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Symmetries and Kähler-Einstein Metrics.

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Sunto. – *Si considerano varietà di Fano M che ammettono un certo numero di rivestimenti di Galois $M \to M_i$, su delle varietà di Fano lisce M_i che ammettono una metrica di Kähler-Einstein. Sotto alcune ipotesi numeriche sui divisori di ramificazione si dimostra che allora anche su M esiste una metrica di Kähler-Einstein.*

Summary. – *We consider Fano manifolds M that admit a collection of finite automorphism groups $G_1, \ldots, G_k$, such that the quotients M/G_i are smooth Fano manifolds possessing a Kähler-Einstein metric. Under some numerical and smoothness assumptions on the ramification divisors, we prove that M admits a Kähler-Einstein metric too.*

1. – Introduction.

The aim of this paper is to provide new examples of Kähler-Einstein metrics of positive scalar curvature. The existence of such a metric on a Fano manifold is a subtle problem, due to the presence of obstructions, that have been discovered during the years, beginning with Matsushima's theorem in 1957, Futaki invariants in 1982, Tian's theorem stating that Kähler-Einstein manifolds of positive scalar curvature are semistable (see [12], Theorem 8.1), up to Donaldson's result [4], Corollary 4, which shows that the existence of Kähler-Einstein metrics (even more generally of a Kähler constant scalar curvature metric) forces the algebraic underlying manifolds to be asympotically stable (see also [2]).

Existence theorems on the other hand are always very hard. The only necessary and sufficient condition, established by Tian, is of a truly analytic character. It says that a Fano manifold M admits a Kähler-Einstein metric, if and only if an integral functional F defined on Kähler metrics in the class $c_1(M)$ is proper (see Theorem 2.1 below). The equivalence of properness of F with the algebraic stability of the underlying manifold, in an appropriate sense, would represent the final solution of the problem, but is still unknown.

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(This has been suggested by Yau, and made precise by Tian, who has also proved that properness implies stability.) Work in progress by Paul and Tian [9] indicates a new stability condition as a candidate for the equivalence with the existence of a Kähler-Einstein metric.

Although by now there is a good deal of examples, the only broad class of manifolds for which the problem is solved is the one of toric Fano manifolds, thanks to a recent theorem of Xujia Wang and Xiaohua Zhu ([14], see also Donaldson's work [5] for related results for extremal metrics). Otherwise, even for manifolds that are deceptively simple from the algebro-geometric point of view, one has often no clue on how to check the properness of F, and finding the metric. The case of Del Pezzo surfaces is quite eloquent from this point of view, as the reader of [11] might verify. Another striking example of the difficulties on which one suddenly runs, is the hypersurface case. Indeed, it is expected that any smooth Fano hypersurface has a Kähler-Einstein metric, nonetheless the only one for which this is known is the Fermat's hypersurface (see [13, p. 85-87]).

The aim of this paper is to use the symmetries of the underlying manifold to prove existence of Kähler-Einstein metrics, inspired by Tian's work on Fermat hypersurfaces. In Section 2 we study the behaviour of properness of F_ω in presence of a Galois covering and find conditions under which the existence of a Kähler-Einstein metric on the base allows one to prove a version of properness, and thus existence, on the covering space. We find algebraic conditions on the covering maps (Theorems 2.3 and 2.5) ensuring that the desired inequalities hold on the covering space. In 3 we show how this can be used to prove the existence of Kähler-Einstein metrics on some classes of Fano manifolds, chosen from the lists of Del Pezzo manifolds, and Fano threefolds with Pic $= \mathbb{Z}$ (see [6], p. 214-215]). Our examples include:

a) hypersurfaces of the form $\{x_0^d + \ldots + x_{k-1}^d + f(x_k, \ldots, x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}$ where f is a homogeneous polynomial of degree d, and $k > n+2-d$;

b) n-dimensional intersections of hypersurfaces of the same form as above, all of the same degree d and with $k > n+2-d$;

c) arbitrary intersections of two (hyper)quadrics;

d) double covers of $\mathbb{P}^n$ ramified along a smooth hypersurface of degree $2d$ with $\dfrac{n+1}{2} < d \leqslant n$;

e) double covers of the n-dimensional quadric $Q_n \subset \mathbb{P}^{n+1}$ with smooth branching locus cut out by a hypersurface of degree $2d$ with $\dfrac{n}{2} < d < n$.

