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Some Results in Lagrangian Mechanics.

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Sunto. – *Associamo ad un'equazione dinamica ξ tre differenti connessioni e quindi consideriamo il significato dell'annullarsi della loro curvatura. Alcune peculiarità del caso di equazione dinamica autonoma polinomiale nelle velocità $\dot{q}$ vengono evidenziate. Finalmente, usando le cosiddette condizioni di Helmholtz, indaghiamo un particolare esempio.*

Summary. – *We associate to a dynamic equation ξ three different connections and then we consider the meaning of the vanishing of their curvatures. Some peculiarities of the case of autonomous dynamic equation polynomial in the velocities $\dot{q}$ are pointed out. Finally, using the so-called Helmholtz conditions, we investigate a particular example.*

1. – Dynamic equations and associated connections.

In this work M will denote a C^∞ connected and noncompact manifold of dimension $n+1$, where $n \geqslant 1$. M will play the role of space-time manifold, and is assumed to be stratified over $\mathbb{R}$, i.e. M is equipped with a global time

$$t : M \rightarrow \mathbb{R}\,, \tag{1}$$

with $dt \neq 0$ everywhere on M; a standard reference is [8]. Thus M is a fibred manifold over $\mathbb{R}$, but not necessarily a bundle. The slices of constant t are closed and regular hypersurfaces of M, called *space slices.*

The fibred structure of M leads naturally to the use of the jet prolongations of M as spaces involved in the Lagrangian formalism; for more details

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see [1, 2, 3]. At the second order the framework is

$$\begin{array}{cl} J^2M & (t, q^i, \dot{q}^i, \ddot{q}^i) \\ \downarrow & \\ J^1M & (t, q^i, \dot{q}^i) \\ \downarrow & \\ M & (t, q^i) \\ \downarrow\ t & \\ \mathbb{R} & (t)\,. \end{array} \tag{2}$$

DEFINITION 1. – *A* dynamic equation *is a section of the bundle* $J^2M \to J^1M$, *i.e.*

$$\xi : J^1M \to J^2M, \qquad \ddot{q}^h \circ \xi = \xi^h, \tag{3}$$

where ξ^h *is a local function on* J^1M.

A solution *of the dynamic equation* ξ *is a section* $c : I \subseteq \mathbb{R} \to M$ *of* (1) *such that*

$$\xi \circ \dot{c} = \ddot{c}\,. \tag{4}$$

Of course, ξ describes a mechanical system whose motions satisfy (4).

The following proposition allows one to associate a certain connection Γ to a dynamic equation ξ and establishes the link between geodesics of Γ and solutions of ξ; for the proof see [7].

PROPOSITION 1. – *Let* ξ *be a dynamic equation. Then*

$$\Gamma^h_i = \frac{1}{2}\,\frac{\partial \xi^h}{\partial \dot{q}^i}\,, \qquad \Gamma^h_0 = \xi^h - \Gamma^h_i\,\dot{q}^i \tag{5}$$

define a connection Γ *on the bundle* $J^1M \to M$.

Conversely, a connection Γ *on* $J^1M \to M$ *is associated to a dynamical equation* ξ *according to* (4) *if and only if the following identity holds*

$$\Gamma^h_i = \frac{\partial \Gamma^h_0}{\partial \dot{q}^i} + \frac{\partial \Gamma^h_j}{\partial \dot{q}^i}\,\dot{q}^j. \tag{6}$$

In this case we obtain ξ *by setting*

$$\xi^k = \Gamma^k_0 + \Gamma^k_i\,\dot{q}^i\,. \tag{7}$$

Geodesics of Γ *are exactly solutions of* ξ.

REMARK 1. – It is important to note that (6) has an intrinsic meaning; in essence it is related to the vanishing of a kind of torsion. Indeed, from (6) it follows the symmetry property

$$\frac{\partial \Gamma^h_j}{\partial \dot q^i} = \frac{\partial \Gamma^h_i}{\partial \dot q^j}. \tag{8}$$

A connection Γ satisfying condition (6) is called *symmetric*. Thus, symmetric connections are in bijection with dynamical equations according to (5) and (7).

