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Curves of Genus Seven That Do Not Satisfy the Gieseker-Petri Theorem.

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Sunto. – Nello spazio dei moduli delle curve di genere $g$, $M_g$, indichiamo con $\mathcal{GP}_g$ il luogo delle curve che non soddisfano il teorema di Gieseker-Petri. In questo lavoro noi proviamo che nel caso di genere sette, $\mathcal{GP}_7$ è un divisore di $M_7$.

Summary. – In the moduli space of curves of genus $g$, $M_g$, let $\mathcal{GP}_g$ be the locus of curves that do not satisfy the Gieseker-Petri theorem. In the genus seven case we show that $\mathcal{GP}_7$ is a divisor in $M_7$.

0. – Introduction

Let $M_g$ be the moduli space of smooth and irreducible projective curves of genus $g$. Let $C \in M_g$ and let $K_C$ be the canonical bundle of $C$. Let $L$ be a line bundle on $C$ and consider the Petri map $\mu_L : H^0(C, L) \otimes H^0(K_C \otimes L^{-1}) \rightarrow H^0(C, K_C)$.

The Gieseker-Petri theorem (see [5], p.285) says that for every line bundle $L$ on a general curve $C \in M_g$, $\mu_L$ is injective. Consider the locus

$$\mathcal{GP}_g := \{ C \in M_g | C \text{ does not satisfy the Gieseker-Petri theorem} \}.$$

By the Gieseker-Petri Theorem, $\mathcal{GP}_g$ is a closed Zariski subset in $M_g$. Let $C$ be a smooth irreducible projective curve of genus $g$ and $L \rightarrow C$ a line bundle of degree $d$ with $r + 1 = h^0(C, L)$. The Brill-Noether number is defined by $\rho(g, r, d) : = h^0(C, K_C) - h^0(C, L)h^0(C, K_C \otimes L^{-1}) = g - (r + 1)(g - d - r)$. So if $\rho(g, r, d) < 0$, the Petri map $\mu_L$ is not injective. Let $M_{g,d}^r := \{ C \in M_g | G^r_d(C) \neq 0 \}$. In [7] Steffen showed that if $\rho(g, r, d) < 0$, each component of $M_{g,d}^r$ has codimension at most $-\rho(g, r, d)$ in $M_g$. When $\rho = -1$, in [3] Eisenbud and Harris showed that $M_{g,d}^r$ has a unique irreducible component of codimension one in $M_g$. M. Teixidor showed (see [8], [9]) that the locus $M_g^1 := \{ C \in M_g | C \text{ has a quasiresidual } g^1_{g-1} \}$ is an irreducible divisor in $M_g$. The above results give us some divisorial components of $\mathcal{GP}_g$.

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We refer the above components as Eisenbud-Harris and Teixidor components respectively. To give all components of $G\mathcal{P}_g$ is a difficult problem, however for specific low genus we can study $G\mathcal{P}_g$ using the projective geometry of curves. For example in [2], the varieties $W_d^r$ for general curves for low genus are described. Using this analysis one can describe all components of $G\mathcal{P}_g$ for $3 \leq g \leq 6$. The genus seven case is a non trivial case for studying $G\mathcal{P}_7$. In this work the main theorem is

