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Simplicity of generic Steiner bundles.

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Sunto. – *Un fibrato di Steiner E su $\mathbb{P}^n$ ha una risoluzione lineare della forma $0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to E \to 0$. In questo lavoro proviamo che il generico fibrato di Steiner E è semplice se e solo se $\chi(\mathrm{End}\, E)$ è minore o uguale a 1. In particolare mostriamo che E è eccezionale oppure soddisfa la disuguaglianza $t \leq \left(\frac{n+1+\sqrt{(n+1)^2-4}}{2}\right)s$.*

Summary. – *A Steiner bundle E on $\mathbb{P}^n$ has a linear resolution of the form $0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to E \to 0$. In this paper we prove that a generic Steiner bundle E is simple if and only if $\chi(\mathrm{End}\, E)$ is less or equal to 1. In particular we show that either E is exceptional or it satisfies the inequality $t \leq \left(\frac{n+1+\sqrt{(n+1)^2-4}}{2}\right)s$.*

1. – Introduction

According to [2] a Steiner bundle E on $\mathbb{P}(V) = \mathbb{P}^{N-1}$ has a linear resolution of the form

$$0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to E \to 0.$$

It is well known that Steiner bundles have rank $t - s \geq N - 1$ and if equality holds then they are stable, in particular they are simple (see [1]). The aim of this paper is to investigate the simplicity of Steiner bundles for higher rank.

MAIN THEOREM -- *Let E be a Steiner bundle on $\mathbb{P}^{N-1}$, with $N \geq 3$, defined by the exact sequence*

$$0 \to \mathcal{O}(-1)^s \xrightarrow{m} \mathcal{O}^t \to E \to 0,$$

where m is a generic morphism in $\mathrm{Hom}\,(\mathcal{O}(-1)^s, \mathcal{O}^t)$*, then the following statements are equivalent:*

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i) *E is simple, i.e.* $\mathrm{h}^0(\operatorname{End} E) = 1$,
ii) $s^2 - Nst + t^2 \leq 1$ *i.e.* $\chi(\operatorname{End} E) \leq 1$,
iii) *either* $s^2 - Nst + t^2 \leq 0$ *i.e.* $t \leq (\frac{N+\sqrt{N^2-4}}{2})s$ *or* $(t,s) = (a_{k+1}, a_k)$, *where* $a_k = \dfrac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}}$.

The generalized Fibonacci numbers appearing in (iii) satisfy a recurrence relation, as it is clear from the proof of Theorem 2.1.

Our result in the case of $\mathbb{P}^2$ is partially contained, although somehow hidden, in [3]. Indeed Drézet and Le Potier find a criterion to check the stability of a generic bundle, given its rank and Chern classes. In the case of a normalized Steiner bundle E on $\mathbb{P}^2$, it is possible to prove that if E satisfies condition (iii) of the main theorem, then the Drézet-Le Potier condition for stability is satisfied. Hence E is stable and, consequently, simple. On the other hand, when E is not normalized, it is very complicated to check the criterion of Drézet-Le Potier, but we can easily prove the simplicity with other techniques. Anyway the proof that we present in this paper is independent of [3], is more elementary and works on $\mathbb{P}^n$ as well.

The genericity assumption cannot be dropped, because when $\mathrm{rk}E = t - s > N - 1$ it is always possible to find a decomposable Steiner bundle, that is in particular non-simple.

Since the equivalence between conditions (ii) and (iii) is an arithmetic statement, our theorem claims that $\chi(\operatorname{End} E)$ is the responsible for the simplicity of a generic Steiner bundle E. Indeed it is easy to check that if E is simple then $\chi(\operatorname{End} E) \leq 1$ (Lemma 3.2) and this is also true for some other bundles, for example for every bundle on $\mathbb{P}^2$. The converse is not true in general, because it is possible to find a non-simple bundle F on $\mathbb{P}^2$ such that $\chi(\operatorname{End} F) < 1$. For example we can consider the cokernel F of a generic map of the form

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^4 \to \mathcal{O}^{16} \to F \to 0,$$

where $\chi(\operatorname{End} F) = -3$, but it can be shown that $\mathrm{h}^0(\operatorname{End} F) = 5$ therefore F is not simple.

In the third statement of our theorem we claim that if E is a simple Steiner bundle, then either E is exceptional or it satisfies a numerical inequality (see Theorem 2.1). We recall that exceptional bundles have no deformations. The name exceptional in this setting is justified by the fact that they are the only simple Steiner bundles which violate the numerical inequality. It is remarkable to note that all the exceptional bundles on $\mathbb{P}^2$ can be constructed by the theory of helices, in particular there exists a correspondence between the exceptional bundles on the projective plane and the solutions of the Markov equation $x^2 + y^2 + z^2 = 3xyz$ (see [6]).