Given a dynamic equation ξ, in the following four steps we are going to define a *linear* connection ∇ on the pull-back vector bundle

$$J^1M \underset{M}{\times} TM \to J^1M. \tag{9}$$

This connection was first introduced in [6] in a rather different way; as we shall see, the vanishing its curvature characterizes a certain property of ξ.

Step 1. Consider the vertical endomorphism v on TJ^1M

$$\begin{array}{ccc} VJ^1M & \xrightarrow{v|_{VJ^1M}} & VJ^1M \\ \downarrow & & \downarrow \\ TJ^1M & \xrightarrow{v} & TJ^1M \end{array}$$

$$v\left(\frac{\partial}{\partial t}\right) = -\dot q^i \frac{\partial}{\partial \dot q^i}, \qquad v\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot q^i}, \qquad v\left(\frac{\partial}{\partial \dot q^i}\right) = 0. \tag{10}$$

Notice that $v^2 = 0$. Let ξ be a dynamic equation and let Γ be its associated connection (5) on $J^1M \to M$. It is possible to introduce the vertical connection $V\Gamma$ of Γ: given the composite bundle

$$V_M J^1M \to J^1M \to M, \tag{11}$$

$V\Gamma$ is the connection on (11) whose coordinate expression is

$$V\Gamma = \left(\Gamma^i_\lambda,\ \frac{\partial \Gamma^i_\lambda}{\partial \dot q^j}\ddot q^j\right). \tag{12}$$

Note that $V\Gamma$ projects over Γ.

Step 2. Since $J^1M \to M$ is affine, from $V\Gamma$ we get the following linear connection $\bar{\Gamma}$ on $V_M J^1 M \to J^1 M$

$$\bar{\nabla}_{\frac{\partial}{\partial q^\lambda}} \frac{\partial}{\partial \dot{q}^j} = -\bar{\Gamma}^h_{\lambda j} \frac{\partial}{\partial \dot{q}^h}, \qquad \bar{\nabla}_{\frac{\partial}{\partial \dot{q}^i}} \frac{\partial}{\partial \dot{q}^j} = 0 . \tag{13}$$

Thus the corresponding parameters are

$$\bar{\Gamma}^h_{\lambda j} = \frac{\partial \Gamma^h_\lambda}{\partial \dot{q}^j}, \qquad \dot{\bar{\Gamma}}^h_{ij} = 0 , \tag{14}$$

where the dot means that the first subscript i refers to vertical fields.

$V\Gamma$ turns out to be the so-called *composite connection* of Γ on the affine bundle $J^1M \to M$ with $\bar{\Gamma}$ on the vector bundle $V_M J^1 M \to J^1 M$.

Step 3. Now consider the splitting associated to Γ of TJ^1M

$$TJ^1M = HJ^1M \bigoplus_{J^1M} V_M J^1 M$$

$$\frac{\partial}{\partial q^\lambda} = h_\lambda - \Gamma^i_\lambda \frac{\partial}{\partial \dot{q}^i}, \qquad \frac{\partial}{\partial \dot{q}^i} = \frac{\partial}{\partial \dot{q}^i}; \tag{15}$$

$$\xi = h_0 + \dot{q}^i h_i .$$

Using ξ and the vertical endomorphism v (10), we get the further splitting

$$HJ^1M = \mathbb{R} \bigoplus_{J^1M} V_M J^1 M ,$$

$$(\xi, h_i) \leftrightarrow \left(\xi, \frac{\partial}{\partial \dot{q}^i}\right). \tag{16}$$

Taking the trivial connection on the line bundle $J^1M \times \mathbb{R} \to J^1M$, the linear connection $\bar{\nabla}$ (13) induces on the horizontal subbundle

$$HJ^1M \to J^1M \tag{17}$$

the following linear connection $\tilde{\nabla}$

$$\tilde{\nabla}_{\frac{\partial}{\partial q^\lambda}} h_i = -\bar{\Gamma}^k_{\lambda i} h_k, \qquad \tilde{\nabla}_{\frac{\partial}{\partial q^\lambda}} \xi = \tilde{\nabla}_{\frac{\partial}{\partial \dot{q}^i}} h_j = \tilde{\nabla}_{\frac{\partial}{\partial \dot{q}^i}} \xi = 0 . \tag{18}$$

