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Curves in Lorentzian Spaces.

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Sunto. – *La nozione di angolo iperbolico tra due qualsiasi direzioni simili al tempo nel piano di Lorentz L^2 è stata appropriatamente definita e studiata da Birman e Nomizu* [1, 2]. *In questo articolo definiamo la nozione di angolo iperbolico tra due qualsiasi direzioni non nulle in L^2 e definiamo una misura sull'insieme di questi angoli iperbolici. Come applicazione, estendiamo il lavoro di Scofield sulle curve euclidee di precessione costante* [9] *all'ambiente di Lorentz, rendendo così esplicite le curve simili allo spazio in L^3 le cui equazioni naturali esprimono la loro curvatura e torsione come autofunzioni elementari del loro Laplaciano.*

Summary. – *The notion of «hyperbolic» angle between any two time-like directions in the Lorentzian plane L^2 was properly defined and studied by Birman and Nomizu* [1,2]. *In this article, we define the notion of hyperbolic angle between any two non-null directions in L^2 and we define a measure on the set of these hyperbolic angles. As an application, we extend Scofield's work on the Euclidean curves of constant precession* [9] *to the Lorentzian setting, thus expliciting space-like curves in L^3 whose natural equations express their curvature and torsion as elementary eigenfunctions of their Laplacian.*

1. – Angles between non-null vectors in the Lorentzian plane.

The Lorentzian n-dimensional space L^n is the standard vector space R^n endowed with the geometrical structure given by *the Lorentzian scalar product* $g(X,Y) := x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n$ for all $X = (x_1, \ldots, x_{n-1}, x_n)$ and $Y = (y_1, \ldots, y_{n-1}, y_n)$ in R^n. A vector $V = (v_1, \ldots, v_{n-1}, v_n)$ in L^n is called *space-like*, *time-like* or *null (light-like)* when respectively $g(V,V) > 0$, $g(V,V) < 0$ or $g(V,V) = 0$ and $V \neq 0 = (0, \ldots, 0, 0)$; a non-null vector V is said to be *future-pointing* or *past-pointing* when respectively $g(V,E) < 0$ or $g(V,E) > 0$ whereby $E = (0, \ldots, 0, 1)$, i.e. when $v_n > 0$ or $v_n < 0$; $\|V\| = \sqrt{|g(V,V)|}$ is called the *norm* or *length* of V, and two vectors V and W in L^n are said to be *orthogonal* when $g(V,W) = 0$ (see e.g. [7], [11]).

We now define *the oriented «hyperbolic» angle* (V,W) for any two vectors V and W in the Lorentzian plane L^2 for which $g(V,V) \neq 0 \neq g(W,W)$. Since such vectors can always be normalized, it suffices to define the oriented hyperbolic angle (X, Y) *for any two unit vectors* X and Y in L^2. Let G be *the proper Lorentz group* of L^2, i.e. the group consisting of all orientation-preserving linear transformations of R^2 which also preserve the Lorentzian scalar product g and the time-orientation: G consists of all matrices of the form

$$R_u = \begin{bmatrix} \cosh(u) & \sinh(u) \\ \sinh(u) & \cosh(u) \end{bmatrix}$$

whereby $u \in R$. For any two unit time-like vectors X and Y in L^2, the oriented hyperbolic angle (X, Y) from X to Y was naturally defined via hyperbolic rotations as follows [1, 2]: in case X and Y are both either future-pointing or past-pointing (0.a) then $(X, Y) := u$ whereby $R_u X = Y$, and in case X and Y have *different time-orientations* (0.b), (then X and the vector $-Y$ obtained from Y by reflection in the origin are unit time-like vectors with the same time-orientation) then $(X, Y) := u$ whereby $R_u X = -Y$.

