \sum_{i=0}^{[\upsilon/2]} \frac{\upsilon!(-1)^i}{(\upsilon-2i)!(2i)!!} \frac{1}{\gamma_n^{2\upsilon-2i}} \left(\frac{k}{n} - x\right)^{\upsilon-2i} \qquad \text{for} \quad \upsilon \in N. \tag{10}$$

Therefore

$$S_{n,1}f(x) =$$

$$= \sum_{l=0}^{r_1} \sum_{j=0}^{l-1} \frac{(-1)^{r_1+l} 2^{2j+1-l-r_1}}{(r_1+l)! n^{2l-2j} \gamma_n^{2l+2r_1}} \binom{r_1+l}{2j} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i (r_1-i)^{2r_1+2l-2j} G^*_{n,2j} f(x) +$$

$$+ (F^*_n f(x))^{(2r_1)}.$$

Consequently, from (9),

$$|(F^*_n f)^{(2r_1)}(x) - n^{2r_1} F^*_n \varDelta^{2r_1}_{1/n} f(x)| \le K_1(r) \sum_{j=0}^{r_1-1} \sum_{l=j+1}^{r_1} \frac{n^{2j}}{(n\gamma_n)^{2l} \gamma_n^{2r_1}} |G^*_{n,2j} f(x)| + |S_{n,2} f(x)|.$$

The condition $n\gamma_n^{r_1+1} \ge cK(r)$ and the boundedness of the sequence (γ_n) lead to

$$|(F^*_n f)^{(2r_1)}(x) - n^{2r_1} F^*_n \varDelta^{2r_1}_{1/n} f(x)| \le K(r,c) \sum_{j=0}^{r_1-1} \gamma_n^{-2j} |G^*_{n,2j} f(x)| + |S_{n,2} f(x)|.$$

Collecting the results we obtain the estimate (7) for even r.

Now, we will prove inequality (7) for odd r. Let $r = 2r_2+1, r_2 \in N$. Then

$$n^r F^*_n (\varDelta^r_{2/n} f(x)) =$$

$$= n^{2r_2+2} \sum_{i=0}^{r_2} \sum_{k=-\infty}^{\infty} \binom{2r_2+1}{i} (-1)^i (p_{n,k-(2r_2+1-2i)}(x;\gamma) - p_{n,k+(2r_2+1-2i)}(x;\gamma)) \int_{k/n}^{(k+1)/n} f(t)dt.$$

It is easy to see that

$$p_{n,k-(2r_2+1-2i)}(x;\gamma) - p_{n,k+(2r_2+1-2i)}(x;\gamma)$$
$$= p_{n,k}(x;\gamma)\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l!}\sum_{j=0}^{[(l-1)/2]}\binom{l}{2j+1}2^{2j+2-l}\left(\frac{k}{n}-x\right)^{2j+1}\frac{n^{2j+1-2l}}{\gamma_n^{2l}(2r_2+1-2i)^{2j-2l+1}}.$$

Consequently

$$(11)\quad n^r F_n^*(\Delta_{2/n}^r f(x))$$
$$= \sum_{l=1}^{2r_2+1} n^{2r_2+2}\sum_{j=0}^{[(l-1)/2]} n^{2j-2l}\gamma_n^{-2l}\frac{(-1)^{l+1}2^{2j+2-l}}{(2j+1)!(l-2j-1)!}$$
$$\cdot\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(2r_2+1-2i)^{2l-2j-1}G_{n,2j+1}^* f(x)$$
$$+\sum_{l=2r_2+2}^{\infty} n^{2r_2+2}\sum_{j=0}^{[(l-1)/2]} n^{2j-2l}\gamma_n^{-2l}\frac{(-1)^{l+1}2^{2j+2-l}}{(2j+1)!(l-2j-1)!}$$
$$\cdot\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(2r_2+1-2i)^{2l-2j-1}G_{n,2j+1}^* f(x)$$
$$= S_{n,1}^* f(x) + S_{n,2}^* f(x).$$

A simple calculation, the Stirling formula and (5) give

$$\|w_{2m}S_{n,2}^* f\|_p \le K_3(r,m,c)\|w_{2m}f\|_p$$

for $n \in N$ such that $(8r^2)/(n\gamma_n) < 1$. Next, in view of (10) and the equality

$$\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(r_2-i+1/2)^{2s-1} = \begin{cases} 0 & \text{if } 0 < s < r_2+1, \\ (2r_2+1)!/2 & \text{if } s = r_2+1 \end{cases}$$

we obtain

$$S_{n,1}^* f(x)$$
$$=\sum_{l=0}^{r_2}\sum_{j=0}^{l-1}\frac{(-1)^{r_2+l}2^{2j+1-l-r_2}}{(2j+1)!(l+r_2-2j)!}n^{2j-2l}\gamma_n^{-2l-2r_2-2}\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(2r_2+1-2i)^{2r_2+2l-2j+1}$$
$$\cdot G_{n,2j+1}^* f(x) + 2^{2r_2+1}(F_n^* f)^{(2r_2+1)}(x).$$

