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Recent Developments in Wavelet Methods for the Solution of PDE's.

SILVIA BERTOLUZZA (*)

Sunto. – *Dopo aver ricordato alcune delle proprietà delle basi di wavelets, ed in particolare la proprietà di caratterizzazione di spazi funzionali tramite coefficienti wavelet, descriviamo due nuovi approcci rispettivamente alla stabilizzazione di problemi numericamente instabili ed alla soluzione nonlineare (adattativa) di equazioni differenziali alle derivate parziali, che sono resi possibili da dette proprietà.*

Summary. – *After reviewing some of the properties of wavelet bases, and in particular the property of characterisation of function spaces via wavelet coefficients, we describe two new approaches to, respectively, stabilisation of numerically unstable PDE's and to non linear (adaptive) solution of PDE's, which are made possible by these properties.*

1. – Introduction.

Wavelet bases were introduced in the late eighties as a tool for signal and image processing. Among the applications considered at the beginning we recall applications in the analysis of seismic signals, the numerous applications in image processing – image compression, edge-detection, denoising, applications in statistics, as well as in physics. Their effectiveness in many of the mentioned fields is nowadays well established: wavelets are actually used by the US *Federal Bureau of Investigation* (or FBI) in their fingerprint database, and they are one of the ingredient of the new MPEG media compression standard. Quite soon it became clear that such bases allowed to represent with a low number of degrees of freedom objects (signals, images, turbulent fields) with singularities of complex structure, a property that is particularly promi-

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sing when thinking of an application to the numerical solution of partial differential equations: many PDEs have solutions which present singularities, and the ability to represent such solution with as little as possible degrees of freedom is essential in order to be able to implement effective solvers for such problems. The first attempts to use such bases in this framework goes back to the late eighties and early nineties, when the first simple adaptive wavelet methods [26] appeared. In those years the problems to be faced were basic ones. The computation of integrals of products of derivative of wavelets – objects which are naturally encountered in the variational approach to the numerical solution of PDEs – was an open problem (solved later by Dahmen and Micchelli in [21]). Moreover, wavelets were defined on \mathbb{R} and on \mathbb{R}^n . Already solving a simple boundary value problem on $(0, 1)$ (the first construction of wavelets on the interval [16] was published in 1993) posed a challenge.

Many step forward have been made since those pioneering works. In particular *thinking in terms of wavelets* gave birth to some new approaches in the numerical solution of PDEs. In this paper we want to show some of these new ideas. In particular we want to show how one key property of wavelets (the possibility of writing equivalent norms for the scale of Besov spaces) allows to write down some new methods.

2. – Hierarchical bases and wavelets.

Let us start by explaining what we mean by wavelets. There are in the literature many definitions of wavelets and wavelet bases, going from the more strict ones (a wavelet is the dilated and translated version of a *mother wavelet* satisfying a suitable set of properties) to more and more general definitions. In this paper we will call *wavelet basis* any basis for $L^2(\Omega)$ satisfying certain properties. The aim of this section is to review what are these properties, or, in other words, to explain what is for us a *wavelet basis*.

Almost all constructions of wavelet bases for $L^2(\Omega)$ ($\Omega \subseteq \mathbb{R}^n$ denoting an open domain) start with a nested sequence $\{V_j\}_{j \geq 0}$ of closed subspaces of $L^2(\Omega)$, whose union is assumed to be dense in $L^2(\Omega)$ ($L^2(\Omega) = \overline{\bigcup_j V_j}$). If we consider a sequence of bounded projectors $P_j: L^2(\Omega) \rightarrow V_j$ – note that $V_j = P_j(L^2(\Omega))$ and that $V_j \subset V_{j+1}$ implies $P_{j+1}P_j = P_j$ – we can introduce the difference spaces

$$W_j = (P_{j+1} - P_j)V_{j+1} = (P_{j+1} - P_j)L^2(\Omega)$$

and we can decompose

$$L^2(\Omega) = V_0 \oplus_j W_j.$$

Now, a hierarchical basis on Ω is constructed by taking a basis $\{\phi_\lambda, \lambda \in K\}$ for V_0 and bases $\{\psi_\lambda, \lambda \in A_j\}$ for each of the W_j and assembling them in order to compose a basis for the whole $L^2(\Omega)$:

$$L^2(\Omega) = \text{span} \langle \phi_\lambda, \lambda \in K, \psi_\lambda, \lambda \in \cup A_j \rangle.$$

The functions ψ_λ for $\lambda \in A_j, j \geq 0$ are the *wavelets*.