The oriented hyperbolic angle (X, Y) between any two unit space-like vectors X and Y or between any two unit vectors X and Y of which one is space-like and the other one is time-like can equally naturally be defined as follows (cfr. also [7], p. 236). When $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are *two unit space-like vectors* in L^2 such that sgn $x_1 =$ sgn y_1 (1.a), respectively sgn $x_1 = -$ sgn y_1 (1.b), i.e. X and Y have the same or opposite orientations with respect to $(1, 0) = E^\perp$, then $(X, Y) := u$ whereby $R_u X = Y$, respectively $R_u X = -Y$. Thus, for two space-like vectors X and Y the angle (X, Y) can be seen as the former angle (DX, DY) of the corresponding time-like vectors DX and DY which are obtained from X and Y by the Euclidean reflection D in the first diagonal $\{(x, x) \mid x \in R\}$ of R^2. And the case of vectors having mixed time-orientations can be dealt within a similar way. When $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are two unit vectors in L^2 such that, say, X is space-like and Y is time-like and such that sgn $x_1 =$ sgn y_2 (2.a), respectively sgn $x_1 = -$ sgn y_2 (2.b), then $(X, Y) := u$ whereby $\overline{R}_u X = Y$, respectively $\overline{R}_u X = -Y$, for

$$\overline{R}_u = \begin{bmatrix} \sinh(u) & \cosh(u) \\ \cosh(u) & \sinh(u) \end{bmatrix} = R_u D, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, for such vectors X and Y the angle (X, Y) can be seen as the angle between the two time-like vectors DX and Y. In terms of the Lorentzian scalar product, the above definitions for the oriented angle $(X, Y) = u$ between unit vectors X and Y amount to the following:

(0.a) $\cosh(u) = -g(X, Y), \qquad \sinh(u) = -g(X, DY);$

(0.b) $\cosh(u) = g(X, Y), \qquad \sinh(u) = g(X, DY);$

(1.a) $\cosh(u) = g(X, Y), \qquad \sinh(u) = g(X, DY);$

(1.b) $\cosh(u) = -g(X, Y)$, $\sinh(u) = -g(X, DY)$;
(2.a) $\cosh(u) = g(X, DY)$, $\sinh(u) = g(X, Y)$;
(2.b) $\cosh(u) = -g(X, DY)$, $\sinh(u) = -g(X, Y)$,

and for vectors X and Y of arbitrary lengths $\|X\| \neq 0 \neq \|Y\|$, for instance in case (2.a):

$$(2.a') \qquad \cosh(u) = \frac{g(X, DY)}{\|X\| \, \|Y\|}, \quad \sinh(u) = \frac{g(X, Y)}{\|X\| \, \|Y\|}.$$

Basic properties concerning this notion of oriented hyperbolic angle and corresponding Lorentzian trigonometry can be found in [1, 6].

The unoriented or *absolute hyperbolic angle* $[X, Y]$ between any two vectors X and Y in L^2, for which $\|X\| \neq 0 \neq \|Y\|$, is defined as $[X, Y] := |(X, Y)|$ where (X, Y) is the oriented hyperbolic angle from X to Y. Consider the following disjoint sets of angles of vectors X and Y in L^2:

$$\mathcal{A}_1 = \{[X, Y] \quad \text{whereby} \quad g(X, X) > 0 \quad \text{and} \quad g(Y, Y) > 0\},$$

$$\mathcal{A}_2 = \{[X, Y] \quad \text{whereby} \quad g(X, X) < 0 \quad \text{and} \quad g(Y, Y) < 0\},$$

$$\mathcal{A}_3 = \{[X, Y] \quad \text{whereby} \quad g(X, X) > 0 \quad \text{and} \quad g(Y, Y) < 0\},$$

and put $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. Then, in analogy with the Euclidean situation [3], a *measure* m on the set $\mathcal{A}$ of unoriented hyperbolic angles is obtained as follows.