Using (11) and the condition $n\gamma_n^{r_2+3/2} \ge cK(r)$ we have

$$|(F_n^* f)^{(2r_2+1)}(x) - (n/2)^{2r_2+1}F_n^*\Delta_{2/n}^{2r_2+1}f(x)| \le K(r_2,c)\sum_{j=0}^{r_2-1}\frac{1}{\gamma_n^{2j+1}}|G_{n,2j+1}^* f(x)| + |S_{n,2}^* f(x)|.$$

Applying (5), we get (7) for odd r. Therefore inequality (7) is proved.

Now we will prove (8). Let $r = 2r_1, r_1 \in N$. Simple calculation and the equality $p_{n,k}(t-(r_1-i)/n;\gamma) = p_{n,k+r_1-i}(t;\gamma)$, give

$$n^r \widetilde{F}_n(\Delta^r_{1/n} f(x))$$

$$= n^{2r_1+1} \sum_{i=0}^{r_1-1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{i} (-1)^i (p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma)) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt$$

$$+ n^{2r_1+1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{r_1} (-1)^i p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt.$$

The estimate (8) follows now on the same way as (7).

3. -- Main results.

As it is known ([5]), the weighted modulus of smoothness $\omega_r(f;t)_{2m,p}$ of $f \in L^p_{2m}(R)$, defined in Section 1, is equivalent to the weighted $\mathcal{K}$-functional

$$\mathcal{K}(f;t^r)_{2m,p} = \inf_g \{\|w_{2m}(f-g)\|_p + t^r \|w_{2m} g^{(r)}\|_p; g \in L^{p,r}_{2m}\},$$

where $L^{p,r}_{2m}(R)$ denotes the class of all functions $g \in L^p_{2m}(R)$ possessing derivative $g^{(r-1)}$ locally absolutely continuous on R and such that $g^{(r)} \in L^p_{2m}(R)$. Then there exists a constant $M > 0$, independent of f and t, such that

$$M^{-1}\omega_r(f;t)_{2m,p} \le \mathcal{K}_r(f;t^r)_{2m,p} \le M\omega_r(f;t)_{2m,p} \qquad \text{for} \quad t > 0.$$

Using this equivalence we present the following

THEOREM. *Let $r \in N, r \ge 2, a \in (0;r)$ and let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive null sequence complying with the following inequality $n\gamma_n^{r/2+1} \ge cK(r)$, for all $n \in N$ with some $c > 0$, where $K(r) = \max_{n \in N}\{\gamma_n^{r/2-1}\}$. Let $f \in L^p_{2m}(R), m \in N_0, 1 \le p \le \infty$ and let $n \in N$ be such that $n\gamma_n > 2(a_r^2)r^2$, where $a_r = 2$ for odd r, $a_r = 1$ for even r. Then, the rate of approximation*

$$\|w_{2m}(F^*_n f - f)\|_p = O(\gamma_n^a) \qquad \text{or} \qquad \|w_{2m}(\widetilde{F}_n f - f)\|_p = O(\gamma_n^a) \tag{12}$$

implies $f \in \mathrm{Lip}_r\,(a;2m,p)$.

PROOF. – It is easy to see that for $0 < h \le 1$ and $0 < \delta \le 1$

$$\|w_{2m}\Delta^r_h f\|_p \le \sum_{i=0}^{r} \binom{r}{i} \left\| w_{2m} f\left(\circ + h\left(\frac{r}{2} - i\right)\right)\right\|_p \le 2^r \max\{2; 1 + r^{2m}/2\} |w_{2m} f|_p$$

and

$$\left\| w_{2m} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} f(\bullet + s_1 + \cdots + s_r) ds_1 \cdots ds_r \right\|_p \le (1 + \max\{1; 2^{2m-1}\} r^{2m}) \delta^r \|w_{2m} f\|_p.$$

Applying these inequalites and (12), we get

$$\begin{aligned}\|w_{2m} \varDelta_h^r f\|_p &\le \|w_{2m} \varDelta_h^r (F_n^* f - f)\|_p + \|w_{2m} \varDelta_h^r F_n^* f\|_p \\ &\le 2^r \max\{2; 1 + r^{2m}/2\} \|w_{2m} (F_n^* f - f)\|_p \\ &\quad + \left\| w_{2m} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} (F_n^* f)^{(r)} (\bullet + s_1 + \cdots + s_r) ds_1 \cdots ds_r \right\|_p \\ &\le K(r,m) \gamma_n^a + K_1(r,m) h^r \|w_{2m} (F_n^* f)^{(r)}\|_p.\end{aligned}$$