In order to simplify the notation in the following it will be useful to set

$$A_{-1} = K, \quad \text{and for } \lambda \in A_{-1}, \quad \psi_\lambda = \phi_\lambda.$$

This will allow us to write

$$L^2(\Omega) = \text{span} \langle \psi_\lambda, \lambda \in A = \bigcup_{j \geq -1} A_j \rangle$$

and

$$f = \sum_{\lambda \in K} f_\lambda \phi_\lambda + \sum_{j \geq 0} \sum_{\lambda \in A_j} f_\lambda \psi_\lambda = \sum_{j \geq -1} \sum_{\lambda \in A_j} f_\lambda \psi_\lambda = \sum_{\lambda \in A} f_\lambda \psi_\lambda.$$

Remark however that, for $\lambda \in A_{-1}$, the functions $\psi_\lambda = \phi_\lambda$ have a different behaviour from the actual wavelets, *i.e.* the functions $\psi_\lambda, \lambda \in A_j$ for $j \geq 0$.

When talking of a wavelet basis, we assume that a number of properties hold: first we assume that we are dealing with a Riesz's basis, that is the $L^2(\Omega)$ norm of a function is equivalent to the ℓ^2 norm of its coefficients. In other word, setting

$$(1) \quad \vec{c} = \{c_\lambda\}_{\lambda \in A} \quad \text{and} \quad \|\vec{c}\|_{\ell^2(A)} := \sqrt{\sum_{\lambda \in A} c_\lambda^2}$$

we have the norm equivalence

$$(2) \quad \left\| \sum_{\lambda \in A} c_\lambda \psi_\lambda \right\|_{L^2(\Omega)} \approx \|\vec{c}\|_{\ell^2(A)},$$

(throughout this paper we will employ the notation $A \leq B$ (resp. $A \geq B$) to say that the quantity A is bounded from above (resp. from below) by cB , with a constant $c > 0$; the expression $A \approx B$ will stand for $A \leq B \leq A$).

Since almost all applications in the field of the numerical solution of PDEs deal with compactly supported wavelets, we will assume that the wavelet ψ_λ are compactly supported.

A key role is played by the so called *dual basis*. The application $f \rightarrow f_\lambda$ that maps each $L^2(\Omega)$ function $f = \sum_{\lambda} f_\lambda \psi_\lambda$ onto its λ -th coefficient f_λ is linear and bounded. Then, by the Riesz representation theorem, there exists an $L^2(\Omega)$ function $\tilde{\psi}_\lambda$ such that for all $f \in L^2(\Omega)$ we have $f_\lambda = \langle f, \tilde{\psi}_\lambda \rangle$, where $\langle \cdot, \cdot \rangle$ denotes

the $L^2(\Omega)$ scalar product. In other words, we have the representation

$$(3) \quad f = \sum_{\lambda \in \mathcal{A}} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda.$$

The set $\{\tilde{\psi}_\lambda, \lambda \in \mathcal{A}\}$ forms a second Riesz's basis for $L^2(\Omega)$ which is *biorthogonal* to original one, that is for $\lambda, \lambda' \in \mathcal{A}$ we have

$$\langle \psi_\lambda, \tilde{\psi}_{\lambda'} \rangle = \delta_{\lambda, \lambda'}, \quad (\text{where } \delta_{\lambda, \lambda'} \text{ is the Kronecker's } \delta).$$

We observe that we can express the projectors P_j in terms of the dual basis. More precisely, for all $f \in L^2(\Omega)$ we can write

$$P_j f = \sum_{m=-1}^{j-1} \sum_{\lambda \in \mathcal{A}_j} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda.$$

2.1. *The classical wavelet for $L^2(\mathbb{R})$: The frequency domain point of view vs. the space domain point of view.*

When handling or dealing with wavelets it is usually useful to think in terms of *frequency localisation* though when Ω is a domain different from \mathbb{R}^n the concept of frequency content of a function (strictly related to its Fourier transform) is not well defined. In order to explain what *thinking in terms of frequency* means, let us first give a brief look at the case $\Omega = \mathbb{R}$.