The plan of the article is as follows: Section 2 is devoted to the case of ex-

ceptional bundles and Section 3 to the proof of the main theorem. At the end of the paper, Theorem 3.8 is a reformulation in terms of matrices of the main theorem. As a basic reference for bundles on $\mathbb{P}^n$ see [5].

I would like to thank Giorgio Ottaviani, for suggesting me the problem and for his continuous assistance, and Enrique Arrondo, for many useful discussions. I also thank very much Jean Vallès, for his helpful comments concerning this work, in particular for his collaboration in simplifying the proof of Lemma 3.7.

2. – Exceptional bundles

In [7] the theory of helices of exceptional bundles is developed in a general axiomatic presentation. Here we give the following result as a particular case of this theory.

THEOREM 2.1. – [6, 7] *Let E_k be a generic Steiner bundle on $\mathbb{P}^{N-1}$, with $N \geq 3$, defined by the exact sequence*

$$0 \to \mathcal{O}(-1)^{a_{k-1}} \to \mathcal{O}^{a_k} \to E_k \to 0,$$

where

$$a_k = \frac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}},$$

then E_k is exceptional (i.e. $\mathrm{h}^0(\operatorname{End} E) = 1$ and $\mathrm{h}^i(\operatorname{End} E) = 0$ for all $i > 0$.)

On $\mathbb{P}(V) = \mathbb{P}^{N-1}$ we define a sequence of vector bundles as follows:

$$F_0 = \mathcal{O}(1), \quad F_1 = \mathcal{O}, \quad F_{n+1} = \ker(F_n \otimes \operatorname{Hom}(F_n, F_{n-1}) \xrightarrow{\psi_n} F_{n-1}), \tag{1}$$

where ψ_n is the canonical map.

The following lemma can be found in [7]. We underline that it is possible to prove it in a straightforward way only by standard cohomology sequences.

LEMMA 2.2. – *Given the definition* (1), *for all $n \geq 1$ the canonical map ψ_n is an epimorphism. Moreover the following properties (A_n), (B_n) and (C_n) are satisfied for all $n \geq 1$:*

$$(A_n) \quad \operatorname{Hom}(F_n, F_n) \cong \mathbb{C}, \quad \operatorname{Ext}^i(F_n, F_n) = 0, \quad \textit{for all} \quad i \geq 1,$$
$$(B_n) \quad \operatorname{Hom}(F_{n-1}, F_n) = 0, \quad \operatorname{Ext}^i(F_{n-1}, F_n) = 0, \quad \textit{for all} \quad i \geq 1,$$
$$(C_n) \quad \operatorname{Hom}(F_n, F_{n-1}) \cong V, \quad \operatorname{Ext}^i(F_n, F_{n-1}) = 0, \quad \textit{for all} \quad i \geq 1.$$

Note that (A_n) means that every F_n is an exceptional bundle.

REMARK 2.3. – *Following* [7] *the previous lemma means that* (F_n, F_{n-1}) *is a left admissible pair and* (F_{n+1}, F_n) *is the left mutation of* (F_n, F_{n-1}) *and that the sequence* (F_n) *forms an exceptional collection generated by the helix* $(\mathcal{O}(i))$ *by left mutations.*

PROOF OF THEOREM 2.1. – Lemma 2.2 states that the bundles F_n, defined as in (1), are exceptional for all $n \geq 0$. Obviously their dual F_n^* are exceptional too. Now we will prove that, for every $n \geq 1$, the bundle F_n^* admits the following resolution

$$0 \to \mathcal{O}(-1)^{a_{n-1}} \to \mathcal{O}^{a_n} \to F_n^* \to 0, \tag{2}$$

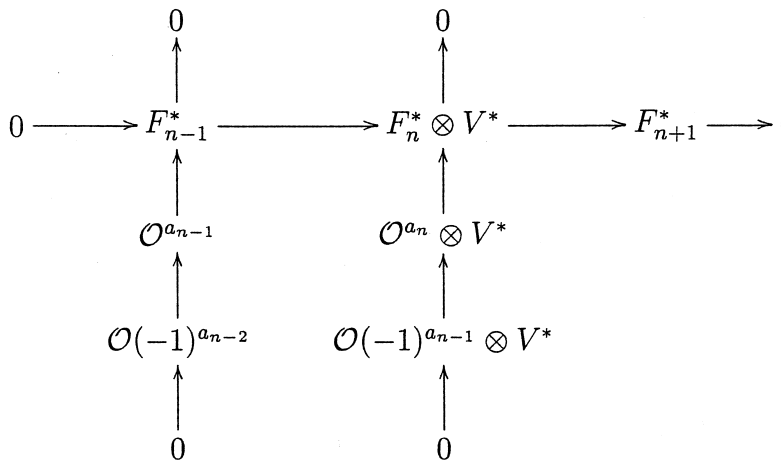
where $\{a_n\}$ is the sequence defined in the statement. This implies that a generic bundle with this resolution is exceptional. We can prove (2) by induction on n. First of all we notice that the sequence $\{a_n\}$ is also defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{n+1} = Na_n - a_{n-1}. \end{cases}$$