THEOREM 1. – *The function* $m : \mathcal{A} \to R^+$ *defined by*

$$m([X, Y]) = \ln\left(\frac{|g(X, Y)| + |g(X, DY)|}{\|X\| \, \|Y\|}\right)$$

satisfies the following properties:

a) *there exist* $[X, Y] \in \mathcal{A}$ *such that* $m([X, Y]) = 1$;
b) *if* $[X, Y] = [V, W]$, *then* $m([X, Y]) = m([V, W])$;
c) *if* $[X, Y] + [Y, Z] = [X, Z]$, *then* $m([X, Y] + m[Y, Z]) = m([X, Z])$.

PROOF. – Hereafter we will restrict attention to angles in the subset $\mathcal{A}_1$; the proofs dealing with the subsets $\mathcal{A}_2$ and $\mathcal{A}_3$ are analogous.

a) For $X = (1, 0)$ and $Y = (\frac{e^2+1}{2e}, \frac{e^2-1}{2e})$, the angle $[X, Y]$ is in $\mathcal{A}_1$ and verification shows that $m([X, Y]) = \ln e = 1$.

b) Next, suppose that $[X, Y], [V, W] \in \mathcal{A}_1$ and that $[X, Y] = [V, W]$. Then $\cosh([X, Y]) = \cosh([V, W])$, $\sinh([X, Y]) = \sinh([V, W])$ and since $\cosh([X, Y])$

$= |g(X,Y)|/\|X\|\,\|Y\|$, $\sinh([X,Y]) = |g(X,DY)|/\|X\|\,\|Y\|$, it follows that

$$\begin{aligned} m([X,Y]) &= \ln\left(\frac{|g(X,Y)| + |g(X,DY)|}{\|X\|\,\|Y\|}\right) \\ &= \ln\left(\frac{|g(V,W)| + |g(V,DW)|}{\|V\|\,\|W\|}\right) \\ &= m([V,W]). \end{aligned}$$

c) Finally, suppose that $[X,Y], [Y,Z], [X,Z] \in \mathcal{A}_1$ and that $[X,Y] + [Y,Z] = [X,Z]$. Then $\cosh([X,Y]+[Y,Z]) = \cosh([X,Z])$, so $\cosh([X,Y])\cosh([Y,Z]) + \sinh([X,Y])\sinh([Y,Z]) = \cosh([X,Z])$. Since $\cosh([X,Y]) = |g(X,Y)|/\|X\|\,\|Y\|$ and $\sinh([X,Y]) = |g(X,DY)|/\|X\|\,\|Y\|$, it follows that

$$\frac{|g(X,Y)|\,|g(Y,Z)|}{\|X\|\,\|Y\|^2\,\|Z\|} + \frac{|g(X,DY)|\,|g(Y,DZ)|}{\|X\|\,\|Y\|^2\,\|Z\|} = \frac{|g(X,Z)|}{\|X\|\,\|Z\|}. \tag{1}$$

By assumption it follows that $\sinh([X,Y]+[Y,Z]) = \sinh([X,Z])$ and therefore $\sinh([X,Y])\cosh([Y,Z]) + \cosh([X,Y])\sinh([Y,Z]) = \sinh([X,Z])$. Hence we get

$$\frac{|g(X,DY)|\,|g(Y,Z)|}{\|X\|\,\|Y\|^2\,\|Z\|} + \frac{|g(X,Y)|\,|g(Y,DZ)|}{\|X\|\,\|Y\|^2\,\|Z\|} = \frac{|g(X,DZ)|}{\|X\|\,\|Z\|}. \tag{2}$$

Consequently, by using the definition of the function m and the relations (1) and (2), it follows that

$$\begin{aligned} & m([X,Y]) + m([Y,Z]) \\ &= \ln\left(\frac{|g(X,Y)| + |g(X,DY)|}{\|X\|\,\|Y\|}\right) + \ln\left(\frac{|g(Y,Z)| + |g(Y,DZ)|}{\|Y\|\,\|Z\|}\right) \\ &= \ln\left(\left(\frac{|g(X,Y)| + |g(X,DY)|}{\|X\|\,\|Y\|}\right)\left(\frac{|g(Y,Z)| + |g(Y,DZ)|}{\|Y\|\,\|Z\|}\right)\right) \\ &= \ln\left(\frac{|g(X,Z)|}{\|X\|\,\|Z\|} + \frac{|g(X,DZ)|}{\|X\|\,\|Z\|}\right) \\ &= m([X,Z]). \qquad \square \end{aligned}$$

2. – An application: curves of constant precession.