In the classical constructions of wavelet bases for $L^2(\mathbb{R})$ [27], all basis functions $\psi_\lambda, \lambda \in \mathcal{A}_j$ with $j \geq 0$, as well as the duals $\tilde{\psi}_\lambda$, are constructed by translation and dilation of a single *mother wavelet* ψ (resp. $\tilde{\psi}$). Here the indexes $\lambda \in \mathcal{A}_j$ are of the form $\lambda = (j, k)$ and

$$\psi_\lambda = \psi_{j,k} = 2^{j/2} \psi(2^j x - k), \quad \tilde{\psi}_\lambda = \tilde{\psi}_{j,k} = 2^{j/2} \tilde{\psi}(2^j x - k).$$

Clearly, the properties of the function ψ will transfer to the functions ψ_λ and will imply properties of the corresponding wavelet basis.

In this framework, the key property of wavelets is their simultaneous good localisation both in space and in frequency. As we already mentioned we will deal with compactly supported wavelets. Therefore we can assume that there exists an $L > 0$ and an $\tilde{L} > 0$ such that

$$(4) \quad \text{supp } \psi \subseteq [-L, L] \Rightarrow \text{supp } \psi_\lambda \subseteq [-(k+L)/2^j, (k+L)/2^j],$$

$$(5) \quad \text{supp } \tilde{\psi} \subseteq [-\tilde{L}, \tilde{L}] \Rightarrow \text{supp } \tilde{\psi}_\lambda \subseteq [-(k+\tilde{L})/2^j, (k+\tilde{L})/2^j].$$

that is, both the wavelet ψ_λ ($\lambda = (j, k)$) and its dual $\tilde{\psi}_\lambda$ will be supported around the point $x_\lambda = k/2^j$, the size of their support will be of the order of 2^{-j} .

Now, for each wavelet $\psi_\lambda \in L^2(\mathbb{R})$ we can consider its Fourier transform

$\widehat{\psi}_\lambda \in L^2(\Omega)$. Since ψ is compactly supported, by Heisenberg's indetermination principle its Fourier transform $\widehat{\psi}$ cannot be itself compactly supported. However it is localised as well as possible around the frequency 1. More precisely the following properties hold: there exist an $M > 0$ and an $R > 0$, with $M > R$, such that for $n = 0, \dots, M$ and for s such that $0 \leq s \leq R$ one has

$$(6) \quad a) \quad \frac{d^n \widehat{\psi}}{d\xi^n}(0) = 0, \quad \text{and} \quad b) \quad \int (1 + |\xi|^2)^s |\widehat{\psi}(\xi)| d\xi \leq 1.$$

Analogously, for $\widetilde{\psi}$ there exist an $\widetilde{M} > 0$ and an $\widetilde{R} > 0$ such that for $n = 0, \dots, \widetilde{M}$ and for s such that $0 \leq s \leq \widetilde{R}$ one has

$$(7) \quad a) \quad \frac{d^n \widehat{\widetilde{\psi}}}{d\xi^n}(0) = 0, \quad \text{and} \quad b) \quad \int (1 + |\xi|^2)^s |\widehat{\widetilde{\psi}}(\xi)| d\xi \leq 1.$$

The frequency localisation property (6) can be rephrased directly in terms of the function ψ_λ , rather than in terms of its Fourier transform:

$$(8) \quad \int x^n \psi(x) dx = 0, \quad n = 0, \dots, M, \quad \text{and} \quad \|\psi\|_{H^s(\mathbb{R})} \leq 1, \quad 0 \leq s \leq R,$$

which, by a simple scaling argument implies

$$(9) \quad \int x^n \psi_\lambda(x) dx = 0, \quad n = 0, \dots, M, \quad \text{and} \quad \|\psi_\lambda\|_{H^s(\mathbb{R})} \leq 2^{js}, \quad 0 \leq s \leq R.$$

Analogously, we can write, for $\widetilde{\psi}_\lambda$

$$(10) \quad \int x^n \widetilde{\psi}_\lambda(x) dx = 0, \quad n = 0, \dots, \widetilde{M}, \quad \text{and} \quad \|\widetilde{\psi}_\lambda\|_{H^s(\mathbb{R})} \leq 2^{js}, \quad 0 \leq s \leq \widetilde{R}.$$

REMARK 1. – *Heisenberg's indetermination principle states that a function cannot be arbitrarily well localised both in space and frequency. More precisely, introducing the position uncertainty Δx and the momentum uncertainty $\Delta \xi$ defined by*

$$\Delta x_\lambda := \left(\int (x - x_\lambda)^2 |\psi_\lambda(x)|^2 dx \right)^{1/2}$$

$$\Delta \xi_\lambda := \left(\int (\xi - \xi_\lambda)^2 |\widehat{\psi}_\lambda(\xi)|^2 d\xi \right)^{1/2}$$

with $x_\lambda = x_{j,k} = k/2^j$ and $\xi_\lambda = \xi_{j,k} \sim 2^j$ defined by $\xi_\lambda = \int \xi |\widehat{\psi}_\lambda(\xi)|^2 d\xi$, one necessarily have $\Delta x \cdot \Delta \xi \geq 1$. In our case $\Delta x \cdot \Delta \xi \leq 1$, that is wavelets are simultaneously localised in space and frequency nearly as well as possible.

