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C. Trombetti

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On a Class of Monge-Ampère Type Equations with Lower Order Terms.

C. TROMBETTI (*)

Sunto. – *Si dimostrano risultati di confronto per soluzioni di equazioni tipo Monge-Ampère in dimensione due, considerando anche il caso delle autofunzioni.*

Summary. – *We prove some comparison results for Monge-Ampère type equations in dimension two. We consider also the case of eigenfunctions and we prove a kind of «reverse» inequalities.*

1. – Introduction.

In this paper we discuss some results presented at the XVII UMI Conference. All the proofs and further developements are contained in [4] and [5].

We consider the following Dirichlet problem

$$\begin{cases} \det D^2 u = f + \sigma u^2 & \text{in } \Omega \\ u \text{ concave} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a convex, bounded open set of $\mathbb{R}^2$, f is a «smooth» and positive function, $\sigma > 0$. Our aim is to compare the solution to problem (1.1) with the solution to a symmetrized problem that is

$$\begin{cases} \det D^2 v = f^{\#} + \sigma v^2 & \text{in } \Omega^{\star} \\ v \text{ concave and continuous in the closure} & \text{of } \Omega^{\star}, \\ v = 0 & \text{on } \partial\Omega^{\star}. \end{cases}$$

where $f^{\#}$ is the spherically decreasing rearrangement of f (see section 2 for

(*) Comunicazione presentata a Milano in occasione del XVII Congresso U.M.I.

the definiton) and $\Omega^\star$ is the ball centered at the origin having the same perimeter L_Ω as Ω.

The first result in this framework is contained in the paper by Talenti (see [19]). In such paper he proved that if u is the solution of

$$\begin{cases} \det D^2 u = f & \text{in } \Omega \\ u \text{ concave} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and v the solution of the symmetrized problem

$$\begin{cases} \det D^2 v = f^{\#} & \text{in } \Omega^\star \\ v \text{ concave and continuous in the closure} & \text{of } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}$$

then

$$u^\star(2\pi|x|) \leqslant v(x)$$

where $u^\star$ is a suitable rearrangement of u (see Section 2). This result has been generalized, in the papers of Tso (see [22]) and Trudinger (see [21]), to any dimension and to the case of general Hessian operators, involving the so called symmetrization by «quermassintegrals». The Monge-Ampère type equation

$$\det D^2 u = g(x, u, Du)$$

has been widely studied. Existence and regularity results can be found for instance in [9], [13], [17], [23], [25] under different assumptions on g and Ω and with different methods. In what follows one of our basic hypotheses will be $f > 0$ in $\overline{\Omega}$.

A preliminary result, that we will need in the following, concerns uniqueness of solutions to problem (1.1). It is related to the eigenvalue $\sigma_1(\Omega)$ of Monge - Ampère operator, introduced by P. L. Lions in [18]. We recall the definition.

DEFINITION 1.1. – $\sigma_1(\Omega) = \inf\{\sigma_1^A : A \in V\}$ *where*

$$V = \{A = (a_{ij}(x)) = (a_{ji}(x)) \in C(\overline{\Omega}),\ a_{ij} > 0 \ \text{ in } \ \overline{\Omega}, \det A \geqslant 1/4\}$$

and σ_1^A *is the first eigenvalue of the linear second order elliptic operator* $-a_{ij}\partial_{ij}$ *with zero boundary conditions.*

There exists also a variational characterization of σ_1^2 (see [23], [25]) that is

$$\sigma_1^2(\Omega) = \inf\left\{ \frac{\int_\Omega u \det D^2 u}{\int_\Omega u^3} : u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}),\right.$$
$$\left. u \text{ is nonzero, concave and vanishes on } \partial\Omega \right\}. \tag{1.3}$$

It is shown (see [18], [23], [25])) that problem (1.3) has a unique (up to a multiplicative constant) minimizer ϕ which satisfies

$$\begin{cases} \det D^2 \phi = (\sigma_1 \phi)^2 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega . \end{cases}$$

The function ϕ is called an eigenfunction of Monge-Ampère operator.