In our opinion, various topics related to hyperbolic angles in semi-Riemannian geometries may be worth-while to be studied: properties concerning angles of particular directions on Lorentzian hypersurfaces in semi-Euclidean space with respect to their indicatrices of Dupin and Euler, the Kaehler-angles for submanifolds in indefinite complex or Sasakian spaces allowing a.o. a study of slant such submanifolds, etcetera. At this stage, as an application of the above, we will mention some results about *space-like curves of constant precession* in the Lorentzian space L^3. Curves of constant precession in the Euclidean space E^3 were studied first by Scofield [9] as the curves whose *Darboux-vector* or *centrode* (i.e. the axis of instantaneous rotation of their Frenet-frame when moving along the curve), makes a fixed angle with a fixed axis and moves about this axis with a constant speed. For their connections with variational problems («k-minimality») and the theory of submanifolds of finite Chen-type, see [4, 5, 8].

A curve β in L^3 is called *space-like* when at every point it has a well-defined space-like tangent direction. Such curves can always be parameterized by an *arclength parameter* s, thus having $\|\beta'\| = 1$ where ' denotes derivation with respect to s and $\beta' = T$ then is a unit space-like tangent vector field along β. In the following, we restrict to space-like curves in L^3 for which the principal normal direction β'' is nowhere a null-direction, i.e. we restrict to space-like curves β in L^3 whose principal normal N is either everywhere *space-like* (I) or is everywhere *time-like* (II).

(I). In standard notations, the Frenet formulae of a space-like curve β in L^3 with space-like principal normals are $T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = \tau N$, whereby $g(T,T) = g(N,N) = -g(B,B) = 1$ and $g(T,N) = g(N,B) = g(B,T) = 0$. Their centrode C is given by $C = \tau T - \kappa B$ (the «rotation»-component of C' with respect to the Frenet-frame $\{T,N,B\}$ is $\tau T' - \kappa B' = 0$). Since the definition of curves of constant precession involves that at each point $\beta(s)$ the centrode $C(s)$ makes a constant angle with a fixed direction, it is implicitely assumed that for each s there holds $\|C(s)\| \neq 0$, i.e. that either everywhere $\kappa^2 > \tau^2$ (I.A) or $\kappa^2 < \tau^2$ (I.B).

We go on here for the case (I.A). Realizing that the fixed axis involved in the definition of curves β of constant precession should be determined by a parallel vector field along β of the form $A(s) = C(s) + \mu N(s)$, $\mu \in R$, and aiming for the natural equations of such curves, we formulate two lemmata, whose proofs are straightforward. 10pt

LEMMA 1. – *The following are equivalent:*

1) $\|C\| = \omega$, $\omega \in R_0^+$;
2) $\|N'\| = \omega$;
3) $\|A\| = a = \sqrt{\mu^2 - \omega^2}$, $a \in R_0^+$;
4) $\sinh{(C,A)} = \omega/a$;
5) $\cosh{(N,A)} = |\mu|/a$.

LEMMA 2. – *Under a condition (1)-(5) of Lemma 1, the following are equivalent:*

1) $\|C'\| = \omega|\mu|$;
2) $A' = 0$.

The following result then characterizes curves of constant precession by their natural equations, i.e. by giving the *curvature* κ and *torsion* τ as concrete functions of an *arclength parameter* s.