Before going on in seeing what the space-frequency localisation properties of the basis function ψ (and consequently of the wavelets ψ_λ 's) imply, let us for a moment consider functions with a stronger frequency localisation. Let us

then for a moment drop the assumption that ψ is compactly supported and assume that its Fourier transform verifies

$$|\text{supp}(\widehat{\psi})| \subset [1, 2], \quad \text{and that} \quad |\text{supp}(\widehat{f})| \subset [0, 1] \quad \forall f \in V_0.$$

(where $|\text{supp}(\cdot)|$ denotes the set $\{|\xi|, \xi \in \text{supp}(\cdot)\}$). Then one can easily deduce several properties of the basis functions ψ_λ : since one can easily check that for $\lambda = (j, k)$, $j \geq 0$, $|\text{supp}(\widehat{\psi})| \subset [2^j, 2^{j+1}]$, observing that on $\text{supp}(\widehat{\psi}_\lambda)$ we have $|\xi| \approx 2^j$ and that, for $\lambda \in \mathcal{A}_j$ and $\mu \in \mathcal{A}_m$ with $m \neq j$, $\text{supp}(\widehat{\psi}_\lambda) \cap \text{supp}(\widehat{\psi}_\mu) = \emptyset$, one immediately obtains the following equivalence: letting $f = \sum_\lambda f_\lambda \psi_\lambda$

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \approx \sum_j 2^{2js} \left\| \sum_{\lambda \in \mathcal{A}_j} f_\lambda \widehat{\psi}_\lambda \right\|_{L^2}^2$$

(remark that for $j = -1$ we have $2^{2js} = 2^{-2s} \approx 1$) which, thanks to the Riesz's basis property (2) implies

$$(11) \quad \|f\|_{H^s}^2 \approx \sum_{j \geq -1} 2^{2js} \sum_{\lambda \in \mathcal{A}_j} |f_\lambda|^2.$$

If we only consider partial sums, we easily derive direct and inverse inequality, namely:

$$(12) \quad \left\| \sum_{j=-1}^J \sum_{\lambda \in \mathcal{A}_j} f_\lambda \psi_\lambda \right\|_{H^s} \leq 2^{Js} \left\| \sum_{j=-1}^J \sum_{\lambda \in \mathcal{A}_j} f_\lambda \psi_\lambda \right\|_{L^2}$$

and

$$(13) \quad \left\| \sum_{j=J+1}^{\infty} \sum_{\lambda \in \mathcal{A}_j} f_\lambda \psi_\lambda \right\|_{L^2} \leq 2^{-Js} \left\| \sum_{j=J+1}^{\infty} \sum_{\lambda \in \mathcal{A}_j} f_\lambda \psi_\lambda \right\|_{H^s}.$$

Properties (12) and (13) – which, as we saw, are easily proven if $\widehat{\psi}$ is compactly supported – go on holding, though their proof is less evident, in the case of ψ compactly supported, provided (6) holds. The same is true for property (11). More precisely we can prove the following inequalities:

Direct Inequality. For all $f \in L^2(\mathbb{R}) \cap H^r(\Omega)$ and for s, r with $-\tilde{R} \leq s \leq r \leq R$

$$(14) \quad \left\| \sum_{m \geq j} \sum_{\lambda \in \mathcal{A}_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_s \leq 2^{-j(r-s)} \left\| \sum_{m \geq j} \sum_{\lambda \in \mathcal{A}_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_r \leq \|f\|_r$$

Inverse Inequality. For all $f \in V_j$ and for all s, r with $-\tilde{R} \leq s \leq r \leq R$ it holds that

$$(15) \quad \|f\|_r \leq \left\| \sum_{m=-1}^j \sum_{\lambda \in \mathcal{A}_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_r \leq 2^{j(r-s)} \left\| \sum_{m=-1}^j \sum_{\lambda \in \mathcal{A}_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_s \leq \|f\|_s.$$