Therefore if $n = 1$ the sequence (2) is $0 \to \mathcal{O}(-1)^{a_0} \to \mathcal{O}^{a_1} \to F_1^* \to 0$, i.e. $0 \to \mathcal{O} \to F_1^* \to 0$, and this is true because $F_1 \cong \mathcal{O}$. Now let us suppose that every F_k^* admits a resolution (2) for all $k \leq n$ and we will prove it for F_{n+1}^*. Let us dualize the sequence

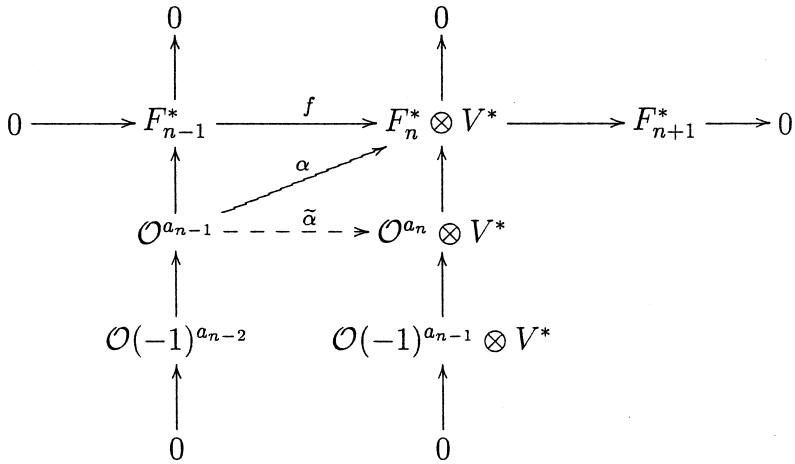
$$0 \to F_{n+1} \to F_n \otimes \mathrm{Hom}\,(F_n, F_{n-1}) \to F_{n-1} \to 0$$

and by induction hypothesis we have:

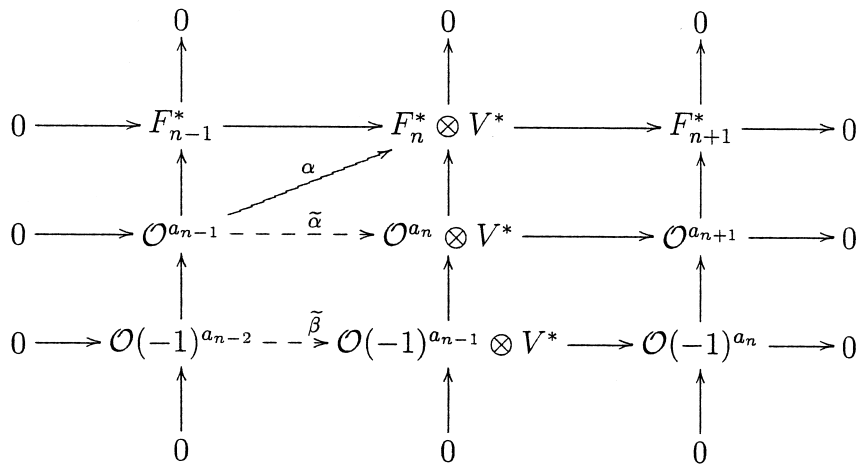


We define the map $a : \mathcal{O}^{a_{n-1}} \to F_n^* \otimes V^*$ as the composition of the known maps. Since $\mathrm{Ext}^1(\mathcal{O}^{a_{n-1}}, \mathcal{O}(-1)^{a_{n-1}} \otimes V^*) \cong \mathrm{H}^1(\mathcal{O}(-1))^{a_{n-1}^2} \otimes V^* = 0$, the map

a induces a map $\widetilde{a} : \mathcal{O}^{a_{n-1}} \to \mathcal{O}^{a_n} \otimes V^*$ such that the following diagram commutes:



We observe that $\widetilde{a}$ is injective if and only if $\mathrm{H}^0(\widetilde{a})$ is injective and, since $\mathrm{H}^0(\widetilde{a}) = \mathrm{H}^0(f)$, they are injective. Obviously the cokernel of $\widetilde{a}$ is $\mathcal{O}^{Na_n - a_{n-1}} = \mathcal{O}^{a_{n+1}}$. Let $\widetilde{\beta}$ be the restriction of $\widetilde{a}$ to $\mathcal{O}(-1)^{a_{n-2}}$. Then we can check that $\widetilde{\beta}$ is injective, its cokernel is $\mathcal{O}(-1)^{Na_{n-1} - a_{n-2}} = \mathcal{O}(-1)^{a_n}$ and the following diagram commutes:



It follows that F^*_{n+1} has the resolution $0 \to \mathcal{O}(-1)^{a_n} \to \mathcal{O}^{a_{n+1}} \to F^*_{n+1} \to 0$ and this completes the proof of our theorem. ∎

3. – Proof of the main theorem

Let E be given by the exact sequence on $\mathbb{P}^{N-1} = \mathbb{P}(V)$

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \xrightarrow{m} W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0, \tag{3}$$

where V, I and W are complex vector spaces of dimension $N \geq 3$, s and t respectively and m is a generic morphism. If we fix a basis in each of the vector spaces I and W, the morphism m can be represented by a $t \times s$ matrix M whose entries are linear forms. Let us consider the natural action of $\mathrm{GL}(I) \times \mathrm{GL}(W)$ on the space

$$H = \mathrm{Hom}\,(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong V \otimes I^{\vee} \otimes W,$$

i.e. the action

$$H \times \mathrm{GL}(I) \times \mathrm{GL}(W) \to H$$

$$(M, A, B) \mapsto A^{-1} M B.$$

When the pair (A, B) belongs to the stabilizer of M, it induces a morphism $\phi : E \to E$, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \xrightarrow{M} & W \otimes \mathcal{O} & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow B & & \downarrow \phi & & \\ 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \xrightarrow{M} & W \otimes \mathcal{O} & \longrightarrow & E & \longrightarrow & 0 \end{array} \tag{4}$$

I. Now we prove the first part of the theorem, i.e. the fact that (i) implies (ii).

REMARK 3.1. – *From the sequence* (3) *it follows that* $\chi(E) = t$ *and* $\chi(E(1)) = (Nt - s)$. *Dualizing* (3) *and tensoring by* E *we get*

$$0 \longrightarrow \mathrm{End}\, E \longrightarrow W^{\vee} \otimes E \longrightarrow I^{\vee} \otimes E(1) \longrightarrow 0, \tag{5}$$

therefore

$$\chi(\mathrm{End}\, E) = t\chi(E) - s\chi(E(1)) = t^2 - s(Nt - s) = t^2 - Nst + s^2.$$

LEMMA 3.2. – *If* E *is a simple Steiner bundle, then* $\chi(\mathrm{End}\, E) \leq 1$.

PROOF. – From the sequences (3) and (5) it is easy to check that $\mathrm{H}^i(\mathrm{End}\, E) = 0$, for all $i \geq 2$. Moreover $\mathrm{h}^0(\mathrm{End}\, E) = 1$ because of the simplicity, and consequently $\chi(\mathrm{End}\, E) = 1 - \mathrm{h}^1(\mathrm{End}\, E) \leq 1$. ∎

II. Now we prove that statement (ii) is equivalent to (iii).

REMARK 3.3. – *Obviously* $s^2 - Nst + t^2 \leq 0$ *is equivalent to* $\left(\frac{N-\sqrt{N^2-4}}{2}\right)s \leq t \leq \left(\frac{N+\sqrt{N^2-4}}{2}\right)s$. *Since* $t > s$ *and* $N > 2$ *this inequality is equivalent to* $t \leq \left(\frac{N+\sqrt{N^2-4}}{2}\right)s$. *Thus we have only to prove that* $s^2 - Nst + t^2 = 1$ *is equivalent to* $(t, s) = (a_{k+1}, a_k)$ *where* a_k *has been defined above.*

LEMMA 3.4. – *All the integer solutions of* $s^2 - Nst + t^2 = 1$, *when* $t > s$, *are exactly* $s = a_k, t = a_{k+1}$, *where* $a_k = \dfrac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}}$.

PROOF. – We already know that the sequence $\{a_k\}$ is defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{k+1} = Na_k - a_{k-1}. \end{cases}$$