**Theorem.** $G\mathcal{P}_7$ is a divisor in $\mathcal{M}_7$.

We show the theorem with a degeneration argument. First we study curves of genus seven with a primitive $g^r_d$, $r = 1, 2$, $d = 1, ..., 6$, for which the Petri map is not injective. In sections 2.1-2.7 we describe two codimension one components of $G\mathcal{P}_7$. These are the Eisenbud-Harris and the Teixidor components. The third codimension one component of $G\mathcal{P}_7$ that we denote by $\mathcal{D}$, is formed by curves of genus seven with a $g^1_5$ for which the Petri map is not injective. To show that $\mathcal{D} \subset \mathcal{M}_7$ has codimension one we proceed as follow: In proposition 2.8 we show that a pentagonal curve $\mathcal{C}$ of genus seven does not satisfy the Gieseker-Petri theorem if and only if it has a $g^1_5$ such that the residual $\tilde{g}^2_6 = |K_C - g^1_5|$ induces a birational morphism on a septic $\Gamma$ in $\mathbb{P}^2$ with eight double points, seven of them lying on a conic. Now consider $\mathcal{V}^{7,7}$ the Severi variety of reduced and irreducible plane curves of degree seven and geometric genus seven. Consider $\mathcal{V}^{7,7}_{8} \subset \mathcal{V}^{7,7}$ the locus consisting of plane curves having eight double points as singularities. In this case, the dimension of $\mathcal{V}^{7,7}_{8}$ is equal to $27$ (see [5], p. 30). The quotient $\mathcal{V} := \mathcal{V}^{7,7}_{8} / \text{PGL}(3, \mathbb{C})$ of $\mathcal{V}^{7,7}_{8}$ with the automorphisms of $\mathbb{P}^2$ is of dimension 19. Consider the subvariety $\mathcal{D}_0$ of $\mathcal{V}$ defined by $\mathcal{D}_0 := \{ \Gamma \in \mathcal{V} \mid \text{seven double points of } \Gamma \text{ lying on a conic} \}$. A consequence of the corollary 2.10 shows that $\mathcal{D}_0$ is irreducible and of dimension 17 in $\mathcal{V}$. In section 3 we consider the natural morphism $\tilde{\phi} : \mathcal{V} \to \mathcal{M}_7$, $\Gamma \to \tilde{\phi}(\Gamma) = \text{normalization of } \Gamma$. Since by excess linear series (see [2], p. 329) a pentagonal curve $\mathcal{C}$ of genus seven has $\dim \mathcal{W}^1_3(C) = 1$, we have that the image $\mathcal{D} := \tilde{\phi}(\mathcal{D}_0)$ has codimension one in $\mathcal{M}_7$ if for each $C \in \mathcal{D}$, the fiber $\tilde{\phi}^{-1}(C) \simeq \mathcal{W}^1_3(C)$ intersects only a finite number of elements of $\mathcal{D}_0$. This means that $\mathcal{D}$ has codimension one in $\mathcal{M}_7$ if for $C \in \mathcal{D}$ we have that for the general element $L \in \mathcal{W}^1_3(C)$, the Petri map $\mu_L$ is injective. To show this, in 3.2 we degenerate a curve $\Gamma \in \mathcal{D}_0$ to a compact type curve $X_0 = \mathbb{P}^1 \cup Z$, $\{p\} := \mathbb{P}^1 \cap Z$, where $Z$ is a sextic with three not collinear double points. By stable reduction ([5], p. 118) the normalization $C_0$ of $Z$ is the stable limit of $X_0$. In Proposition 3.3 we show that for the general linear series $|D| = g^1_6$ on $C_0$ the Petri map $\mu_L$ is injective. This implies that for each $C \in \mathcal{D}$ and for a general element $L$ of $\mathcal{W}^1_3(C)$, the Petri map $\mu_L$ is injective. So we have that $\mathcal{D}$ is an irreducible component of $G\mathcal{P}_7$ of codimension one in $\mathcal{M}_7$. Thus the components of $G\mathcal{P}_7$ are $\mathcal{M}^1_7$, $\mathcal{M}^4_{7,4}$, $\mathcal{D}$. Thus of this way we will prove the theorem.

It is an interesting open problem to prove that every irreducible component of $G\mathcal{P}_g$ is divisorial in $\mathcal{M}_g$. 

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1. – Preliminaries.

1.1. – Let $C$ be a smooth projective irreducible curve of genus $g$, $D$ a divisor on $C$ and $K_C$ the canonical bundle on $C$. We say that the linear series $|D|$ is primitive if $|D|$ and $|K_C - D|$ are free of base points.

Let $D$ be a divisor on $C$. We write $r = r(D) := h^0(C, D) - 1$. Suppose that $|D|$ is not primitive and let $p \in C$ be a base point of $|K_C - D|$, by Riemann-Roch we have that $R(D + p) = \dim |D + p| = \deg D + 1 - g + h^0(K - (D + p))$, now since $h^0(K_C - D - p) = h^0(K_C - D) = r(D) + g - \deg D$, then $R(D + p) = R(D) + 1$, that is, $p$ is not base point of $|D + p|$. In this way we transferred a base point of $|K_C - D|$ to the series $|D|$ obtaining two series $|K_C - (D + p)|$ and $|D + p|$, residual one to the other with respect to the canonical series and of dimension $d - g$ and $r + 1$ respectively. Iterating this process we can obtain from a pair of non primitive series $(|D|, |K_C - D|)$ a pair of primitive series $(|D'|, |K_C - D'|)$.