In essence, $\tilde{\nabla}$ is a natural extension of $\overline{\nabla}$ (13) to $HJ^1M \to J^1M$. However, since

$$\overline{\Gamma}^k_{\lambda i} = \frac{\partial \Gamma^k_\lambda}{\partial \dot{q}^i}$$

we have lost information with respect to the original connection Γ. We can recover this information by adding to $\tilde{\nabla}$ the canonical soldering form associated to the vertical projection of the splitting (15), i.e.

$$(d\dot{q}^i - \Gamma^i_\lambda \, dq^\lambda) \otimes dt \otimes h_i . \tag{19}$$

The resulting linear connection ∇ on $HJ^1M \to J^1M$ reads

$$\nabla_{\frac{\partial}{\partial q^\lambda}} h_i = -\overline{\Gamma}^k_{\lambda i} h_k, \qquad \nabla_{\frac{\partial}{\partial \dot{q}^i}} h_i = 0\,, \qquad \nabla_{\frac{\partial}{\partial q^\lambda}} \xi = -\Gamma^i_\lambda h_i, \qquad \nabla_{\frac{\partial}{\partial \dot{q}^i}} \xi = h_i . \tag{20}$$

Step 4. Finally, consider the isomorphism

$$HJ^1M \leftrightarrow J^1M \underset{M}{\times} TM$$

$$h_\lambda \leftrightarrow \frac{\partial}{\partial q^\lambda} \tag{21}$$

and note that we have

$$\xi \leftrightarrow \lambda = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} . \tag{22}$$

Thus the connection ∇ (using the same symbol) on (9) induced by this isomorphism reads

$$\nabla_{\frac{\partial}{\partial q^\lambda}} \frac{\partial}{\partial q^i} = -\overline{\Gamma}^h_{\lambda i} \frac{\partial}{\partial q^h}\,, \qquad \nabla_{\frac{\partial}{\partial q^\lambda}} \frac{\partial}{\partial t} = -(\Gamma^h_\lambda - \dot{q}^j \overline{\Gamma}^h_{\lambda j}) \frac{\partial}{\partial q^h}\,, \qquad \nabla_{\frac{\partial}{\partial \dot{q}^i}} \frac{\partial}{\partial q^\lambda} = 0\,. \tag{23}$$

REMARK 2. – From the identity $\overline{\Gamma}^h_{0i} = \Gamma^h_i - \dot{q}^j \overline{\Gamma}^h_{ij}$ we see that the following symmetry property holds:

$$\nabla_{\frac{\partial}{\partial \dot{q}^\lambda}} \frac{\partial}{\partial q^\mu} = \nabla_{\frac{\partial}{\partial \dot{q}^\mu}} \frac{\partial}{\partial q^\lambda} . \tag{24}$$

REMARK 3. – Note that ∇ restricts to a linear connection $\overline{\nabla}$ on the subbundle

$$J^1M \underset{M}{\times} VM \to J^1M \tag{25}$$

of $J^1M \underset{M}{\times} TM \to J^1M$ given by

$$\overline{\nabla}_{\frac{\partial}{\partial q^\lambda}} \frac{\partial}{\partial q^j} = -\overline{\Gamma}^h_{\lambda j} \frac{\partial}{\partial q^h}, \qquad \overline{\nabla}_{\frac{\partial}{\partial \dot{q}^i}} \frac{\partial}{\partial q^j} = 0 ; \tag{26}$$

recalling that $V_M J^1M = J^1M \underset{M}{\times} VM$, we see that the connections (13) and (26) coincide (indeed, we have used the same symbol).

Let us summarize the main objects we have obtained up to now, starting with the original connection Γ and its associated vertical connection $V\Gamma$:

- Γ defined in (5) on the affine bundle $J^1M \to M$,
- ∇ defined in (23) on the pull-back vector bundle $J^1M \underset{M}{\times} TM \to J^1M$ and
- its restriction $\overline{\nabla}$ defined on (26) on the subbundle $J^1M \underset{M}{\times} VM \to J^1M$.