So we prove by induction on k that $(s = a_k, t = a_{k+1})$ is a solution of

$$s^2 - Nst + t^2 = 1. \tag{6}$$

If $k = 0$, obviously $(s = 0, t = 1)$ is a solution. Let the pair (a_{k-1}, a_k) satisfy (6), then, using the recursive definition, we check that (a_k, a_{k+1}) is a solution too. Hence we have to prove that there are no other solutions. By the change of coordinates $\{r = 2t - Ns, s = s\}$ our equation becomes the following Pell-Fermat equation $r^2 - (N^2 - 4)s^2 = 4$. By Number Theory results (see for example [8], page 77, or [4]), we know that all the solutions (r, s) are given by the sequence (r_k, s_k) defined by

$$r_k + s_k\sqrt{N^2 - 4} = \frac{1}{2^{k-1}}(N + \sqrt{N^2 - 4})^k,$$

for all $k \geq 0$. Now we have only to prove that these solutions are exactly those already known. We can easily check that the pair of sequences (s_k, t_k) can be recursively defined by

$$\begin{cases} r_0 = 2, \\ s_0 = 0, \\ r_{k+1} = \frac{(N^2-4)s_k + Nr_k}{2}, \\ s_{k+1} = \frac{Ns_k + r_k}{2}. \end{cases}$$

By a change of coordinates we define $t_k = \frac{Ns_k + r_k}{2}$ and we check that the pair (s_k, t_k) is exactly (a_k, a_{k+1}), for all $k \geq 0$. In fact $(s_0, t_0) = (0, 1) = (a_0, a_1)$ and, moreover, $t_k = s_{k+1}$ and $t_{k+1} = \frac{Ns_{k+1} + r_{k+1}}{2} = \frac{(N^2-2)s_k + Nr_k}{2} = \frac{(N^2-2)s_k + N(2t_k - Ns_k)}{2} = Nt_k - t_{k-1}$. ∎

III. Now we prove the last implication, i.e. (iii) implies (i). In the case $(t, s) = (a_{k+1}, a_k)$, the generic E is an exceptional bundle by Theorem 2.1,

therefore it is in particular simple. So suppose $s^2 - Nst + t^2 \leq 0$ and recall that H denotes $\mathrm{Hom}\,(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong V \otimes I^\vee \otimes W$. Let S be the set

$$\{A, B, M : A^{-1}MB = M\} \subset \mathrm{GL}(I) \times \mathrm{GL}(W) \times H$$

and π_1 and π_2 the projections on $\mathrm{GL}(I) \times \mathrm{GL}(W)$ and on H respectively. Notice that, for all $M \in H$, $\pi_1(\pi_2^{-1}(M))$ is the stabilizer of M with respect to the action of $\mathrm{GL}(I) \times \mathrm{GL}(W)$. Obviously $(\lambda\mathrm{Id}, \lambda\mathrm{Id}) \in \mathrm{Stab}(M)$, therefore $\dim \mathrm{Stab}\,(M) \geq 1$.

LEMMA 3.5. – *If E is defined by the sequence*

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \xrightarrow{M} W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0 \tag{7}$$

and $\dim \mathrm{Stab}\,(M) = 1$, *then E is simple.*

PROOF. – If by contradiction E is not simple, then there exists $\phi : E \to E$ non-trivial. Applying the functor $\mathrm{Hom}\,(-, E)$ to the sequence (7) we get that ϕ induces $\widetilde{\phi}$ non-trivial in $\mathrm{Hom}\,(W \otimes \mathcal{O}, E)$. Now applying the functor $\mathrm{Hom}\,(W \otimes \mathcal{O}, -)$ again to the same sequence we get $\mathrm{Hom}\,(W \otimes \mathcal{O}, W \otimes \mathcal{O}) \cong \mathrm{Hom}\,(W \otimes \mathcal{O}, E)$ because

$$\mathrm{Hom}\,(W \otimes \mathcal{O}, I \otimes \mathcal{O}(-1)) \cong W \otimes I \otimes \mathrm{H}^0(\mathcal{O}(-1)) = 0$$

and

$$\mathrm{Ext}^1(W \otimes \mathcal{O}, I \otimes \mathcal{O}(-1)) \cong W \otimes I \otimes \mathrm{H}^1(\mathcal{O}(-1)) = 0.$$

It follows that there exists $\widetilde{\phi}$ non-trivial in $\mathrm{End}\,(W \otimes \mathcal{O})$, i.e. a matrix $B \neq \mathrm{Id}$ in $\mathrm{GL}(W)$. Restricting $\widetilde{\phi}$ to $I \otimes \mathcal{O}(-1)$ and calling A the corresponding matrix in $\mathrm{GL}(I)$, we get the commutative diagram (4). Therefore $(A, B) \neq (\lambda\mathrm{Id}, \lambda\mathrm{Id})$ belongs to $\mathrm{Stab}\,(M)$ and consequently $\dim \mathrm{Stab}\,(M) > 1$. ∎

Finally it suffices to prove that for all generic $M \in H$, the dimension of the stabilizer is exactly 1. In other words we have to prove the following

PROPOSITION 3.6. – *Let $H = V \otimes I^\vee \otimes W$ as above and suppose $s^2 - Nst + t^2 \leq 0$. Then the generic orbit in H with respect to the natural action of* $\mathrm{GL}(I) \times \mathrm{GL}(W)$ *has dimension exactly* $(s^2 + t^2 - 1)$.