All functions $f \in W_j$ verify both direct and inverse inequality:

$$\|f\|_{H^r(\Omega)} \approx 2^{jr} \|f\|_{L^2(\Omega)}.$$

In particular, the wavelets themselves satisfy

$$(16) \quad \|\psi_\lambda\|_{H^r(\Omega)} \approx 2^{jr}, \quad \lambda = (k, j), \quad j \geq 0 \quad r \in]-\tilde{R}, R[.$$

REMARK 2. – Note that an inequality of the form (12) is satisfied by all functions whose Fourier transform is supported in the interval $[-2^j, 2^j]$, while an inequality of the form (15) is verified by all functions whose Fourier transform is supported (away from 0) in $(-\infty, -2^j] \cup [2^j, \infty)$. Such inequalities are inherently bound to the frequency localisation of the functions considered, or, to put it in a different way, to their more or less oscillatory behaviour. Saying that a function is «low frequency» means that such function does not oscillate too much. This translates in an inverse type inequality. On the other hand, saying that a function is «high frequency» means that it is oscillating, that is that it is locally orthogonal to polynomials (where the meaning of «locally» is related to the frequency) and this translates in a direct inequality. In many applications (14) and (13) can actually replace the informations on the localisation of the Fourier transform. In particular this will be the case when we deal with functions defined on a bounded set Ω , for which the concept of Fourier transform does not make sense. Many of the things that can be proven for the case $\Omega = \mathbb{R}$ by using Fourier transform techniques, can be proven in an analogous way for bounded Ω by suitably using inequalities of the form (14) and (15).

2.2. The general case: Ω domain of \mathbb{R}^n .

Let us go back to the general case of Ω being a (possibly bounded) Lipschitz domain of \mathbb{R}^n . For the sake of simplicity let us not take into account the problem of boundary conditions (which certainly needs to be faced when aiming at an application in the framework of the numerical solution of PDEs), and let us consider wavelet bases that do not satisfy any kind of boundary conditions. We will briefly comment on this issue later on, in subsection 2.4.

The properties of being localised in space can be easily state also for wavelet bases on general domains.

Localisation in space. For each $\lambda \in A_j$ we have that

$$\text{diam}(\text{supp } \psi_\lambda) \leq 2^{-j}, \quad \text{and} \quad \text{diam}(\text{supp } \psi_\lambda) \leq 2^{-j}.$$

and for all $\mathbf{k} = (k_1, k_2, \dots, k_n)$ there are at most K (resp. \tilde{K}) values of $\lambda \in A_j$

such that

$$\text{supp } \psi_\lambda \cap \square_{j,k} \neq \emptyset \quad (\text{resp. } \text{supp } \tilde{\psi}_\lambda \cap \square_{j,k})$$

(where $\square_{j,k}$ denotes the cube of centre $k/2^j$ and side 2^{-j}). This last requirement is equivalent to asking that the basis functions at j fixed are uniformly distributed over the domain of definition. It avoids, for instance, that they accumulate somewhere.

Clearly, the concept of frequency and the definition of Fourier transform do not make sense in such framework. Still, we can ask that the basis functions have the same property in term of oscillations. We will then assume that they satisfy (9). More precisely we assume that the basis function ψ_λ verify

$$(17) \quad \int_{\Omega} x^n \psi_\lambda(x) dx = 0, \quad n = 0, \dots, M, \quad \text{and} \quad \|\psi_\lambda\|_{H^s(\Omega)} \leq 2^{js}, \quad 0 \leq s \leq R.$$

Analogous relations hold for the dual basis:

$$(18) \quad \int_{\Omega} x^n \tilde{\psi}_\lambda(x) dx = 0, \quad n = 0, \dots, \tilde{M}, \quad \text{and} \quad \|\tilde{\psi}_\lambda\|_{H^s(\Omega)} \leq 2^{js}, \quad 0 \leq s \leq \tilde{R}.$$

Additionally we will assume that the functions in the spaces V_j do not oscillate «too much» (*i.e.*, that they verify a suitable inverse type inequality) and that the remainders $f - P_j f$ are actually oscillating functions (*i.e.*, they satisfy a suitable direct inequality). In other words, we will ask that the following to properties hold.