**Lemma 1.2.** – If there exists $|D|$ such that $\mu_D : H^0(C, D) \otimes H^0(C, K_C - D) \to H^0(C, K_C)$ is not injective then there exists a primitive series $|D'|$ such that $\mu_{D'}$ is not injective.

**Proof.** – Let $|D|$ be a non primitive series such that $\mu_{0,D}$ is not injective. We can write $|D| = p_1 + \cdots + p_n + |D_1|$ and $|K_C - D| = q_1 + \cdots + q_m + |D_2|$, where $|D_1|$ and $|D_2|$ are free of base points. Let $L_i, i = 1, 2$ be the line bundles defined by $D_i$.

Consider the map $\mu_1 : H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2)$. There exists an isomorphism $\psi : \text{Kernel } \mu_D \to \text{Kernel } \mu_1$ given by $\psi(\sum s_i \otimes t_i) = \sum \frac{s_i}{f_1} \otimes \frac{t_i}{f_2}$, where $s_i \in H^0(C, D)$, $t_i \in H^0(C, K_C - D)$, $(f_1) = p_1 + \cdots + p_n$, $(f_2) = q_1 + \cdots + q_m$. Applying the above process of transferring base points we obtain $D'_1 = D_1 + q_1 + \cdots + q_m$ and $D'_2 = D_2 + p_1 + \cdots + p_n$ such that $D'_2 = K - D'_1$ and $|D'_1|$, $|D'_2|$ are primitive. Since $D_1 \subset D'_1$, $D_2 \subset D'_2$ then $H^0(C, L_i) \subset H^0(C, L'_i)$, $i = 1, 2$, where $L'_i$ is the line bundle defined by $D'_i$. We have the following commutative diagram

\[
\begin{array}{ccc}
H^0(C, L_1) \otimes H^0(C, L_2) & \xrightarrow{\mu_1} & H^0(C, L_1 \otimes L_2) \\
\downarrow \pi & & \downarrow \pi \\
H^0(C, L'_1) \otimes H^0(C, K_C \otimes (L'_1)^{-1}) & \xrightarrow{\mu_{D'_1}} & H^0(C, K_C)
\end{array}
\]
Since Kernel $\mu_1 \neq 0$, then Kernel $\mu^{L_1} \neq 0$, that is, Kernel $\mu^{D_1} \neq 0$. □

Thus we consider primitive linear series for which the multiplication map is not injective. Since $W^{g_{r-d}}_n(C) \cong W^{g_{r-d-1}}_{2g_{r-d}}(C)$, $|D| \rightarrow |K_C - D|$, we only consider special primitive linear series $|D|$ such that $r = h^0(C, D) - 1 > 0$, and $g - d + r = h^0(C, K_C - D) > 0$, with $d = \mathrm{deg} \ D < g$. By Clifford theorem ([2]), a linear series $|D|$ of degree $d \leq 2g - 1$ satisfies $2r \leq d$ with equality if and only if $D = 0$, $D$ is the canonical divisor or $C$ is hyperelliptic and $D$ is a multiple of the hyperelliptic involution. Then for our analysis we only need to consider special linear series $|D|$ such that $r > 0$ and $2r \leq d < g$.

Also we use The Base point free pencil trick ([2], p.126): if $|D|$ is a pencil free of base points, we have that Ker $\mu_D \cong H^0(C, K_C( -2D))$.

2. – The locus $\mathcal{G}_7$.

Let $C \in \mathcal{M}_7$. We study primitive linear series on $C$ of dimension $r$ and degree $d$ such that $2r \leq d \leq g - 1 = 6$, for which the Petri map is not injective.

2.1. $r = 1$, $d = 2, 3, 4$. In this case we have $\rho(7, 1, d) < 0$, and by gonality, $\mathcal{M}_{1,2} \subseteq \mathcal{M}_{1,3} \subseteq \mathcal{M}_{1,4}$. For $d = 4$, $\rho(7, 1, 4) = -1$, so by [3], $\mathcal{M}_{1,4}$ is an irreducible divisor in $\mathcal{M}_7$.

2.2. $r = 1$, $d = 5$. We postpone this case.

2.3. $r = 1$, $d = 6$. By the base point free pencil trick, the multiplication map for a $g^0_6$ on $C$ is not injective if and only if $g^0_6$ is autoresidual. By [8, 9], the locus $\mathcal{M}_7^1$ is an irreducible divisor in $\mathcal{M}_7$.