(See section 3.) Example (a) generalises Tian's theorem about Fermat's hypersurfaces. Both (a) and (b) give positive-dimensional *families* of Kähler-

Einstein manifolds, instead of just isolated examples. This becomes even more striking in examples (c), (d) and (e), since every element in the moduli of such manifolds has a Kähler-Einstein metric. (c) had been proved previously, for two special quadrics in $\mathbb{P}^5$, by Alan Nadel (see [8], p. 589]).

This paper is an abridged version of [1]. Full proofs can be found there.

We wish to thank Gang Tian for many helpful conversations and for his interest in this work.

2. – Existence theorems on covering spaces.

Let M be a Fano manifold, G a compact subgroup of $\mathrm{Aut}(M)$, and ω a G-invariant Kähler metric in the class $2\pi c_1(M)$. Put

$$P_G(M, \omega) = \{\varphi \in C^\infty(M) : \omega_\varphi > 0, \text{ and } \varphi \text{ is } G\text{-invariant}\} \tag{1}$$

where $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi$, and by $\omega_\varphi > 0$ we mean that ω_φ is a Kähler metric. If $G = \{1\}$ we simply write $P(M, \omega)$. Recall the definition of the following functionals on the space $P_G(M, \omega)$

$$\begin{gathered} I_\omega(\varphi) = \frac{1}{V}\int_M \varphi(\omega^n - \omega_\varphi^n) \qquad J_\omega(\varphi) = \int_0^1 \frac{I_\omega(s\varphi)}{s}\,ds \\ F_\omega^0(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^n. \end{gathered} \tag{2}$$

Here $V = \langle [M], [\omega]^n \rangle = n!\,\mathrm{vol}(M, \omega)$. Let $f = f(\omega)$ be the unique function on M satisfying $\mathrm{Ric}(\omega) = \omega + i\partial\bar{\partial}f(\omega)$ and $\int_M e^{f(\omega)}\omega^n = V$. Then we define

$$F_\omega(\varphi) = F_\omega^0(\varphi) - \log\left[\frac{1}{V}\int_M e^{f(\omega) - \varphi}\omega^n\right]. \tag{3}$$

We say that F_ω is *proper* on $P_G(M, \omega)$ if there is a proper increasing function $\mu : \mathbb{R} \to \mathbb{R}$, such that the inequality

$$F_\omega(\varphi) \geq \mu(J_\omega(\varphi))$$

holds for any $\phi \in P_G(M, \omega)$. The importance of this notion is mainly due to the following theorem (see [12], Theorem 1.6] and [13] Chapter 7).

THEOREM 2.1 (Tian). – *Let M be a Fano manifold and G a maximal compact subgroup of* $\mathrm{Aut}(M)$. *Then M admits a Kähler-Einstein metric if and only if F_ω is proper on $P_G(M, \omega)$. Moreover, in this case F_ω is bounded from below on all $P(M, \omega)$.*

We consider the following set of «normalised» potentials:

$$Q_G(M,\omega)=\{\phi\in P_G(M,\omega): A_\omega(\varphi)=0\}.$$

For any $\varphi\in P_G(M,\omega)$, $\varphi+A_\omega(\varphi)\in Q_G(M,\omega)$ is the corresponding normalised potential.

The following proposition gives a sufficient condition for the existence of Kähler-Einstein metrics on Fano manifolds.

PROPOSITION 2.1. – *Let M be a Fano manifold, and ω a Kähler metric in the class $2\pi c_1(M)$. If there are constants $C_1, C_2>0$ such that*

$$F_\omega(\varphi)\geqslant C_1\sup_M \varphi - C_2 \tag{4}$$

for any $\phi\in Q_G(M,\omega)$, then M admits a Kähler-Einstein metric.

PROPOSITION 2.2. – *If there are constants $C_1, C_2>0$ and $\beta>0$ such that*

$$F_\omega(\varphi)\geqslant C_1\log\left[\frac{1}{V}\int_M e^{f(\omega)-(1+\beta)\varphi}\omega^n\right]-C_2 \tag{5}$$

for any $\phi\in Q_G(M,\omega)$, then M admits a Kähler-Einstein metric.