THEOREM 2. – *A unit speed space-like curve $\beta(s)$ with space-like principal normal in the Lorentzian space L^3 is a curve of constant precession if and only if*

$$(*) \qquad \kappa(s) = \omega \cosh(\mu s), \quad \tau(s) = \omega \sinh(\mu s),$$

for some $\omega \in R_0^+$ and $\mu \in R$.

PROOF. – If (*) holds, then $\tau' = \mu\kappa$ and $\kappa' = \mu\tau$, which implies that $A' = 0$ and that $\|A\|$ is constant. Then Lemma 1 and Lemma 2 show that β is a curve of constant precession.

Conversely, if β is a curve of constant precession, then from $A' = 0$ it follows that $\tau' = \mu\kappa$ and $\kappa' = \mu\tau$. Thus κ and τ satisfy the differential equation $f'' = \mu^2 f$ whose integration, by appropriate choice of the arclength parameter, essentially yields (*). □

The next purpose is to obtain explicit parameter-equations for the Lorentzian co-ordinates (x, y, z) of curves of constant precession in L^3 from (*). In the Euclidean situation, this was done based on the known parameter-equations of the spherical helices [10] which turn out to be the tangential indicatrices of curves of constant precession in E^3. Correspondingly, in our situation, we also first look at the tangential indicatrix $\gamma(s) := T(s) = \beta'(s)$ of β. Since $g(T, T) = 1$, the curve γ lies on the *pseudo-sphere* S_1^2 with equation $x^2 + y^2 - z^2 = 1$ in L^3. Since $\gamma' = T' = \kappa N$, by Lemma 1 (5), γ' makes a constant oriented hyperbolic angle θ with the constant vector A: the tangential indicatrix γ of β therefore is a *space-like pseudo-spherical helix*.

Since A is a space-like constant vector, we may take $\tilde{A} = \frac{A}{\|A\|} = (1, 0, 0)$. Denote by s_γ an arclength parameter of the curve γ. Then the parameter-equations of γ are given by

$$(3) \qquad x_\gamma = s_\gamma \cosh\theta, \quad \upsilon_\gamma = \upsilon(\sigma_\gamma), \quad \zeta_\gamma = \zeta(\sigma_\gamma).$$

If we project γ onto the plane $\pi \equiv Oyz$ perpendicular to $\tilde{A}$, then its orthogonal projection $\gamma_\pi = \gamma_\pi(s_\pi)$ has parameter-equations of the form

$$(4) \qquad x_\pi = 0, \quad y_\pi = y(s_\gamma), \quad z_\pi = z(s_\gamma),$$

and the curves γ and γ_π are related by

$$(5) \qquad \gamma_\pi(s_\gamma) = \gamma(s_\gamma) - g(\gamma(s_\gamma), \tilde{A})\tilde{A}.$$

Differentiating with respect to s_γ, and s_π being an arclength-parameter of γ_π, we get

$$\frac{d\gamma_\pi}{ds_\pi}\frac{ds_\pi}{ds_\gamma} = T_\gamma - \cosh(\theta)\tilde{A}, \tag{6}$$

or equivalently

$$T_\pi \frac{ds_\pi}{ds_\gamma} = T_\gamma - \cosh(\theta)\tilde{A}. \tag{7}$$

Hence

$$\left(\frac{ds_\pi}{ds_\gamma}\right)^2 = \sinh^2(\theta), \tag{8}$$

and we may therefore proceed by taking

$$s_\pi = \sinh(\theta) s_\gamma. \tag{9}$$

From (6) we have

$$T_\pi = \frac{d\gamma_\pi}{ds_\pi} = \frac{1}{\sinh(\theta)} T_\gamma - \coth(\theta)\tilde{A}, \tag{10}$$

and consequently

$$T'_\pi = \frac{dT_\pi}{ds_\pi} = \frac{d^2\gamma_\pi}{ds_\pi^2} = \frac{dT_\pi}{ds_\gamma}\frac{ds_\gamma}{ds_\pi} = \left(\frac{1}{\sinh(\theta)} T'_\gamma\right)\frac{1}{\sinh(\theta)} = \frac{1}{\sinh^2(\theta)}\kappa_\gamma N_\gamma. \tag{11}$$