2.4. $r = 2$, $d = 4$. By the genus formula, a genus seven curve $C$ with a $g^2_4$ is hyperelliptic, then $C \in \mathcal{M}_{1,2}^1$.

2.5. $r = 2$, $d = 5$. By the genus formula a curve of genus seven has no primitive $g^2_5$.

2.6. $r = 2$, $d = 6$. Let $C \in \mathcal{M}_7$ be a non hyperelliptic curve with a $g^2_6$ that induces a map $\psi : C \rightarrow \mathbb{P}^2$, $X := \psi(C)$. If the degree of $X$ is two or three, either $X$ is trigonal or bielliptic, so $C \in \mathcal{M}_{1,4}^1$. If the degree of $X$ is six, either $X$ has a triple point or it has three double points, in any case $X$ is either trigonal or tetragonal. So a curve $C$ with a $g^2_6$ belongs to $\mathcal{M}_{1,4}^1$.

2.7. $r = 3$, $d = 6$. By Castelnuovo’s bound ([2], p. 116) a curve of genus seven with a $g^3_6$ is hyperelliptic.

Now we study the case 2.2. Let $C$ be a pentagonal curve. The residual of a primitive $g^3_3$ on $C$ is a base point free $g^2_7$. This $g^2_7$ defines a birational map of $C$ onto
a plane curve in \( \mathbb{P}^2 \). Since a septic curve of genus seven with a triple point has a \( g_3^1 \) cut out by lines through the triple point, we only consider septic curves with double points. Let \( \Gamma \) be such a curve and \( f : C \to \Gamma \) the normalization of \( \Gamma \). We denote by \( \Delta_\Gamma \) the scheme of singular points of \( \Gamma \) and \( \Delta := f^*(\Delta_\Gamma) \), note that \( \Delta \) is a divisor of degree sixteen. By the genus formula the length of \( (\Delta_\Gamma) = 8 \), i.e. \( \Delta_\Gamma \) consists of eight points which can be infinitely near. However by our assumption that \( \Gamma \) has only double points the scheme \( \Delta_\Gamma \) is in any case curvilinear.

**Proposition 2.8.**

\( a) \) Let \( \Gamma \) be a plane curve of degree seven and genus seven with only double points and let \( f : C \to \Gamma \) be its normalization. Suppose that there is a conic \( Q \) such that the scheme theoretic intersection of \( Q \) with \( \Delta_\Gamma \) has length equal to seven, i.e. \( f^*(Q) \) contains a divisor of degree fourteen contained in \( \Delta \), then \( C \) does not satisfy the Gieseker-Petri theorem.

\( b) \) Conversely if \( C \) is a pentagonal curve of genus seven such that there is a \( |D| = g_5^1 \) on \( C \) for which \( \mu_4 \) is not injective, then there is in \( \mathbb{P}^2 \) a birational model \( \Gamma \) of \( C \) of degree seven with only double points such that the \( g_5^1 \) is cut out by lines passing through a double point \( p \) and there is a conic \( Q \) such that \( Q \) contains \( \Delta_\Gamma - \{ p \} \).

**Proof.** – First I will prove the part \( a) \). Also I will only consider the most complicated case in which the support of \( \Delta_\Gamma = \{ x \} \). The other cases are easier and can be left to the reader.

If the support of \( \Delta_\Gamma = \{ x \} \), then \( \Gamma \) has eight infinitely near double points. Let \( \eta := f^*(x) \), so that \( \eta \) is a divisor of degree two and \( \Delta = 8\eta \). Our hypothesis means that the pullback \( f^*Q \) on \( C \) contains \( 7\eta \). Consider the \( |D| = g_5^1 \) cut out on \( C \) by the lines through \( x \). Let \( \ell_1, \ell_2 \) be general such lines, cutting out on \( C \) two effective divisors \( D_1, D_2 \in |D| \). The pullback of \( Q + \ell_1 + \ell_2 \) contains \( 9\eta + D_1 + D_2 \sim 9\eta + 2D \). By adjunction formula ([2], p. 53), one has \( K_C \sim O_C(4)(-\Delta) \), and therefore \( K_C - 2D \) is effective. Since \( \ker \mu_4 \simeq H^0(C,K_C - 2D) \), we have the assertion.