2. – Notation and Preliminaries.

Given a measurable function $u : \Omega \to \mathbb{R}$, we recall the definition of decreasing rearrangement of u. If

$$\mu(t) = \mathcal{L}^2(\{x \in \Omega : |u(x)| > t\}), \qquad t \geqslant 0,$$

denotes the distribution function of u, then the decreasing rearrangement of u is the distribution function of μ that is

$$u^*(s) = \sup\{t \geqslant 0 : \mu(t) \geqslant s\}, \qquad s \in [0, |\Omega|].$$

By the spherically decreasing rearrangement of u we mean

$$u^\#(x) = u^*(\pi|x|^2), \qquad x \in \Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin, having the same area as Ω.

If Ω is a convex set of $\mathbb{R}^2$ and u has convex level sets, we also define $\lambda(t)$ as the perimeter of the level sets of u, $\{x \in \Omega : |u(x)| > t\}$. The rearrangement of u with respect to the perimeter is defined as

$$u^\star(s) = \sup\{t \geqslant 0 : \lambda(t) \geqslant s\}, \qquad s \in [0, L_\Omega],$$

where L_Ω is the perimeter of Ω.

From now on we consider only functions belonging to the class

$$\Phi_0(\Omega) = \{u : \Omega \to \mathbb{R} : u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}),\ u \text{ concave in } \Omega,\ u = 0 \text{ on } \partial\Omega\};$$

we easily infer that $u \geqslant 0$ in Ω. Moreover, the perimeter $\lambda(t)$ of the level set

$\{u > t\}$ equals

$$\lambda(t) = \text{length}\,\{x \in \Omega : u(x) = t\}$$

for $0 < t < \max u$.

The following statements hold for $u \in \Phi_0(\Omega)$ (see [19], [22]):

i) $\lambda(t)$ is a non increasing function on $[0, \max u]$;

ii) $-\lambda'(t) = \int\limits_{u=t} k|Du|^{-1}$, where

$$k = -|Du|^{-3}\left(\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} Du, Du\right) \geqslant 0$$

is the curvature of the level line $\{u=t\}$;

iii) $u^\star$ is a non increasing, concave function on $[0, L]$ and $u^\star(0) = \max u$ and $u^\star(L_\Omega) = 0$;

iv) $u^\star(\lambda(t)) = t$;

v) $u^\star(2\pi|x|) \in \Phi_0(\Omega^\star)$, where $\Omega^\star$ is the ball centered at the origin, having the same perimeter L_Ω as Ω.

Isoperimetric inequality ensures that (see [21])

$$\|u\|_{L^p(\Omega)} \leqslant \|u^\star(2\pi|x|)\|_{L^p(\Omega^\star)}, \qquad p \geqslant 1\,. \tag{2.1}$$

We also recall the Gauss-Bonnet theorem

$$\int\limits_{u=t} k = 2\pi$$

and the following rapresentation formula for the Hessian determinant of u

$$\det D^2 u = -\frac{1}{2}\operatorname{div}(A(u)Du),$$

where $A(u)$ is the matrix given by

$$A(u) = \begin{pmatrix} -u_{yy} & u_{xy} \\ u_{xy} & -u_{xx} \end{pmatrix}$$

which is positive definite for every $u \in \Phi_0(\Omega)$, solution to (1.1).

Furthermore we consider the integral functional (see [23], [21])

$$I(u, \Omega) = \int\limits_\Omega u \det D^2 u$$

and we recall the following extension of the Polya-Szegö principle (see [21]).

THEOREM 2.1. – *Let Ω be convex and $u \in \Phi_0(\Omega)$. Then*

$$I(u, \Omega) \geqslant I(u^\star(2\pi|x|), \Omega^\star). \tag{2.2}$$

As an immediate consequence of (1.3), (2.2) and (2.1) we have a Faber-Krahn type inequality

$$\sigma_1(\Omega) \geqslant \sigma_1(\Omega^\star).$$

Finally the following existence and uniqueness result holds true (see [18]).