REMARK 4. – In particular, consider the case in which ξ is quadratic in $\dot{q}^i$, for which

$$\Gamma^h_i = K^h_{ij}\dot{q}^j + K^h_{i0}, \qquad \Gamma^h_0 = K^h_{0i}\dot{q}^i + K^h_{00}, \qquad K^h_{0i} = K^h_{i0}. \tag{27}$$

Then Γ is the restriction of a linear connection K on M compatible with t, i.e.

$$K^0_{\lambda\mu} = 0 . \tag{28}$$

It is clear from (23) that in this case ∇ is merely the pull-back of K and $\overline{\nabla}$ is the pull-back of the restriction of K to $VM \to M$.

2. – Vanishing of curvature.

The curvature R of the connection Γ (5) is the map

$$R = \frac{1}{2} R^h_{\lambda\mu}\, dx^\lambda \wedge dx^\mu \otimes \frac{\partial}{\partial q^h} : J^1M \to \overset{2}{\wedge} T^*M \otimes VM ,$$

where

$$R^h_{\lambda\mu} = \frac{\partial \Gamma^h_\mu}{\partial q^\lambda} - \frac{\partial \Gamma^h_\lambda}{\partial q^\mu} + \Gamma^k_\lambda \frac{\partial \Gamma^h_\mu}{\partial \dot{q}^k} - \Gamma^k_\mu \frac{\partial \Gamma^h_\lambda}{\partial \dot{q}^k} . \tag{29}$$

In particular we have

$$R_{ij}^{h} = \frac{\partial \Gamma_j^h}{\partial q^i} - \frac{\partial \Gamma_i^h}{\partial q^j} + \Gamma_i^k \frac{\partial \Gamma_j^h}{\partial \dot{q}^k} - \Gamma_j^k \frac{\partial \Gamma_i^h}{\partial \dot{q}^k} .$$

$$R_{i0}^{h} = \frac{\partial \Gamma_0^h}{\partial q^i} - \frac{\partial \Gamma_i^h}{\partial q^0} + \Gamma_i^k \frac{\partial \Gamma_0^h}{\partial \dot{q}^k} - \Gamma_0^k \frac{\partial \Gamma_i^h}{\partial \dot{q}^k} .$$

The curvature ϱ of the linear connection ∇ (23) is the map

$$\varrho : J^1 M \to \overset{2}{\wedge} T^* J^1 M \otimes T^* M \otimes VM ,$$

with

$$\varrho^{\,h}_{\lambda\mu\alpha} = \frac{\partial \gamma^{\,h}_{\mu\alpha}}{\partial q^\lambda} - \frac{\partial \gamma^{\,h}_{\lambda\alpha}}{\partial q^\mu} + \gamma^{\,k}_{\lambda\alpha}\gamma^{\,h}_{\mu k} - \gamma^{\,k}_{\mu\alpha}\gamma^{\,h}_{\lambda k},$$

$$\dot{\varrho}^{h}_{\lambda i\alpha} = -\frac{\partial \gamma^{\,h}_{\lambda\alpha}}{\partial \dot{q}^i} ,$$

$$\ddot{\varrho}^{h}_{ij\alpha} = 0$$

and

$$\gamma^{\,h}_{\lambda i} = \frac{\partial \Gamma^h_\lambda}{\partial \dot{q}^i} , \qquad \gamma^{\,h}_{\lambda 0} = \Gamma^h_\lambda - \frac{\partial \Gamma^h_\lambda}{\partial \dot{q}^j}\, \dot{q}^j .$$

In particular, we have

$$\varrho^{\,h}_{\lambda\mu 0} = \frac{\partial \gamma^{\,h}_{\mu 0}}{\partial q^\lambda} - \frac{\partial \gamma^{\,h}_{\lambda 0}}{\partial q^\mu} + \gamma^{\,k}_{\lambda 0}\gamma^{\,h}_{\mu k} - \gamma^{\,k}_{\mu 0}\gamma^{\,h}_{\lambda k},$$

$$\begin{aligned}\varrho^{\,h}_{\lambda\mu i} &= \frac{\partial \gamma^{\,h}_{\mu i}}{\partial q^\lambda} - \frac{\partial \gamma^{\,h}_{\lambda i}}{\partial q^\mu} + \gamma^{\,k}_{\lambda i}\gamma^{\,h}_{\mu k} - \gamma^{\,k}_{\mu i}\gamma^{\,h}_{\lambda k} = \\ &= \frac{\partial R^h_{\lambda\mu}}{\partial \dot{q}^i} + \Gamma^k_\mu \frac{\partial^2 \Gamma^h_\lambda}{\partial \dot{q}^i\,\partial \dot{q}^k} - \Gamma^k_\lambda \frac{\partial^2 \Gamma^h_\mu}{\partial \dot{q}^i\,\partial \dot{q}^k} .\end{aligned}$$