Let $n \in N$ and let g_n be a function of class $L_{2m}^{p,r}(R)$ for which

$$\|w_{2m}(f - g_n)\|_p \le 2\mathcal{K}_r(f; \gamma_n^r) \qquad \text{and} \qquad \gamma_n^r \|w_{2m} g_n^{(r)}\|_p \le 2\mathcal{K}_r(f; \gamma_n^r).$$

Using (10), we have

$$\begin{aligned}&\|w_{2m}(F_n^* f)^{(r)}\|_p \\ &= \left\| w_{2m} \sum_{i=0}^{[r/2]} \frac{r!(-1)^i}{(r-2i)!(2i)!!} \frac{1}{\gamma_n^{2r-2i}} n \sum_{k=-\infty}^{\infty} p_{n,k}(\circ;\gamma) \left(\frac{k}{n} - \circ\right)^{r-2i} \int_{k/n}^{(k+1)/n} f(t) dt \right\|_p.\end{aligned}$$

In view of (5),

$$\begin{aligned}\|w_{2m}(F_n^* f)^{(r)}\|_p &\le \left\| \sum_{i=0}^{[r/2]} \frac{r!}{(r-2i)!(2i)!!} \frac{1}{\gamma_n^{2r-2i}} w_{2m} G_{n,r-2i}^* \right\|_p \\ &\le r![r/2] \sum_{i=0}^{[r/2]} \frac{1}{\gamma_n^{2r-2i}} \|w_{2m} G_{n,r-2i}^*\|_p \le K_3(r,m,c) \gamma_n^{-r} |w_{2m} f|_p.\end{aligned}$$

Next, in view of (7) and the above inequality,

$$\begin{aligned}\|w_{2m}(F_n^* f)^{(r)}\|_p &\le \|w_{2m} F_n^* (f - g_n)^{(r)}\|_p + \|w_{2m}((n/a_r)^r F_n^*(\varDelta_{a_r/n}^r g_n))\|_p \\ &\quad + \|w_{2m}((F_n^* g_n)^{(r)} - (n/a_r)^r F_n^*(\varDelta_{a_r/n}^r g_n)\|_p \\ &\le K_3(r,m,c) \gamma_n^{-r} \|w_{2m}(f - g_n)\|_p + K(r,m) \|w_{2m} F_n^* g_n^{(r)}\|_p \\ &\quad + K(r,m,c) \|w_{2m}(f - g_n)\|_p + K(r,m,c) \|w_{2m} f\|_p.\end{aligned}$$

Using (5) for $j = 0$, we have

$$\|w_{2m}(F_n^* f)^{(r)}\|_p \le K_4(r,m,c)\gamma_n^{-r}(\|w_{2m}(f-g_n)\|_p + \gamma_n^r\|w_{2m}g_n^{(r)}\|_p + \gamma_n^r\|w_{2m}f\|_p)$$

$$\le K_5(r,m,c)\gamma_n^{-r}(\mathcal{K}_r(f;\gamma_n^r)_{2m,p} + \gamma_n^r\|w_{2m}f\|_p)$$

$$\le K_6(r,m,c)\gamma_n^{-r}(\omega_r(f;\gamma_n)_{2m,p} + \gamma_n^r\|w_{2m}f\|_p).$$

Consequently

$$\|w_{2m}\Delta_h^r f\|_p \le K(r,m)\gamma_n^a + K_1(r,m,c)h^r\gamma_n^{-r}(\omega_r(f;\gamma_n)_{2m,p} + \gamma_n^r\|w_{2m}f\|_p)$$

$$\le K_7(r,m,c)(\gamma_n^a + (h/\gamma_n)^r\omega_r(f;\gamma_n)_{2m,p} + \gamma_n^r).$$

Applying Lemma 2, we can prove the Theorem for $F_n^* f$.

On the same way we can prove the Theorem for $\widetilde{F}_n f$, using (6) and (8).

From this Theorem and [8], we have

COROLLARY. *Let $\gamma = (\gamma_n)_{n=1}^\infty$ be a positive null sequence satisfying $n\gamma_n^2 \ge c$ for all $n \in N$ with some $c > 0$, $f \in L_{2m}^p(R), m \in N_0, 1 \le p \le \infty$. Then, for $n \in N$ such that $n\gamma_n > 8$ and $a \in (0;2)$ the following assertions are equivalent:*

a) $\|w_{2m}(F_n^* f - f)\|_p = O(\gamma_n^a)$ or $\|w_{2m}(\widetilde{F}_n f - f)\|_p = O(\gamma_n^a)$

b) $f \in \mathrm{Lip}_2\,(a;2m,p)$.

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