THEOREM 2.2. – *Let $H:(x,t) \in \Omega \times \mathbb{R} \to]0, +\infty[$ be a smooth function such that $\dfrac{\partial H}{\partial t} \leqslant \sigma_0 < \sigma_1$ for $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Then the problem*

$$\begin{cases} (\det D^2 u)^{1/2} = H(x, u) & \text{in } \Omega \\ u \text{ concave} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique solution in $C^{1,1}(\overline{\Omega}) \cap C^2(\Omega)$.

REMARK 2.1. – *We need to observe that solvability of problem* (1.1) *has been proved, for example, by Lions* [18], *Tso* [23], *Wang* [24]. *For various proofs of the uniqueness we refer the reader to* [15], [18], [23], [25].

REMARK 2.2. – *Returning to problem* (1.1) *we observe that, if $H(x,t) = \sqrt{f(x) + \sigma t^2}$, the above condition $\dfrac{\partial H}{\partial t} \leqslant \sigma_0 < \sigma_1$ is satisfied if $\sigma \leqslant \sigma_0^2$.*

3. – Main result.

In this section we will prove that the rearrangement of the solution u to problem (1.1) can be estimated by the solution of the conveniently symmetrized problem (3.1). In order to prove this comparison result we will follow an argument which can be found in [12] in the case of p-laplacian operator. We need that there exists a unique solution to problem (3.1), which is decreasing and spherically symmetric, and this is true, for example, if $\sigma \leqslant \sigma_0^2 < \sigma_1^2(\Omega^\star)$. Now we can state our main result.

THEOREM 3.1. – *Let u be a classical solution to problem* (1.1) *and let v be the solution to problem*

$$(3.1)\qquad \begin{cases} \det D^2 v = f^{\#} + \sigma v^2 & \text{in } \Omega^\star \\ v \text{ concave and continuous in the closure} & \text{of } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star. \end{cases}$$

Then

$$(3.2)\qquad u^\star(2\pi|x|) \leqslant v(x), \qquad x \in \Omega^\star.$$

PROOF. – Let $0 < t < \max u$ and let us integrate the equation in (1.1) on the level set $\{u > t\}$. By divergence theorem we get

$$(3.3)\qquad \int_{u>t} \det D^2 u = -\frac{1}{2}\int_{u=t}\left(\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx}\end{pmatrix} Du, \frac{Du}{|Du|}\right) = $$
$$= \frac{1}{2}\int_{u=t} k|Du|^2 = \int_{u>t} f + \sigma\int_{u>t} u^2.$$

Since

$$(3.4)\qquad \int_{u>t} u^2 = \int_t^{+\infty} s^2(-\mu'(s))\, ds = t^2\mu(t) + \int_t^{+\infty} 2s\mu(s)\, ds$$

by (3.3), (3.4), Hardy-Littlewood inequality and isoperimetric inequality we obtain

$$(3.5)\qquad \frac{1}{2}\int_{u=t} k|Du|^2 \leqslant \int_0^{\frac{\lambda^2(t)}{4\pi}} f^*(s)\, ds + \sigma t^2\frac{\lambda^2(t)}{4\pi} - \sigma\int_0^{\lambda(t)} \frac{2}{4\pi} s^2 u^\star(s)\, u^\star(s)'\, ds\, .$$

Let us consider the left hand side of (3.5). By Hölder inequality and Gauss-Bonnet theorem we have

$$(3.6)\qquad \frac{1}{2}\int_{u=t} k|Du|^2 \geqslant \frac{1}{2}\,\frac{\left(\int_{u=t} k\right)^3}{\left(\int_{u=t} k|Du|^{-1}\right)^2} = \frac{4\pi^3}{(-\lambda'(t))^2}\, .$$

From (3.5) and (3.6), with $s = \lambda(t)$, we deduce the following inequality in terms of the rearrangement $u^\star$ of u

$$(3.7)\ (-u^\star(s)')^2 \leqslant \frac{\sigma}{8\pi^4}\int_0^s r u^\star(r)^2\, dr + \frac{1}{4\pi^3}\int_0^{s^2/4\pi} f^*(r)\, dr, \qquad s \in (0, L_\Omega)\, .$$