Direct Inequality. For all $f \in L^2(\Omega)$ and for s, r with $-\tilde{R} \leq s \leq r \leq R$

$$(19) \quad \left\| \sum_{m \geq j} \sum_{\lambda \in \Lambda_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_s \leq 2^{-j(r-s)} \left\| \sum_{m \geq j} \sum_{\lambda \in \Lambda_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_r \leq \|f\|_r.$$

Inverse Inequality. For all $f \in V_j$ and for all s, r with $-\tilde{R} \leq s \leq r \leq R$ it holds that

$$(20) \quad \|f\|_r \leq \left\| \sum_{m=-1}^j \sum_{\lambda \in \Lambda_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_r \leq 2^{j(r-s)} \left\| \sum_{m=-1}^j \sum_{\lambda \in \Lambda_m} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda \right\|_s \leq \|f\|_s.$$

Before going on in describing the properties of wavelets bases, let us spend some word on some examples. For $\Omega = \mathbb{R}$ and $\Omega = \mathbb{R}^n$, different constructions verifying all the properties mentioned here for arbitrary values of the parameters R and M are available in the literature [23]. Just to mention some of them, let us recall orthonormal ($\tilde{\psi} = \psi$) Daubechies wavelets, orthonormal Coiflets (built in such a way that particularly simple quadrature formulas are available for computing $\langle \cdot, \tilde{\psi}_\lambda \rangle$), biorthogonal spline wavelets (the ψ_λ are compactly supported splines of order r , and the $\tilde{\psi}_\lambda$ are constructed in an implicit way (no closed form representation is available) in order to satisfy biorthogonality). A general strategy to build bases with the required characteristics for $]0, 1[$ out of

the bases for \mathbb{R}^n has been proposed in several papers [16], [1]. To actually build wavelet bases for general bounded domains, several strategies have been followed. Following the same strategy as for the construction of wavelet bases for cubes, *wavelet frames* (all the property mentioned here hold but, for each j the elements set $\{\psi_\lambda, \lambda \in \Lambda_j\}$ are not linearly independent) for $L^2(\Omega)$ (Ω Lipschitz domain) can be constructed according to [14]. The most popular approach nowadays is domain decomposition: the domain Ω is split as the disjoint union of tensorial subdomains Ω_ℓ and a wavelet basis for Ω is constructed by suitably assembling wavelet bases for the Ω_ℓ 's [13], [22], [17]. The construction is quite technical, since it is not trivial to retain in the assembling procedure the properties of the wavelets.

2.3. Characterisation of Function spaces.

A consequence of the validity of properties (19) and (20) is the characterisation, through the wavelet coefficients, of the scale of Besov spaces [19]. We first observe that, since all the functions $\tilde{\psi}_\lambda$ have a certain regularity, namely $\tilde{\psi}_\lambda \in H^{\tilde{R}}(\Omega)$, the Fourier development (3) makes sense (at least formally), provided f has enough regularity for $\langle f, \tilde{\psi}_\lambda \rangle$ to make sense, at least as a duality product, that is provided $f \in (H^{\tilde{R}}(\Omega))'$. The properties of wavelets imply that by looking at the absolute values of the coefficients $\langle f, \tilde{\psi}_\lambda \rangle$ of such formal Fourier development, it is possible to establish whether or not a function belongs to certain function spaces, and it is possible to write an equivalent norm for such function spaces in terms of the wavelet coefficients. More precisely, letting $B_q^{s,p}(\Omega) = B_q^s(L^p(\Omega))$ denote the Besov space of regularity index s , summability index p (q being the third index involved in the definition of Besov spaces, see [28]), given $f \in (H^{\tilde{R}}(\Omega))'$, then for $s \in [-\tilde{R}, R]$, f belongs to $B_q^{s,p}$ if and only if

$$(21) \quad \|f\|_{s,p,q} = \left(\sum_j 2^{j(s+n/2-n/p)q} \left(\sum_{\lambda \in \Lambda_j} |\langle f, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q}.$$

Moreover the norm $\|\cdot\|_{s,p,q}$ is an equivalent norm for such a space:

$$\text{for all } f \text{ in } B_q^{s,p}(\Omega) \quad \|f\|_{s,p,q} \approx \|f\|_{B_q^{s,p}(\Omega)}$$

Two cases are of particular interest for us. For $p = q = 2$ the Besov space $B_q^{s,p} = B_2^{s,2}$ coincides with the Sobolev space $H^s(\Omega)$ (or $(H^{-s}(\Omega))'$ for negative s). We have then the following characterisation of the Sobolev norms: for positive s

$$(22) \quad \|f\|_{H^s} \approx \|f\|_{s,2,2} = \left(\sum_j 2^{2js} \sum_{\lambda \in \Lambda_j} |\langle f, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2}$$

while for $s < 0$

$$(23) \quad \|f\|_{(H^{-s})'} \simeq \|f\|_{s, 2, 2} = \left(\sum_j 2^{2js} \sum_{\lambda \in A_j} |\langle f, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2}.$$