Below we will need a slight extension of the integral functionals defined above. Let M be a compact complex manifold and γ a continuous hermitian form on M. A closed positive current T of bidegree (1,1) is called a *Kähler current* if for some constant $c>0$ one has $T\geqslant c\gamma$ in the sense of currents. The definition does not depend on the choice of γ, since M is compact. If M is a Fano manifold, $G\subset \mathrm{Aut}(M)$ is a compact subgroup, and ω is a G-invariant Kähler form in the class $2\pi c_1(M)$, we put

$$P^0_G(M,\omega)=\{\psi\in C^0(M): \omega+\mathrm{i}\partial\bar\partial\psi \text{ is a Kähler current}\}.$$

This means that ψ belongs to $P^0_G(M,\omega)$ if and only if $\omega+\mathrm{i}\partial\bar\partial\psi\geqslant c\omega$ in the sense of currents for some $c>0$.

PROPOSITION 2.3. – *The functionals I_ω, J_ω, F^0_ω and F_ω can be extended to $P^0_G(M,\omega)$. The extensions are continuous with respect to the C^0-topology.*

In the proof of the next Theorem we will need the following density result.

PROPOSITION 2.4. – *Any $\psi\in P^0_G(M,\omega)$ is the C^0-limit of a sequence $\varphi_n\in P_G(M,\omega)$.*

This is a straightforward application of a result due to Richberg ([10]).

THEOREM 2.2. – *Let M and N be Fano manifolds, $\pi : M \to N$ a ramified Galois covering of degree d with structure group G, ω_N a Kähler-Einstein metric on N and $\omega \in 2\pi c_1(M)$ a G-invariant Kähler metric. Denote by $R(\pi)$ be the ramification divisor of π, and assume that numerically (i.e. in homology) $R(\pi) = \beta K_M^{-1}$ for some $\beta \in \mathbb{Q}$. (Since $R(\pi)$ is effective and K_M^{-1} is ample, $\beta > 0$.) Then there is a constant constant $C_1 > 0$ such that for any $\varphi \in P_G(M, \omega_M)$*

$$F_\omega^0(\varphi) \geq \log\left[\frac{1}{V}\int_M e^{-(1+\beta)\varphi}\pi^*\omega_N^n\right] - C_1. \tag{6}$$

PROOF. – The classical Hurwitz formula for the canonical bundle of a ramified covering, $\pi^* K_N = K_M - R(\pi)$, yields that

$$\pi^*[\omega_N] = (1+\beta)[\omega].$$

Denote by G the Galois group of the covering, and choose a G-invariant $u \in C^\infty(M)$ such that $\pi^*\omega_N = (1+\beta)\omega + i\partial\bar{\partial}u$. We claim that any $\varphi \in P_G(M, \omega)$ is of the form $\varphi = (u + \pi^*\psi)/(1+\beta)$ for some $\psi \in P^0(N, \omega_N)$. Indeed $(1+\beta)\varphi - u$ is G-invariant, so $(1+\beta)\varphi - u = \pi^*\psi$ for some continuous function ψ, because $N = M/G$ has the quotient topology.

LEMMA 2.1. – *If $\pi : M \to N$ is a finite holomorphic map of compact complex manifolds, the direct image via π of a Kähler current on M is a Kähler current on N.*

PROOF OF THE LEMMA. – Let $R \subset M$ and $B \subset N$ denote ramification and branching locus of π, and d its degree. Let γ_M and γ_N be continuous hermitian forms on M and N respectively. Since $\pi^*\gamma_N$ is continuous and γ_M is positive definite, there is $c_1 > 0$ such that $\gamma_M \geq c_1\pi^*\gamma_N$. If T is a Kähler current on M, by definition $T \geq c_2\gamma_M$ for some $c_2 > 0$, so that $T \geq c\pi^*\gamma_N$ with $c = c_1 c_2 > 0$. Given a positive form $\eta \in \wedge^{n-1,n-1}(N)$ we have

$$\langle \pi_* T, \eta\rangle = c \cdot d\langle \gamma_N, \eta\rangle$$

so that $T \geq c \cdot d\gamma_N$. This proves the lemma.