Setting $c_1 = \dfrac{\sigma}{8\pi^4}$ and $c_2 = \dfrac{1}{4\pi^3}$, for $v(|x|) = v^\star(2\pi|x|)$ the following equality holds

$$(-v^\star(s)')^2 = c_1 \int_0^s r v^\star(r)^2\, dr + c_2 \int_0^{s^2/4\pi} f^*(r)\, dr\,. \tag{3.8}$$

Let

$$U(s) = \int_0^s r u^\star(r)^2\, dr \qquad V(s) = \int_0^s r v^\star(r)^2\, dr\,,$$

we want to prove that $U(s) \leqslant V(s)$, $s \in (0, L_\Omega)$. By choosing

$$\varphi_1(s) = \frac{\left(U(s)^{3/2} - V(s)^{3/2}\right)^+}{U(s)^{1/2}}\,, \qquad \varphi_2(s) = \frac{\left(U(s)^{3/2} - V(s)^{3/2}\right)^+}{V(s)^{1/2}}$$

as test functions in (3.7) and (3.8) respectively, and integrating by parts one can deduce that $|U > V| = 0$. Hence $U(s) \leqslant V(s)$ $s \in (0, L_\Omega)$. Then by (3.7) and (3.8) we have

$$-u^\star(s)' \leqslant -v^\star(s)' \qquad s \in (0, L_\Omega)$$

and integrating between s and L_Ω

$$u^\star(s) \leqslant v^\star(s) \qquad s \in (0, L_\Omega)\,. \qquad \blacksquare$$

4. – The case of eigenfuntions.

Let us consider a fixed eigenfunction u of problem

$$\begin{cases} \det D^2 u = (\sigma_1 u)^2 & \text{in } \Omega \\ u \text{ concave} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

and the ball B centered at the origin such that $\sigma_1(B) = \sigma_1(\Omega)$. Let v_q be the corresponding eigenfunction such that

$$\int_0^{L_\Omega} r(u^\star(r))^q\, dr = \int_0^{L_B} r(v_q^\star(r))^q\, dr \qquad 0 < q < +\infty\,, \tag{4.2}$$

where L_Ω and L_B are the perimeters of Ω and B respectively, and let v_∞ be the corresponding eigenfunction having the same L^∞-norm as u. In other words

v_q, $0 < q \leqslant +\infty$, solves the following problem

$$\begin{cases} \det D^2 v_q = (\sigma_1 v_q)^2 & \text{in } B \\ v_q \text{ concave} & \text{in } B \\ v_q = 0 & \text{on } \partial B . \end{cases}$$

By Faber-Krahn inequality, $\sigma_1(B) \geqslant \sigma_1(\Omega^\star)$ and so $L_B \leqslant L_\Omega$. The following comparison results hold.

THEOREM 4.1. – *Let u and v_q be defined as above, we have*

i) *if* $0 < q < +\infty$, *then*

$$\int_0^s r(u^\star(r))^q\,dr \leqslant \int_0^s r(v_q^\star(r))^q\,dr, \qquad s \in [0, L_B];$$

ii) *if* $q = +\infty$, *then*

$$u^\star(s) \geqslant v_\infty^\star(s), \qquad s \in [0, L_B].$$

A consequence of Theorem 4.1 *is the following reverse inequality.*

THEOREM 4.2. – *Let u be an eigenfunction of problem* (4.1). *Then, for* $0 < q < p \leqslant +\infty$, *we have*

$$\left(\int_0^{L_\Omega} r(u^\star(r))^p\,dr \right)^{1/p} \leqslant c(p, q, \sigma_1) \left(\int_0^{L_\Omega} r(u^\star(r))^q\,dr \right)^{1/q}. \tag{4.3}$$

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Dipartimento di Matematica e Applicazioni «R. Caccioppoli»
Università degli Studi di Napoli «Federico II», Complesso Monte S. Angelo
via Cintia, 80126 Napoli, Italy. E-mail: cristina@unina.it

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