The proof of part \( b) \) is as follow: Let \( |D| = g_5^1 \) be a primitive linear series on \( C \) for which \( \mu_4 \) is not injective. Consider \( g_7^2 = K_C - D \). This \( g_7^2 \) determines a birational morphism \( C \to \Gamma \subset \mathbb{P}^2 \) and \( \Gamma \) has only double points. Since \( C \) fails the Gieseker-Petri theorem for the \( g_5^1 \), we have that \( \ker \mu_4 \simeq H^0(C,K_C - 2D) \), but \( K_C - 2D \sim g_7^2 - g_5^1 \) is effective, so necessarily the \( g_5^1 \) is cut out by a pencil of lines through a singular point \( p \) of \( \Gamma \). The existence of the conic \( Q \) is now clear.

**Remark 1.** – Suppose we have a curve \( \Gamma \) like in proposition 2.8. Then we claim that the conic \( Q \) cannot be singular at any point of \( \Delta_\Gamma \). Suppose in fact that \( p \in \Delta_\Gamma \) is singular for \( Q \). Suppose that \( Q = L_1 + L_2 \), where \( L_1, L_2 \) are lines through \( p \). Suppose that \( L_1 \) contains \( i \) double points infinitely near to \( p \).
Therefore $Q$ has to contain $7 - i$ more double points of $I$ which can be distinct or infinitely near. If $j$ of such points are on $L_1$ one has $2i + 2j \leq 7$. If $L_1 = L_2$ we must have $i + j = 7$ which gives a contradiction. Suppose then that $L_1 \neq L_2$ and suppose that $k$ double points lie on $L_2$ off $p$. Then $2 + 2k \leq 7$ and moreover $i + j + k = 7$ which again gives a contradiction. Let $V_8^{7,7}$ be the family of reduced, irreducible plane curves of degree seven with geometric genus seven. This is an irreducible variety (see [5] p.30). Let $V_8^{7,7'}$ be the Zariski open subset of $V_8^{7,7}$ defined by all irreducible curves in $V_8^{7,7}$ with only eight double points. The dimension of $V_8^{7,7}$ is 27 (see [5] p.30). We will denote by $\mathcal{Z}$ the Zariski open subset of the Hilbert Scheme of locally complete intersection zero-dimensional subschemes in $\mathbb{P}^2$ of length 8, formed by curvilinear subschemes. Now consider the subvariety $\mathcal{I} \subseteq \mathcal{Y}_8^{7,7} \times \mathcal{U} \times \mathcal{X}$, where $\mathcal{U} \subset |O_{\mathbb{P}^2}(2)|$ is the open set of smooth conics in $\mathbb{P}^2$. The variety $\mathcal{I}$ consists of all triples $(\Gamma, Q, \triangle)$ such that $\triangle = \triangle_\Gamma$ and $Q \cap \triangle_\Gamma$ contains seven points. Note that by proposition 2.8, the image of the projection map $\text{Pr}_1 : \mathcal{I} \rightarrow \mathcal{Y}_8^{7,7}$ is the subvariety $\Sigma : = \text{Pr}_1(\mathcal{I})$ of $\mathcal{Y}_8^{7,7}$ consisting of curves for which the Gieseker-Petri fails for a $g^1_5$.

**Proposition 2.9.** $\mathcal{I}$ is irreducible of dimension 25.

**Proof.** Consider the projection $\pi_3 : \mathcal{I} \rightarrow \mathcal{X}$. Let $S \in \pi_3(\mathcal{I})$ be any point. Namely $S$ is a curvilinear scheme of length 8, seven points of which lie on a irreducible conic. Therefore, if $(\Gamma, Q, S) \in \pi_3^{-1}(S)$, then $Q$ is uniquely determined by $S$. Moreover $\Gamma$ belongs to the linear system of plane curves of degree seven which are singular at $S$. Let $\mathcal{Z}_S$ be this linear system.

**First Claim:** The dimension of $\mathcal{Z}_S = 11$ and the general element of $\mathcal{Z}_S$ is irreducible of geometric genus seven.