Recall that we have defined the following diagram

$$\begin{array}{ccc} & S = \{A, B, M : A^{-1}MB = M\} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathrm{GL}(I) \times \mathrm{GL}(W) & & H \end{array}$$

Let (A, B) be two fixed Jordan canonical forms in $\mathrm{GL}(I) \times \mathrm{GL}(W)$. We define $G_{AB} \subset \mathrm{GL}(I) \times \mathrm{GL}(W)$ as the set of couples of matrices similar respectively to A

and B. Note that $\pi_2\pi_1^{-1}(G_{AB}) = \{C^{-1}MD : A^{-1}MB = M, C \in \mathrm{GL}(I), D \in \mathrm{GL}(W)\}$. Moreover $G_{\mathrm{IdId}} = \{(\lambda\mathrm{Id}, \lambda\mathrm{Id}), \lambda \in \mathbb{C}\}$ and $\pi_2\pi_1^{-1}(G_{\mathrm{IdId}}) = H$.

LEMMA 3.7. – *If $s^2 - Nst + t^2 \leq 0$ and (A, B) are Jordan canonical forms different from $(\lambda\mathrm{Id}, \lambda\mathrm{Id})$ for any λ, then $\pi_2\pi_1^{-1}(G_{AB})$ is contained in a Zariski closed subset strictly contained in H.*

PROOF. – Suppose that the assertion is false. Then there exist two Jordan canonical forms A and B, different from $(\lambda\mathrm{Id}, \lambda\mathrm{Id})$, such that $\pi_2\pi_1^{-1}(G_{AB})$ is not contained in any closed subset. This implies that we can take a general $M \in H$ such that $AM = MB$ and in particular we can suppose the rank of M maximum.

Now we prove that A and B have the same minimal polynomial. First, if p_B is the minimal polynomial of B, i.e. $p_B(B) = 0$, then it follows that $p_B(A)M = Mp_B(B) = 0$ and since M is injective we get $p_B(A) = 0$, hence the minimal polynomial of B divides that of A. Now if we denote by λ_i $(1 \leq i \leq q)$ the eigenvalues of A and by μ_j $(1 \leq j \leq q')$ those of B, we obtain that $\mu_j \in \{\lambda_1, \ldots, \lambda_q\}$ for all $1 \leq j \leq q'$. Let us define $A' = (A - x\mathrm{Id}_s)$ and $B' = (B - x\mathrm{Id}_t)$: obviously we obtain $A'M = MB'$. We denote by $\overline{B'}$ the matrix of cofactors of B' and we know that $B'\overline{B'} = \det(B')\mathrm{Id}_t = P_B(x)\mathrm{Id}_t$, where P_B is the characteristic polynomial of B. Therefore

$$A'M\overline{B'} = P_B(x)M$$

and developing this expression we see that $q' = q$. In fact if there exists a $\lambda_i \neq \mu_j$ for all $j = 1, \ldots, q'$, then there is a row of zeroes in M and consequently M is not generic. Then we get A and B with the same eigenvalues λ_i $(1 \leq i \leq q)$ with multiplicity respectively $a_i \geq 1$ and $b_i \geq 1$. The hypothesis that $(A, B) \neq \neq (\lambda\mathrm{Id}, \lambda\mathrm{Id})$ means that either A and B have more than one eigenvalue or at least one of them is non-diagonal.

Now consider the first case, i.e. $q \geq 2$. Since $\dim I = s$ and $\dim W = t$, obviously $\sum_{i=1}^{q} a_i = s$ e $\sum_{i=1}^{q} b_i = t$. Now we denote $M = (M_{ij})$, where M_{ij} has dimension $a_i \times b_j$. Since $AM = MB$, every block M_{ij} is zero for all $i \neq j$, i.e. it is possible to write M with the form

$$M = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

In particular we can define $n_1 = a_1, n_2 = \sum_{i=2}^{p} a_i, m_1 = b_1, m_2 = \sum_{i=2}^{p} b_i$ and thus the matrix M becomes

$$M = \begin{pmatrix} (*)_{n_1 \times m_1} & (0)_{n_1 \times m_2} \\ (0)_{n_2 \times m_1} & (*)_{n_2 \times m_2} \end{pmatrix} \tag{8}$$

where $n_1 + n_2 = s$ and $m_1 + m_2 = t$ and $n_i, m_i \geq 1$ for $i = 1, 2$. Thus it only suffices to show that a matrix in the orbit

$$O_M = \{C^{-1}MD : C \in \mathrm{GL}(s), D \in \mathrm{GL}(t), M \text{ with the form } (8)\},$$

is not generic in H if $s^2 - Nst + t^2 \leq 0$. This fact contradicts our assumption and completes the proof.