Thanks to these norm equivalences we can then evaluate the Sobolev norms for spaces with negative and/or fractionary indexes by using simple operations, namely the evaluation of $L^2(\Omega)$ scalar products and the evaluation of an (infinite) sum. Moreover, it is easy to realize that the norm $\|\cdot\|_{s, 2, 2}$ is an hilbertian norm, induced by the scalar product

$$(24) \quad (f, g)_s = \sum_j 2^{2js} \sum_{\lambda \in A_j} \langle f, \tilde{\psi}_\lambda \rangle \langle g, \tilde{\psi}_\lambda \rangle.$$

Equation (24) provides us with an equivalent scalar product for the Sobolev spaces $H^s(\Omega)$, $s \geq 0$ and, for $s < 0$, for the dual space $(H^{-s}(\Omega))'$, which is more easy evaluated than the original one.

Another case of the norm equivalence (21) which will be of interest is the case in which the norm on the right hand side is the ℓ^τ norm of the wavelet coefficients. This happens provided s , p and q are related by the relation

$$(25) \quad p = q = \tau, \quad s = n/p - n/2.$$

If we consider the space $B_\tau^{s, \tau}$ we have the equivalent norm

$$(26) \quad \|f\|_{s, \tau, \tau} = \left(\sum_{\lambda \in A} |\langle f, \tilde{\psi}_\lambda \rangle|^\tau \right)^{1/\tau}.$$

In the following Section we will see that the Besov spaces $B_\tau^{s, \tau}$ play a key role in the analysis on *nonlinear approximation* in $L^2(\Omega)$.

2.4. The issue of boundary conditions.

When aiming at using wavelet bases for the numerical solution of PDE's, one has to take into account the issue of boundary conditions. If, for instance, in the equation considered, essential boundary conditions (for example $u = g$) need to be imposed on a portion Γ_e of the boundary $\Gamma = \partial\Omega$ of the domain Ω of definition of the problem, we will want that the basis functions ψ_λ , $\lambda \in A$, satisfy themselves the corresponding homogeneous boundary conditions on Γ_e (that is, in the example mentioned above, $\psi_\lambda = 0$ on Γ_e). Depending on the projectors P_j , the dual wavelets $\tilde{\psi}_\lambda$ will not however need to satisfy themselves the same homogeneous boundary conditions, though this might be the case (if for instance the projector P_j is chosen to be the $L^2(\Omega)$ orthogonal projector). Depending on whether the ψ_λ and the $\tilde{\psi}_\lambda$ satisfy or not some homogeneous boundary conditions, the same boundary conditions will be incorporated in the spaces that we will be able to characterise through such functions. It is not the aim of this paper to go into details but only to give an idea on the kind of results that hold. To fix the ideas let us then consider the case of $\Gamma_e = \Gamma$ and of

Dirichlet boundary condition of order zero, namely $u = 0$ on Γ and let us concentrate on the characterisation of Sobolev spaces. If, for all $\lambda \in \mathcal{A}$, $\psi_\lambda = 0$ on Γ , then (20) will hold provided f belongs to the H^s closure of $H^s \cap H_0^1$, that we will denote $\mathcal{H}_0^s(\Omega)$. If all the $\tilde{\psi}_\lambda$'s satisfy $\tilde{\psi}_\lambda = 0$, we cannot hope to characterise (through scalar products with such functions) the space $(H^s(\Omega))'$, but only the space $(\mathcal{H}_0^s(\Omega))'$. In particular, provided for all $\lambda \in \mathcal{A}$ $\tilde{\psi}_\lambda = 0$, for $s = -1$ we will have a characterisation of the form

$$(27) \quad \|f\|_{H^{-1}(\Omega)}^2 \simeq \left(\sum_j 2^{-2j} \sum_{\lambda \in \mathcal{A}_j} |\langle f, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2}.$$

Then, again, an equivalent $H^{-1}(\Omega)$ scalar product can be defined as

$$(28) \quad (f, g)_{-1} = \sum_j 2^{-2j} \sum_{\lambda \in \mathcal{A}_j} \langle f, \tilde{\psi}_\lambda \rangle \langle g, \tilde{\psi}_\lambda \rangle.$$

Clearly, if for all $\lambda \in \mathcal{A}$ the ψ_λ 's satisfy an homogeneous boundary condition, we can expect a direct inequality of the form (19) to hold only if we assume that the function f to approximate satisfy itself the same homogeneous boundary conditions.