**Proof of the First Claim:** Let $\mathcal{Z}$ be the proper transform of $\mathcal{Z}_S$ on the surface $X$ which is $\mathbb{P}^2$ blown up at $S$. Let $\bar{Q}$ the proper transform of $Q$ on $X$. Consider the exact sequence:

$$0 \rightarrow \mathcal{Z}(-\bar{Q}) \rightarrow \mathcal{Z} \rightarrow \mathcal{Z}|_{\bar{Q}} \rightarrow 0$$

we remark that $\mathcal{Z}|_{\bar{Q}} = O_Q$. Moreover $\mathcal{Z}(-\bar{Q})$ is the proper transform on $X$ of the linear system of quintics of $\mathbb{P}^2$ with a double point off $Q$ and seven points on $Q$. It is easily seen that $h^1(\mathcal{Z}(-\bar{Q})) = 0$ and $h^0(\mathcal{Z}(-\bar{Q})) = 11$. Hence $h^0(\mathcal{Z}) = 12$. The proof of the irreducibility of the general element of $\mathcal{Z}_S$ is easily obtained by Bertini theorem. We omit the details. In order to finish the proof it is sufficient to show that:

**Second Claim:** $\pi_3(\mathcal{I})$ is irreducible of dimension 14.

**Proof of the Second Claim.** Let $\pi_{23} : \mathcal{I} \rightarrow \mathcal{U} \times \mathcal{X}$ be the projection on the
second and third factor and let $\mathcal{Y} := \pi_2(\mathcal{F})$ be. The above discussion implies
that $\mathcal{Y}$ is the variety formed by pairs $(Q, \triangle)$ such that $Q \cap \triangle$ consists of seven
points. Let $\pi_1 : \mathcal{Y} \to \mathcal{H}$ be the projection to the first factor which is dominant.
The fiber is of course irreducible of dimension 9. This shows that $\dim \mathcal{Y} = 14$. On
the other hand $\pi_3 : \mathcal{Y} \to \mathcal{X}$ is finite, so the assertion follows. □

**Corollary 2.10.** – The subvariety $\Sigma := \text{Pr}_1(\mathcal{F}) \subset \mathcal{V}^{7,7}_8$ is irreducible of
dimension 25.

**Proof.** – The map $\text{Pr}_1 : \mathcal{F} \to \mathcal{V}^{7,7}_8$ is generically finite. By proposition 2.9, we
have that $\Sigma$ is irreducible of dimension 25.

3. – Proof of the theorem.

In this section we will prove that $GP_7$ is a divisor in $\mathcal{M}_7$.

Consider the natural morphism $\phi : \Sigma \to \mathcal{M}_7$ where the general fiber of this
map has dimension at least 8, because $PGL(3, \mathbb{C})$ acts on $\Sigma$ and any orbit lies in a
fiber of $\phi$. Let $V := \mathcal{V}^{7,7}_8/PGL(3, \mathbb{C})$. Note that $\mathcal{D}_0 := \Sigma/PGL(3, \mathbb{C}) \subset V$ is of
dimension 17. Now we will prove that the general fiber of $\phi : \mathcal{D}_0 \to \mathcal{M}_7$ is zero-
dimensional, that is, $\mathcal{D} := \phi(\mathcal{D}_0)$ has codimension one in $\mathcal{M}_7$. This will prove that
$\mathcal{D} := \phi(\mathcal{D}_0)$ is an irreducible component of $GP_7$ of codimension one in $\mathcal{M}_7$. We will
prove the theorem with a degeneration argument following the next steps:

3.1. Consider the conic $Q(x, y, u) = y^2 - txw$. When $t \to 0$ we obtain that $Q$
tends to the double line $y^2 = 0$. In $\mathbb{CP}^2$ consider the points $[t : t : 1]$,
$[4t : 2t : 1], [9t : 3t : 1] \in Q$. Restricting to $\mathbb{C}^2$ we have the points $p_1(t) = (t, t),$
$p_2(t) = (4t, 2t), p_3(t) = (9t, 3t)$ on the conic $y^2 - tx$. Let $I_1(x) = \langle x - t, y - t,\rangle,$
$I_2(x) = \langle x - 4t, y - 2t, \rangle$, $I_3(x) = \langle x - 9t, y - 3t \rangle$ the ideals that define $p_1(t), p_2(t),$
$p_3(t)$ respectively. The schemes Spec $\mathbb{C}[x, y]/I_k^2(t)$ define $p_k(t)$ as double points for
$k = 1, 2, 3$. Set $J(t) := \bigcap_{k=1}^3 (I_k^2(t))$. For $t \neq 0$, the scheme $S_t := \text{Spec} \mathbb{C}[x, y]/J(t)$ is
the union of these three double points. Using ([4]) we have that a Groebner basis for
$J(t)$ is given by the polynomials $x^3 - 12x^2y + 47xy^2 - 60y^3 + 11x^2t - 84xyt +$
$157yt^2 + 36x^2t^2 + 132yt^2 - 36t^3$, $y^4 - 2xy^2t + x^2t^2, xy^3 - x^2yt - 6xy^2t + 11y^3t +$
$6x^2t^2 - 11yt^2 - 6y^2t^2 + 6xt^3, x^2y^2 - 12x^2yt + 22xy^2t + 36x^2t^2 - 144xy^2t + 121y^2t^2 +$
$72xt^3 - 132yt^3 + 36t^4$. So we have that $J(0) = \langle f_1, f_2, f_3, f_4 \rangle$, where $f_1 = x^3 - 12x^2y +$
$47xy^2 - 60y^3, f_2 = y^4, f_3 = xy^3, f_4 = x^2y^2$. Note that $J(0)$ defines the flat limit for
t $\to 0$ of the scheme $S_t$. Remark that $f_1 = (x - 4y)(x - 3y)(x - 5y)$. It is then clear
that $J(0)$ consists of all polynomials $f(x, y)$ such that $f = 0$ defines a curve with an
ordinary triple point at the origin with tangent lines $x = 3y, x = 4y, x = 5y$. In
Conclusion, the limit of the three double points, at $p_1(t), p_2(t), p_3(t)$ is an ordinary
triple point with fixed tangent lines. In a similar way when we take the points
\[\frac{1}{t} : 1 : 1, \frac{t}{2} : 4 : 1, \frac{3}{4} : 9 : 1\] and \(t \to 0\), the limit of these three points as double points will be another ordinary triple point. Finally, we can let another double point \(p(t)\) on \(Q\) tend for \(t = 0\) to the point \([1:0:1]\). For instance, take \(p(t) = [1 : \sqrt{t} : 1]\)

3.2. If we apply the above specialization to a conic \(Q\) on which we have seven double points of an irreducible curve \(\Gamma\) of degree seven and genus seven, we have that we can specialize this curve \(\Gamma\) to a curve \(\Gamma_0\) of degree seven with two triple points and one double point on a line \(\ell\), so that the line \(\ell\) splits off \(\Gamma_0\), that is, \(\Gamma_0 = \ell \cup Z\), where \(Z\) is a sextic curve with three double points. Notice that we can make the above limit in such a way that the three double points of \(Z\) are not collinear. Let \(\psi_t : C_t \to \Gamma_t\) be the normalization of \(\Gamma_t\), \(t \neq 0\). \(\{C_t\}\) form the fibers of a family \(\pi : \mathcal{X}^* \to D(0,1) - \{0\}\), where \(D(0,1) := \{t \in \mathbb{C} : |t| < 1\}\). By stable reduction ([5], p. 118), we can make a base change and complete the family \(\pi : \mathcal{X}^* \to D(0,1) - \{0\}\) to a family \(\pi : \mathcal{X} \to D(0,1)\) of stable curves. In this case \(\mathcal{X}\) is smooth, and the stable limit of the \(C_t\) is the central fiber of the family \(\pi : \mathcal{X} \to D(0,1)\) which is the normalization \(C_0\) of \(Z\). The dimension of \(W_5^1(C_0)\) is one: We apply Martens’s theorem ([2, p. 191]) and the proof of the Mumford theorem ([2, p. 193]) to the case \(d = 5, g = 7\) to deduce that \(\dim W_5^1(C_0) = 1\).

Remark 2. — We remark that \(C_0\) has only three \(g_4^1\), i.e. the ones cut out by the lines through the double points of \(Z\). We recall that the double points of \(Z\) are not collinear. It is clear that \(C_0\) is not trigonal. Let \(g_4^1\) be on \(Z\) and \(D = q_1 + q_2 + q_3 + q_4 \in g_4^1\) a general divisor. Note that \(D\) imposes only three conditions to cubics through the double points \(p_1, p_2, p_3\) of \(Z\). Consider the conic \(Q\) passing through \(p_1, p_2, p_3, q_1, q_2\), we claim that \(q_3, q_4 \in Q\), otherwise, by monodromy \(q_3, q_4\) both do not lie on \(Q\). Let \(\ell_0\) be a general line through \(q_3\) so that \(q_4 \notin \ell_0\). Then \(Q + \ell_0\) contains \(p_1, p_2, q_3, q_1, q_2, q_3\) but not \(q_4\) a contradiction. Now I claim that \(Q\) splits in the line \(\ell_12\) through \(p_1, p_2\) and a line \(\ell\) containing \(p_3, q_1, q_2, q_3, q_4\). In fact, if one uses the Cremona transformation based at \(p_1, p_2, p_3\), then \(Z\) is mapped to another sextic curve with three double points and the \(g_6^2\) is now contained in the \(g_6^2\) cut out by the lines, hence it is cut out by the lines through a double point. This implies that also on \(Z\) the same happens.