LEMMA 1. – *We have the following relations between the two curvatures R and ϱ:*

$$R^h_{\lambda\mu} = \varrho^{\,h}_{\lambda\mu 0} + \varrho^{\,h}_{\lambda\mu l}\dot{q}^l, \tag{30}$$

$$\dot{\varrho}^h_{\lambda i 0} + \dot{\varrho}^h_{\lambda i l}\dot{q}^l = 0 . \tag{31}$$

REMARK 5. – Suppose that $\overline{\varrho} = 0$. Then

$$\dot{\overline{\varrho}}^h_{\lambda ij} = -\frac{\partial^2 \Gamma^h_\lambda}{\partial \dot{q}^i \partial \dot{q}^j} = 0\,,$$

i.e. Γ is affine in $\dot{q}^i$ and (27) holds. Thus

$$\gamma^h_{\lambda\mu} = K^h_{\lambda\mu}$$

are local functions on M. Note that

$$\dot{\varrho}^h_{\lambda i0} = 0\,, \tag{32}$$

which follows from the identity (31). Moreover, from (30) we get

$$R^h_{\lambda\mu} = \varrho^h_{\lambda\mu 0}. \tag{33}$$

In conclusion:

- if $\overline{\varrho} = 0$ the connection Γ is affine and we have

$$R = \varrho = \frac{1}{2}\varrho^h_{\lambda\mu 0}\, dx^\lambda \wedge dx^\mu \otimes \frac{\partial}{\partial q^h} : M \to \overset{2}{\wedge} T^* M \otimes VM\,;$$

- of course, $\varrho = 0$ implies $\overline{\varrho} = 0$ and hence $R = 0$ because Γ is affine. Note that this fact also follows directly from (30);

- the converse is not true: $R = 0$ does not implies $\varrho = 0$, of course when Γ is not affine; see Proposition 5.

The following propositions and remarks clarify the meaning of the curvatures $\overline{\varrho}$ and ϱ.

PROPOSITION 2. – *Let Γ be a connection on $J^1 M \to M$ with $\overline{\varrho} = 0$. Then Γ is affine and there exists an atlas of charts (t, q^i) on M such that*

$$K^h_{\lambda i} = K^h_{i\lambda} = 0\,, \tag{34}$$

where K is the connection introduced in (27), (28).

REMARK 6. – In charts (t, q^i) in which (34) holds, we can take t as the standard coordinate on $\mathbb{R}$. In these charts, only the following components of the curvature ϱ do not vanish:

$$\varrho^h_{i00} = -\varrho^h_{0i0} = \frac{\partial K^h_{00}}{\partial q^i}\,. \tag{35}$$

Note that the dynamic equation ξ reads simply

$$\xi^h = K^h_{00}, \tag{36}$$

i.e. *ξ does not depend on $\dot{q}^i$.*

Conversely, if there exists an atlas of charts (t, q^i) in which the dynamic equation ξ reads as in (36), then $K^h_{\lambda i} = 0$ and $\overline{\varrho} = 0$; note that such a ξ is necessarily quadratic in $\dot{q}^i$.

PROPOSITION 3. – *Let Γ be a connection on $J^1 M \to M$ with $\varrho = 0$. Then Γ is affine and there exists an atlas of charts (t, q^i) on M such that*

$$K^h_{\lambda\mu} = K^h_{\mu\lambda} = 0\,, \tag{37}$$

where K is the connection introduced in (27), (28).

REMARK 7. – In these new charts the dynamic equation ξ reads as a *free motion equation*

$$\xi^h = 0\,. \tag{38}$$

Conversely, if there exists an atlas of charts (t, q^i) in which the dynamic equation ξ reads as in (38), then $K^h_{\lambda\mu} = 0$ and $\varrho = 0$; note that such a ξ is necessarily quadratic in $\dot{q}^i$.