Q.D.E.

Since

$$\omega_N + \mathrm{i}\partial\bar{\partial}\psi = \frac{1+\beta}{d}\pi_*(\omega + \mathrm{i}\partial\bar{\partial}\varphi)$$

(as currents) the lemma implies that $\omega_N + \mathrm{i}\partial\bar{\partial}\psi$ is a Kähler current, i.e. that $\psi \in P^0(N, \omega_N)$. We have shown that to any potential $\varphi \in P_G(M, \omega)$ corresponds a *continuous* potential $\psi \in P^0(N, \omega_N)$ such that $\pi^*(\omega_N + \mathrm{i}\partial\bar{\partial}\psi) = (1+\beta)(\omega + \mathrm{i}\partial\bar{\partial}\varphi)$. Since N is Kähler-Einstein by hypothesis, Tian's Theorem 2.1 implies that there is a constant C_3 such that $F_{\omega_N}(\eta) \geqslant -C_3$ for any $\eta \in P(N, \omega_N)$. By Proposition 2.3 the functional F_{ω_N} can be extended continuously to $P^0(N, \omega_N)$, and by Proposition 2.4 $P(N, \omega_N)$ is dense in $P^0(N, \omega_N)$, so we can conclude that

$$F_{\omega_N}(\psi) \geqslant -C_3 \tag{7}$$

for ψ as above. To finish the proof we need to «lift» this inequality from N to M.

$$(1+\beta)^n F^0_{\omega_N}(\psi) = F^0_{\pi^*\omega_N}(\pi^*\psi)\,. \tag{8}$$

Moreover

$$\frac{1}{V_N}\int_N e^{-\psi}\omega_N^n = \frac{1}{(1+\beta)^n}\frac{1}{V}\int_M e^{-(1+\beta)\varphi}e^u(\pi^*\omega_N)^n \tag{9}$$

where

$$V_N = \langle [N], [\omega_N]^n \rangle = \frac{(1+\beta)^n}{d}V\,.$$

The homogeneity of F^0 and the cocycle relation it satisfies (see [13], pp. 60-61]) yield finally

$$F^0_\omega(\varphi) \geqslant C_5 + \log\left[\frac{1}{V}\int_M e^{-(1+\beta)\varphi}(\pi^*\omega_N)^n\right] - C_4.$$

Q.D.E.

The first criterion for the existence of Kähler-Einstein metrics is the following

THEOREM 2.3. – *Let M be an n-dimensional Fano manifold. Assume that ramified coverings $\pi_i\colon M \to M_i$ are given for $i = 1, \dots, k$, satisfying the following assumptions*:

1. *M_i is a Fano manifold and admits a Kähler-Einstein metric*;
2. *the coverings are Galois, i.e. $M_i = M/G_i$ for some finite group G_i*,

3. *the groups* G_i *are contained in some* compact *subgroup* $G \subset \operatorname{Aut}(M)$;

4. *the intersection of the ramification* $E(\pi_i)$ *divisors is empty*;

5. *the divisors* $R(\pi_i)$ *are all proportional to the anticanonical divisor of* M, *i.e. there are some (necessarily positive) rational numbers* β_i *such that numerically (i.e. in homology)* $R(\pi_i) = \beta_i K_M^{-1}$.

Then M *has a Kähler-Einstein metric.*

PROPOSITION 2.5. – *Let* M *be an* n*-dimensional Fano manifold. Assume that ramified coverings* $\pi_i \colon M \to M_i$ *are given for* $i = 1, \dots, k$, *satisfying the following assumptions*:

1. M_i *is a Fano manifold and admits a Kähler-Einstein metric*;

2. *the coverings are Galois, i.e.* $M_i = M/G_i$ *for some finite group* G_i;

3. *the groups* G_i *are contained in some* compact *subgroup* $G \subset \operatorname{Aut}(M)$;

4. *there are (positive) rational numbers* β_i *such that numerically* $R(\pi_i) = \beta_i K_M^{-1}$.