Now note that one component \(W_1\) of \(W_5^1(C_0)\) is formed by the family of \(g_4^1\) cut out by lines through a general point of the sextic \(Z\). A second component \(W_2\) is formed by the \(g_6^2\) cut out by conics through the three double points \(p_1, p_2, p_3\) and a general point of \(Z\). We can go from \(W_1\) to \(W_2\) via the quadratic Cremona transformation based at the double points \(p_1, p_2, p_3\) of \(Z\). A third component \(W_{p_i}\) is formed by the \(g_6^1\)‘s given by \(g_4^1 + q, q \in Z\) general, where the \(g_4^1\) is cut out by lines through the double point \(p_1\). In analogous way we have the components \(W_{p_2}, W_{p_3}\). Now take a \(g_6^2\) not belonging either to \(W_1\) or \(W_{p_i}\), \(i = 1, 2, 3\). Let \(D = q_1 + \cdots + q_5 \in g_6^2\) be a general divisor. We have that no three points of \(D\) are no collinear, then \(D\) lies on an irreducible conic \(Q_D\). Supose that \(Q_D\) does not
contain $p_1, p_2, p_3$. Since the linear system $\mathcal{A} := |K_C - D|$ is cut out by of cubics through $p_1, p_2, p_3, q_1, q_5$ and has dimension two, we can split off $Q_D$ for a cubic of $\mathcal{A}$, and the residual $\tilde{D}$ would be a line containing $p_1, p_2, p_3$ which is not possible. In a similar way we see that it cannot be the case that $Q_D$ does not contain some of the points $p_1, p_2, p_3$, in other words $p_1, p_2, p_3 \in Q_D$ and therefore $g^1_5 \subset W_2$.

**Proposition 3.3.** – Let $f : C_0 \to Z$ be the normalization of $Z$. $\mu_D$ is not injective only for a finite number of pencils $g^1_5$ on $C_0$.

**Proof.** – We have that for a $|D| = g^1_5$, ker $\mu_D \simeq H^0(C_0, K_{C_0} - 2D)$, where $K_{C_0} - 2D$ is the pullback under $f$ of the linear system of cubics through $p_1, p_2, p_3$ and $D_1, D_2$ with $D_1, D_2 \in |D|$. Note that $H^0(C, K_{C_0} - 2D) = 0$ if $D$ belongs either $W_1$ or $W_2$. Let $D \in W_{p_i}$ for some $i = 1, 2, 3$. By simplicity assume that $D \in W_{p_1}$. Thus we have that $|D| = g^1_5 + q$, $q \in Z$ general and the $g^1_5$ is cut out by the lines through the double point point $p_1$. Every divisor $D \in |D|$ has four points lying on a line through $p_1$. A section of $H^0(C, K_{C_0} - 2D)$ will be a cubic $G$ that has four intersection points with two lines $\ell_1, \ell_2$ where the four points of $D_1, D_2 \in |D|$ respectively lie. Thus $G$ splits in $G = \ell_1 \cdot \ell_2 \cdot \ell_3$, where $\ell_3$ is the line through $p_2, p_3$. If $G \neq 0$, $q$ must lie on $\ell_3$, that is, $q$ must be one of the other two points $z_1, z_2$ on $Z$ where $\ell_3$ intersects $Z$. So for $D \in W_{p_1}$, $\mu_D$ is not injective only for $|D| = g^1_4 + z_j$ $j = 1, 2$.  

The following corollary is now clear.

**Corollary 3.4.** – Let $C \in \mathcal{D}$. Let $L$ be a general point in $W^1_3(C)$, then the Petri map $\mu_L : H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$ is injective.

So we have shown that $\mathcal{D}$ is an irreducible component of $\mathcal{G}P_7$ of codimension one in $\mathcal{M}_7$. Thus $\mathcal{M}_7^1, \mathcal{M}^1_{7, 4}, \mathcal{D}$ are the components of $\mathcal{G}P_7$, then our theorem is proved.

**References**


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