In order to show this, we introduce the following diagrams

$$\begin{array}{ccc} & \{\phi, I_1, W_1 : \phi(I_1 \otimes V^\vee) \subseteq W_1\} & \\ {\scriptstyle \alpha_1}\swarrow & & \searrow{\scriptstyle \beta_1} \\ H = \mathrm{Hom}(I \otimes V^\vee, W) & & \mathcal{G}_1 = \mathcal{G}(\mathbb{C}^{n_1}, \mathbb{C}^s) \times \mathcal{G}(\mathbb{C}^{m_1}, \mathbb{C}^t) \end{array}$$

where $\mathcal{G}(\mathbb{C}^k, \mathbb{C}^h)$ denotes the Grassmannian of $\mathbb{C}^k \subset \mathbb{C}^h$ and

$$\begin{array}{ccc} & \{\phi, I_2, W_2 : \phi(I_2 \otimes V^\vee) \subseteq W_2\} & \\ {\scriptstyle \alpha_2}\swarrow & & \searrow{\scriptstyle \beta_2} \\ H = \mathrm{Hom}(I \otimes V^\vee, W) & & \mathcal{G}_2 = \mathcal{G}(\mathbb{C}^{n_2}, \mathbb{C}^s) \times \mathcal{G}(\mathbb{C}^{m_2}, \mathbb{C}^t) \end{array}$$

It is easy to check that the matrices of the set O_M live in the subvariety

$$\widetilde{H} = a_1(\beta_1^{-1}(\mathcal{G}_1)) \cap a_2(\beta_2^{-1}(\mathcal{G}_2)) \subseteq H,$$

then, in order to prove that these matrices are not generic, it suffices to show that $\dim \widetilde{H} < \dim H$. Since $\dim(\mathcal{G}_i) = (n_1 n_2 + m_1 m_2)$ for $i = 1, 2$, we obtain

$$\dim(a_1(\beta_1^{-1}(\mathcal{G}_1))) \leq \dim(\beta_1^{-1}(\mathcal{G}_1)) = n_1 n_2 + m_1 m_2 + N(n_1(m_1 + m_2) + n_2 m_2)$$

and

$$\dim(a_2(\beta_2^{-1}(\mathcal{G}_2))) \leq \dim(\beta_2^{-1}(\mathcal{G}_2)) = n_1 n_2 + m_1 m_2 + N(n_1 m_1 + n_2(m_1 + m_2)).$$

Therefore, since $\dim H = Nst = N(n_1 + n_2)(m_1 + m_2)$ we only need to show that either $(n_1 n_2 + m_1 m_2 - N n_2 m_1) < 0$ or $(n_1 n_2 + m_1 m_2 - N n_1 m_2) < 0$. In other words we have to prove that the system

$$\begin{cases} n_1 n_2 + m_1 m_2 - N n_1 m_2 \geq 0 \\ n_1 n_2 + m_1 m_2 - N n_2 m_1 \geq 0 \end{cases}$$

has no solutions in our hypotesis $s^2 - Nst + t^2 \leq 0$, i.e. if

$$\frac{N - \sqrt{N^2 - 4}}{2} t \leq s \leq \frac{N + \sqrt{N^2 - 4}}{2} t.$$

This is equivalent to prove that the system

$$\begin{cases} n_1n_2 + m_1m_2 - Nn_1m_2 \geq 0 \\ n_1n_2 + m_1m_2 - Nn_2m_1 \geq 0 \\ n_1 + n_2 \geq \frac{N-\sqrt{N^2-4}}{2}(m_1 + m_2) \\ n_1 + n_2 \leq \frac{N+\sqrt{N^2-4}}{2}(m_1 + m_2) \end{cases}$$

has no solutions. In order to do it, consider n_1 and m_1 as parameters and write the previous system as a system of linear inequalities in two unknowns n_2 and m_2:

$$\begin{cases} n_1n_2 \geq (Nn_1 - m_1)m_2 \\ (n_1 - Nm_1)n_2 \geq -m_1m_2 \\ n_2 \geq a_-m_2 + (a_-m_1 - n_1) \\ n_2 \leq a_+m_2 + (a_+m_1 - n_1) \end{cases}$$

where we denote $a_- = \frac{N-\sqrt{N^2-4}}{2}$ and $a_+ = \frac{N+\sqrt{N^2-4}}{2}$. Notice that $(a_- + a_+) = N$ and $a_-a_+ = 1$, because they are solutions of the equation $s^2 - Nst + t^2 = 0$. Now let us consider three cases:

- if $0 < n_1 - a_+m_1$ the system

$$\begin{cases} n_2 \geq \frac{(Nn_1-m_1)}{n_1}m_2 \\ n_2 \leq a_+m_2 + (a_+m_1 - n_1) \end{cases}$$

has no solutions because $(a_+m_1 - n_1) < 0$ and $a_+ < \frac{(Nn_1-m_1)}{n_1}$, since $(N - a_+)n_1 - m_1 = a_-n_1 - m_1 = (a_+)^{-1}(n_1 - a_+m_1) > 0$;

- if $n_1 - a_+m_1 < 0 < n_1 - a_-m_1$ the system is

$$\begin{cases} n_2 \geq \frac{(Nn_1-m_1)}{n_1}m_2 \\ n_2 \leq \frac{m_1}{(Nm_1-n_1)}m_2 \end{cases}$$

because $Nm_1 - n_1 > a_+m_1 - n_1 > 0$ and there is no solution because $\frac{m_1}{(Nm_1-n_1)} < \frac{(Nn_1-m_1)}{n_1}$, since $N(Nn_1m_1 - m_1^2 - n_1^2) > 0$;

- if $n_1 - a_-m_1 < 0$ then the system

$$\begin{cases} n_2 \leq \frac{m_1}{(Nm_1-n_1)}m_2 \\ n_2 \geq a_-m_2 + (a_-m_1 - n_1) \end{cases}$$

has no solutions because $(a_-m_1 - n_1) > 0$ and $a_- > \frac{m_1}{(Nm_1-n_1)}$ i.e. $a_+ < \frac{(Nm_1-n_1)}{m_1}$, since $(N - a_+)m_1 - n_1 = a_-m_1 - n_1 > 0$.

Thus the proof in the case $q \geq 2$ is complete.

In the second case we consider $q = 1$ and the two matrices are

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_h \end{pmatrix}, \quad \text{where} \quad J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

and c_i denotes the order of J_i and

$$B = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_k \end{pmatrix}, \quad \text{where} \quad L_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

and d_i is the order of L_i. We suppose that $c_1 \geq 2$ or $d_1 \geq 2$ i.e. $h < s$ or $k < t$. Then a matrix M such that $AM = MB$ has the form $M = (M_{ij})$ and M_{ij} is a $c_i \times d_j$ matrix such that

$$M_{ij} = \begin{cases} T_c & \text{if} \quad c_i = d_j = c \\ (0|T_c) & \text{if} \quad c = c_i < d_j \\ \left(\frac{T_d}{0}\right) & \text{if} \quad c_i > d_j = d \end{cases}$$

and T_c is a $c \times c$ upper-triangular Toeplitz matrix. It is easy to see that M has at least k columns in which there are at least $(c_1 - 1) + (c_2 - 1) + \ldots + (c_h - 1) = (s - h)$ zeroes in such a way that we can order the basis so as to write M in the following form

$$\begin{pmatrix} (*)_{h\times k} & (*)_{h\times(t-k)} \\ (0)_{(s-h)\times k} & (*)_{(s-h)\times(t-k)} \end{pmatrix}.$$

Analogously M has at least h rows with at least $(t - k)$ zeroes such that it is possible to write the matrix in the form

$$\begin{pmatrix} (*)_{h\times k} & (0)_{h\times(t-k)} \\ (*)_{(s-h)\times k} & (*)_{(s-h)\times(t-k)} \end{pmatrix}.$$

Hence there exist non-trivial subspaces I_1, I_2, W_1, W_2 such that $M(I_i \otimes V^\vee) \subseteq W_i$, for $i = 1, 2$, and $\dim I_1 = s - h$, $\dim W_1 = k$, $\dim I_2 = h$, $\dim W_2 = t - k$. Therefore exactly the same argument used in the first case gives that M is not generic and completes the proof. ∎

The previous lemma proves Proposition 3.6 and the main theorem follows. This theorem can also be reformulated as follows:

THEOREM 3.8. – *Let M a $(s \times t)$ matrix whose entries are linear forms in N variables and consider the system*

$$XM = MY, \tag{9}$$

where $X \in \mathrm{GL}(s)$ *and* $Y \in \mathrm{GL}(t)$ *are the unknowns. Then if* $s^2 + t^2 - Nst \leq 1$, *there is a dense subset of the vector space* $\mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^N$, *where* M *lives, such that the only solutions of* (9) *are trivial, i.e.* $(X, Y) = (\lambda \mathrm{Id}, \lambda \mathrm{Id}) \in \mathrm{GL}(s) \times \mathrm{GL}(t)$ *for* $\lambda \in \mathbb{C}$. *Conversely if* $s^2 + t^2 - Nst \geq 2$, *then for all* M *there are non-trivial solutions.*

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