Consider $M = \mathbb{R} \times \mathbb{R}$ coordinated by (t, q). Consider also the following *autonomous* dynamic equation written formally

$$\xi(q, \dot{q}) = \sum_{n \geqslant 0} a_n(q)\, \dot{q}^n, \qquad q, \dot{q} \in \mathbb{R}\,. \tag{39}$$

Note that in this case we only have

$$\Gamma = \frac{1}{2}\frac{\partial \xi}{\partial \dot{q}}, \qquad \Gamma_0 = \xi - \Gamma \dot{q}\,. \tag{40}$$

From (29) we see that the curvature R of Γ reduces to

$$R = \frac{\partial \Gamma_0}{\partial q} + \Gamma \frac{\partial \Gamma_0}{\partial \dot{q}} - \Gamma_0 \frac{\partial \Gamma}{\partial \dot{q}}\,. \tag{41}$$

The following proposition partially clarify the meaning of the curvature R.

PROPOSITION 4. – *Suppose that ξ as in* (39) *is polynomial in $\dot{q}$, i.e. there is $N \geqslant 0$ such that*

$$a_n = 0 \quad \text{if } n \geqslant N+1$$

and that its curvature R vanishes identically.

Then ξ is necessarily quadratic in $\dot{q}$, i.e. $N = 2$,

$$\xi(q, \dot{q}) = a_0(q) + a_1(q)\, \dot{q} + a_2(q)\, \dot{q}^2,$$

with

$$a_0(q) = \exp\left[\int_{q_0}^{q} a_2(u)\, du\right] \cdot \left\{ a_0(q_0) - \frac{1}{4} a_1^2 \int_{q_0}^{q} \exp\left[- \int_{q_0}^{u} a_2(v)\, dv \right] du \right\},$$

$a_1(q) = a_1$ *arbitrary constant,*

$a_2(q) =$ *arbitrary.*

REMARK 8. – Consider the following quadratic dynamic equation:

$$\xi(q, \dot{q}) = -\dot{q}^2 \tag{42}$$

For this ξ we have $R = 0$ and $\varrho = 0$. In the following proposition, by adding to (42) a perturbative term, we produce a (nonpolynomial) example of ξ with $R = 0$ but $\varrho \neq 0$.

PROPOSITION 5. – *The dynamic equation*

$$\begin{aligned} \xi(q, \dot{q}) &= -\dot{q}^2 + 6 \cdot \sum_{n \geqslant 3} (-1)^{n+1} \frac{(2n-5)!!}{n!} \varepsilon^{n-2} e^{2(n-2)q} \dot{q}^n = \\ &= 2\dot{q}^2 + \frac{6}{\varepsilon e^{2q}} \dot{q} + \frac{2}{\varepsilon^2 e^{4q}} - \frac{2}{\varepsilon^2 e^{4q}} (1 + 2\varepsilon e^{2q} \dot{q})^{\frac{3}{2}} \end{aligned} \tag{43}$$

verifies $R = 0$ but, obviously, $\varrho \neq 0$ since it is not quadratic in $\dot{q}$ because of the last term in (43).

3. – The role of the metric and Helmholtz conditions.

Let ξ be a dynamic equation and let Γ be its associated connection on $J^1 M \to M$ as given in (5). For our purposes it is convenient to introduce the nonholonomic basis of $TJ^1 M$

$$\xi, \qquad h_i, \qquad \frac{\partial}{\partial \dot{q}^i} \tag{44}$$

and the dual basis of T^*J^1M

$$dt\,, \qquad \theta^i, \qquad \Gamma^i \tag{45}$$

where

$$h_\lambda = \frac{\partial}{\partial q^\lambda} + \Gamma^i_\lambda \frac{\partial}{\partial \dot{q}^i}\,, \qquad \xi = h_0 + \dot{q}^i h_i,$$

$$\theta^i = dq^i - \dot{q}^i\, dt\,, \qquad \Gamma^i = d\dot{q}^i - \Gamma^i_\lambda\, dq^\lambda.$$