On the other hand

$$T'_\pi = \kappa_\pi N_\pi \tag{12}$$

which together with (11) implies that

$$\kappa_\gamma = \sinh^2(\theta)\kappa_\pi, \quad N_\gamma \| N_\pi. \tag{13}$$

Differentiating the relation

$$\cosh(\theta) = g(T_\gamma, \tilde{A}) = \text{constant}, \tag{14}$$

with respect to s_γ, we obtain that

$$g(N_\gamma, \tilde{A}) = 0. \tag{15}$$

It follows that

$$\tilde{A} = \cosh(\theta) T_\gamma + \sinh(\theta) B_\gamma, \tag{16}$$

and differentiation with respect to s_γ yields

$$\cosh(\theta)T'_\gamma + \sinh(\theta)B'_\gamma = 0. \tag{17}$$

Using the corresponding Frenet-equations, we get

$$\frac{\kappa_\gamma}{\tau_\gamma} = -\tanh(\theta) = \text{constant}. \tag{18}$$

Moreover, differentiating $g(\gamma(s_\gamma), \gamma(s_\gamma)) = 1$ with respect to s_γ and using the corresponding Frenet-equations, we find that

$$\gamma(s_\gamma) = -\frac{1}{\kappa_\gamma}N_\gamma + \frac{1}{\tau_\gamma}\left(\frac{1}{\kappa_\gamma}\right)' B_\gamma \tag{19}$$

and therefore that

$$g(\gamma(s_\gamma), \gamma(s_\gamma)) = \left(\frac{1}{\kappa_\gamma}\right)^2 - \left(\frac{1}{\tau_\gamma}\left(\frac{1}{\kappa_\gamma}\right)'\right)^2 = 1. \tag{20}$$

Put $\tilde{\kappa}_\gamma = 1/\kappa_\gamma$ and $\tilde{\tau}_\gamma = 1/\tau_\gamma$. Then the previous equation becomes

$$\tilde{\kappa}_\gamma^2 - (\tilde{\tau}_\gamma \tilde{\kappa}'_\gamma)^2 = 1.$$

By (18) and integration, we get

$$\tilde{\kappa}_\gamma^2 - s_\gamma^2 \coth^2(\theta) = 1. \tag{21}$$

Put $\tilde{\kappa}_\pi = 1/\kappa_\pi$. Then by (9) and (13), (21) becomes

$$\tilde{\kappa}_\pi^2 - s_\pi^2 \cosh^2(\theta) = \sinh^4(\theta), \tag{22}$$

or equivalently

$$\kappa_\pi^2 = \frac{1}{\sinh^4(\theta) + s_\pi^2 \cosh^2(\theta)}. \tag{23}$$

Denote by ϕ the oriented hyperbolic angle from $e_2 = (0, 1, 0)$ to the vector T_π, which by (10) is seen to be a unit timelike vector in π. Then

$$T_\pi = \sinh(\phi)e_2 + \cosh(\phi)e_3. \tag{24}$$

Moreover, since $\gamma'_\pi(s_\pi) = T_\pi(s_\pi)$, we get

$$\gamma_\pi = \int \frac{1}{\kappa_\pi(\phi)}(\sinh(\phi)e_2 + \cosh(\phi)e_3)d\phi.$$

From $\phi'(s_\pi) = \kappa_\pi(s_\pi)$ and (22) it follows that

$$\phi(s_\pi) = \int \frac{ds_\pi}{\sqrt{\sinh^4(\theta) + s_\pi^2 \cosh^2(\theta)}}. \tag{26}$$