Define $\eta \in C^\infty(M)$ *by* $\frac{1}{k} \sum_{i=1}^{k} \pi_i^* \omega_i^n = \eta \omega^n$, *and put* $c := \sup\{\lambda \geq 0 : \eta^{-\lambda} \in L^1(M, \omega^n)\}$ *and* $\beta := \min_i \beta_i$. *If* $\frac{1}{c} < \beta$, *then* M *admits a Kähler-Einstein metric.*

It is clear that the last proposition is of some use only if c can be computed or at least bounded from below. This number is an instance of an interesting invariant of a singularity studied – among others – by Demailly and Kollár (see [3] and [7]). We present below two cases in which it can be computed very easily.

THEOREM 2.4. – *Let* M *be an* n*-dimensional Fano manifold, and let* $\pi : M \to N$ *be a Galois covering with group* G *onto a Kähler-Einstein manifold* N. *Assume that homologically* $R(\pi) = \beta K_M^{-1}$, *and that*

$$d - 1 < \beta \tag{10}$$

where $d = \#G = \deg(\pi)$. *Then* M *has a Kähler-Einstein metric.*

THEOREM 2.5. – *Let* M *be an* n*-dimensional Fano manifold. Assume that ramified coverings* $\pi_i \colon M \to M_i$ *are given for* $i = 1, \dots, k$, *satisfying the following assumptions*:

1. M_i *is a Fano manifold and admits a Kähler-Einstein metric*;

2. *the coverings are Galois, i.e.* $M_i = M/G_i$;

3. *the groups* G_i *are all contained in some fixed* compact *subgroup* $G \subset \mathrm{Aut}(M)$;

4. *if* V_i *denotes the reduced divisor of* M *associated to the ramification divisor of* π_i, *then the* V_i*'s are smooth hypersurfaces, that intersect transversally in a smooth submanifold* V;

5. *there are (positive) rational numbers* β_i *such that* $R(\pi_i) = \beta_i K_M^{-1}$, *and they satisfy*

$$\frac{1}{d_1 - 1} + \ldots + \frac{1}{d_k - 1} > \frac{1}{\beta} \tag{11}$$

where $\beta := \min \beta_i$ *and* $d_i = \# G_i$.

Then M *has a Kähler-Einstein metric.*

3. – Examples.

Consider the hypersurface

$$M = \{ x_0^d + \ldots + x_{k-1}^d + f(x_k, \ldots, x_{n+1}) = 0 \} \subset \mathbb{P}^{n+1}$$

where f is any homogeneous polynomial of degree d such that M is smooth. Note that this is equivalent to saying that

$$V = M \cap \{ x_0 = \ldots = x_{k-1} = 0 \} \cong \{ f = 0 \} \subset \mathbb{P}^{n+1-k}$$

be smooth.

PROPOSITION 3.1. – *If* $k > n + 2 - d$ *then* M *admits a Kähler-Einstein metric.*

PROPOSITION 3.2. – *Let* $M \subset \mathbb{P}^{n+m}$ *be a complete intersection of* m *hypersurfaces of degree* d, *given by equations of the form*

$$F_j(x_0, \ldots, x_{n+m}) = a_0^j x_0^d + \ldots a_{k-1}^j x_{k-1}^d + f_j(x_k, \ldots, x_{n+m}) = 0, \quad j = 1, \ldots, m.$$

I.e. the equations are diagonal in the first k *coordinates. If* $n + 2 - d < k$, *then* M *admits a Kähler-Einstein metric.*

THEOREM 3.1. – *Any smooth intersection of two quadrics* $M = Q_1 \cap Q_2$ *in* $\mathbb{P}^{n+2}$ *has a Kähler-Einstein metric.*

THEOREM 3.2. – *Let* M *be an* n*-dimensional Fano manifold that admits a double covering* π *over* $\mathbb{P}^n$ *with branching divisor a smooth hypersurface of degree* $2d$, *with* $\dfrac{n+1}{2} < d \leq n$. *Then* M *admits a Kähler-Einstein metric.*

THEOREM 3.3. – *Let M be an n-dimensional Fano manifold that is a double cover of the quadric $Q_n \subset \mathbb{P}^n$ ramified along a smooth divisor cut out by a hypersurface of degree $2d$, with $\frac{n}{2} < d < n$. Then M admits a Kähler-Einstein metric.*

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