Using (45) the curvature R (29) of Γ reads

$$R = R^h \otimes \frac{\partial}{\partial \dot{q}^h}$$

where

$$R^h = R^h_i\, \theta^i \wedge dt + \frac{1}{2} R^h_{ij}\, \theta^i \wedge \theta^j, \qquad R^h_i = R^h_{i0} + \dot{q}^j R^h_{ij}.$$

The two following projections over J^1M will play a role

$$\theta, \;\; \Gamma : TJ^1M \to V_M J^1M \tag{46}$$

$$\theta = \theta^i \otimes \frac{\partial}{\partial \dot{q}^i}\,, \qquad \Gamma = \Gamma^i \otimes \frac{\partial}{\partial \dot{q}^i}\,.$$

REMARK 9. – Let g be a metric on the vertical bundle $V_M J^1M \to J^1M$, that is

$$g = g_{ij}\, d\dot{q}^i \otimes d\dot{q}^j, \qquad g_{ij} = g_{ji},$$

where g_{ij} are local functions over J^1M. Recall that $V_M J^1M = J^1M \underset{M}{\times} VM$. Then we define the following 2-form ω over J^1M

$$\omega(u, v) = g(\Gamma u, \theta v) - g(\Gamma v, \theta u)\,, \tag{47}$$

where u and v are vector fields over J^1M. Its local expression looks particularly simple using the nonholonomic basis (45):

$$\omega = g_{ij}\, \Gamma^i \wedge \theta^j.$$

It is easily seen that ω is of maximal rank $2n$. The main facts about this ω are collected in the two following propositions; similar results appear in [4, 5, 9].

PROPOSITION 6. – *It can be shown that the closure condition $d\omega = 0$ is equivalent to the so-called Helmholtz conditions*

$$\frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^j}, \tag{48}$$

$$\xi \cdot g_{ij} + g_{ih} \Gamma^h_j + g_{jh} \Gamma^h_i = 0, \tag{49}$$

$$g_{ih} R^h_j = g_{jh} R^h_i. \tag{50}$$

PROPOSITION 7. – *The two following facts are equivalent*

(i) $d\omega = 0$;

(ii) *there exists a sheaf of local Lagrangians $\mathcal{L}$ for the dynamic equation ξ.*

REMARK 10. – The dynamic equation $\xi(q, \dot{q}) = -\dot{q}^2$ (42) verifies Helmholtz conditions with metric $g(q, \dot{q}) = e^{2q}$; a possible Lagrangian is $L(q, \dot{q}) = \frac{1}{2} e^{2q} \dot{q}^2$. We want to investigate whether the dynamic equation given in (43) verifies Helmholtz conditions and, if so, we want to find a Lagrangian for it.

Note that the only condition to be checked in this case is (49), which reduces to

$$\dot{q} \frac{\partial g}{\partial q} + \xi \frac{\partial g}{\partial \dot{q}} + g \frac{\partial \xi}{\partial \dot{q}} = 0. \tag{51}$$

We look for a solution g of the form

$$g(q, \dot{q}) = \sum_{n \geq 0} b_n(q)\, \dot{q}^n,$$

where the functions b_n have the form

$$b_n(q) = \beta_n \varepsilon^n e^{2(n+1)q}, \qquad n \geq 0.$$

PROPOSITION 8. – *The metric*

$$\begin{aligned} g(q, \dot{q}) &= e^{2q} \sum_{n \geq 0} (-1)^n \frac{(2n+1)!}{2^n (n!)^2} (\varepsilon e^{2q} \dot{q})^n = \\ &= e^{2q} (1 + 2\varepsilon e^{2q} \dot{q})^{-\frac{3}{2}} \end{aligned} \tag{52}$$

verifies Helmholtz condition (51). *A possible Lagrangian is*

$$L(q, \dot{q}) = \frac{1}{2} e^{2q} \dot{q}^2 - \sum_{n \geqslant 3} \binom{\frac{1}{2}}{n} \varepsilon^{n-2} e^{2q(n-1)} (2\,\dot{q})^n = \tag{53}$$

$$= \frac{e^{-2q}}{\varepsilon^2} \left[1 - (1 + 2\varepsilon e^{2q} \dot{q})^{\frac{1}{2}}\right] + \frac{\dot{q}}{e}\,.$$

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