Therefore

$$\phi(s_\pi) = \frac{1}{\cosh(\phi)} \sinh^{-1}\left(\frac{s_\pi \cosh(\theta)}{\sinh^2(\theta)}\right), \tag{27}$$

such that

$$s_\pi = \frac{\sinh^2(\theta)}{\cosh(\theta)} \sinh(\phi \cosh(\theta)). \tag{28}$$

Substituting (28) into relation (22), we find that

$$\kappa_\pi = \frac{1}{\sinh^2(\theta)\cosh(\phi\cosh(\theta))}. \tag{29}$$

Substituting (29) into (25) and integrating, we find that the parameter-equations of γ_π are given by

$$\begin{cases} x_\pi = 0, \\ y_\pi = \dfrac{\sinh^2\theta}{2}\left(\dfrac{1}{1+\cosh\theta}\cosh(\phi(1+\cosh\theta)) + \dfrac{1}{1-\cosh\theta}\cosh(\phi(1-\cosh\theta))\right), \\ z_\pi = \dfrac{\sinh^2\theta}{2}\left(\dfrac{1}{1+\cosh\theta}\sinh(\phi(1+\cosh\theta)) + \dfrac{1}{1-\cosh\theta}\sinh(\phi(1-\cosh\theta))\right). \end{cases} \tag{30}$$

Further, any arclength-parameters s and s_γ of the curves β and its tangent indicatrix γ being related by

$$\frac{ds_\gamma}{ds} = \kappa(s), \tag{31}$$

and by (*) having $\kappa(s) = \omega \cosh(\mu s)$, organizing a choice for s and s_γ such that $s_\gamma = 0$ when $s = 0$, we have

$$s_\gamma = \frac{\omega}{\mu}\sinh(\mu s). \tag{32}$$

By Lemma 1, $\cosh(\theta) = |\mu|/a$ and $\sinh(\theta) = \omega/a$. Using (28), the relation (9) becomes

$$s_\gamma = \frac{\omega}{\mu}\sinh(\phi\cosh(\theta)). \tag{33}$$

Then (32) and (33) imply

$$\phi = as. \tag{34}$$

Therefore, since $\sinh(\theta) = \omega/a$, $\cosh(\theta) = |\mu|/a$ and by using (34), the parameter-equations (30) become

$$(35)\qquad \begin{cases} x_\pi = 0, \\ y_\pi = \dfrac{\omega^2}{2a}\left(\dfrac{1}{a+\mu}\cosh((a+\mu)s) + \dfrac{1}{a-\mu}\cosh((a-\mu)s)\right), \\ z_\pi = \dfrac{\omega^2}{2a}\left(\dfrac{1}{a+\mu}\sinh((a+\mu)s) + \dfrac{1}{a-\mu}\sinh((a-\mu)s)\right). \end{cases}$$

Hence (3),(4),(32) imply that the tangent indicatrix γ of β has parameter-equati\-ons of the form

$$(36)\qquad \begin{cases} x_\gamma = \dfrac{\omega}{a}\sinh(\mu s), \\ y_\gamma = \dfrac{\mu-a}{2a}\cosh((a+\mu)s) - \dfrac{a+\mu}{2a}\cosh((a-\mu)s), \\ z_\gamma = \dfrac{\mu-a}{2a}\sinh((a+\mu)s) - \dfrac{a+\mu}{2a}\sinh((a-\mu)s). \end{cases}$$

Finally, integrating (36) we obtain the parameter-equations of β that we were aiming for.

THEOREM 3. – *The parameter-equations of a unit speed space-like curve $\beta = \beta(s)$ of constant precession with space-like principal normal and for which $\tau^2(s) > \kappa^2(s)$ are given by*

$$(37)\qquad \begin{cases} x(s) = \dfrac{\omega}{\mu a}\cosh(\mu s), \\ y(s) = \dfrac{\mu-a}{2a(a+\mu)}\sinh((a+\mu)s) - \dfrac{a+\mu}{2a(a-\mu)}\sinh((a-\mu)s), \\ z(s) = \dfrac{\mu-a}{2a(a+\mu)}\cosh((a+\mu)s) - \dfrac{a+\mu}{2a(a-\mu)}\cosh((a-\mu)s), \end{cases}$$

whereby $\omega \in R_0^+$, $\mu \in R$, $\mu^2 > \omega^2$ and $a = \sqrt{\mu^2 - \omega^2}$.

REMARK 1. – From (37) it may be observed that these curves β lie on the quadric $\frac{\mu^2}{\omega^2}x^2 + y^2 - z^2 = -\frac{4\mu^2}{\omega^4}$ and are non-closed curves.

The cases (I.B) and (II) can be dealt with in a perfectly similar way so as to lead to the following results. For case (I.B), corresponding to Theorem 3 we have the next theorem.

THEOREM 4. – *The parameter-equations of a unit speed space-like curve $\beta = \beta(s)$ of constant precession with space-like principal normal N and for which $\tau^2(s) < \kappa^2(s)$ are given by*

$$x(s) = \frac{\omega}{\mu a} \sinh(\mu s),$$

$$y(s) = \frac{a-\mu}{2a(a+\mu)} \sinh((a+\mu)s) - \frac{a+\mu}{2a(\mu-a)} \sinh((\mu-a)s),$$

$$z(s) = \frac{a-\mu}{2a(a+\mu)} \cosh((a+\mu)s) + \frac{a+\mu}{2a(\mu-a)} \cosh((\mu-a)s),$$

whereby $\omega \in R_0^+$, $\mu \in R$ and $a = \sqrt{\omega^2 + \mu^2}$.

REMARK 2. – Such curves β lie on the quadric $\frac{\mu^2}{\omega^2}x^2 - y^2 + z^2 = \frac{4\mu^2}{\omega^4}$ and are non-closed curves.

For case (II), i.e. for *space-like curves β* whose *principal normal N* is everywhere *time-like*, the Frenet formula's are given by $T' = \kappa N$, $N' = \kappa T + \tau B$, $B' = \tau N$ whereby $g(T,T) = g(B,B) = -g(N,N) = 1$ and $g(T,N) = g(N,B) = g(B,T) = 0$. Their *centrode C* is given by $C = -\tau T + \kappa B$ and thus everywhere has a well-defined non-null direction. Corresponding to Theorems 2 and 3, for case (II) we have the following.

THEOREM 5. – *A unit-speed space-like curve $\beta = \beta(s)$ with time-like principal normal in the Lorentzian space L^3 is a curve of constant precession if and only if*

(**) $$\kappa(s) = \omega \cos(\mu s), \quad \tau(s) = \omega \sin(\mu s),$$

for some $\omega \in R_0^+$ and $\mu \in R$.

THEOREM 6. – *The parameter-equations of a unit-speed space-like curve $\beta = \beta(s)$ of constant precession with time-like principal normal are given by*

$$x(s) = \frac{a-\mu}{2a(a+\mu)} \cos((a+\mu)s) + \frac{a+\mu}{2a(a-\mu)} \cos((a-\mu)s),$$

$$y(s) = \frac{a-\mu}{2a(a+\mu)} \sin((a+\mu)s) + \frac{a+\mu}{2a(a-\mu)} \sin((a-\mu)s),$$

$$z(s) = -\frac{\omega}{\mu a} \cos(\mu s),$$

whereby $\omega \in R_0^+$, $\mu \in R$, $\omega^2 < \mu^2$ and $a = \sqrt{\mu^2 - \omega^2}$.

REMARK 3. – Such curves β lie on the quadric $x^2 + y^2 - \frac{\mu^2}{\omega^2} z^2 = \frac{4\mu^2}{\omega^4}$ and are closed if and only if μ/a is rational number.

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