We would finally like to point out that the real challenge which is presented by essential boundary conditions in this framework is the actual construction of wavelet bases satisfying such boundary conditions. In simple cases, for example Ω equals a square or a cube (or the union of squares or cubes) and Γ_e equals to the union of edges or faces of the squares or cubes respectively, this can be done quite easily. More complicated situations may pose problems and a general solution is not yet available.

3. – Wavelet Stabilisation of unstable problems.

Let us then see how the properties of wavelets can be exploited in order to design new numerical methods for the solution of partial differential equations. Throughout the paper we will concentrate on simple model problems, though in general the ideas that we are going to present can be easily generalised to a wide class of differential equation (provided a wavelet basis for the domain of definition of the problem can be constructed).

As we saw in the previous section, wavelet bases give us a way of practically realizing equivalent scalar products for Sobolev spaces of negative and/or fractionary index. This is the key ingredient in the *wavelet stabilisation* technique.

To fix the ideas, let us consider the example of the Stokes problem. Given

$f \in (H^{-1}(\Omega))^3$ (Ω bounded domain of \mathbb{R}^3) find $u : \Omega \rightarrow \mathbb{R}^3$ and $p : \Omega \rightarrow \mathbb{R}$ such that

$$(29) \quad \begin{cases} -\Delta u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases}$$

$$(30) \quad u = 0, \quad \text{on } \partial\Omega, \quad \int_{\Omega} p = 0$$

or, in variational formulation,

Stokes Problem. Find $u \in U = (H_0^1(\Omega))^2$ and $p \in Q = L_0^2(\Omega)$ such that for all $v \in U$ and $q \in Q$ one has

$$(31) \quad \int_{\Omega} (\nabla u \cdot \nabla v - p \nabla \cdot v) = \int_{\Omega} f \cdot v,$$

$$(32) \quad \int_{\Omega} \nabla \cdot u q = 0,$$

where $L_0^2(\Omega) \subset L^2(\Omega)$

$$(33) \quad L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \right\}.$$

denotes the space of L^2 functions with zero mean value. It is well known that the bilinear form $a : (U \times Q) \times (U \times Q) \rightarrow \mathbb{R}$

$$(34) \quad a(u, p; v, q) = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} \nabla \cdot v p + \int_{\Omega} \nabla \cdot u q,$$

corresponding to such a problem is not coercive. Existence and uniqueness of the solution of such a problem are ensured by the *inf-sup* condition

$$(35) \quad \inf_{q \in Q} \sup_{v \in U} \frac{\int_{\Omega} \nabla \cdot v q}{\|v\|_{1, \Omega} \|p\|_{0, \Omega}} \geq \alpha > 0.$$

As a consequence, in characterisation such a problem, an arbitrary choice of the discretisation spaces for the velocity u and for the pressure p can lead to an unstable discrete problem. In order to have stable discretisations, the velocity and pressure approximation spaces U_h and Q_h need to be coupled in such a way that they satisfy a *discrete inf-sup condition* [12]:

$$(36) \quad \inf_{q_h \in Q_h} \sup_{u_h \in U_h} \frac{\int_{\Omega} \nabla \cdot u_h q_h}{\|u_h\|_{1, \Omega} \|q_h\|_{0, \Omega}} \geq \alpha_1 > 0,$$

with α_1 independent of the discretisation step h .

The idea of the *wavelet stabilised method* [3] is to introduce the following equivalent formulation of the Stokes problem:

Regularized Stokes problem. Find $u \in (H_0^1(\Omega))^3$ and $p \in L_0^2(\Omega)$ such that for all $v \in (H_0^1(\Omega))^3$ and $q \in L_0^2(\Omega)$ we have

$$(37) \quad \int_{\Omega} (\nabla u \cdot \nabla v - p \nabla \cdot v) = \int_{\Omega} f \cdot v,$$

$$(3.8) \quad \int_{\Omega} \nabla \cdot u q + \gamma(-\Delta u + \nabla p, \nabla q)_{-1} = \gamma(f, \nabla q)_{-1},$$

where $(\cdot, \cdot)_{-1} : (H^{-1}(\Omega))^3 \times (H^{-1}(\Omega))^3$ is the equivalent scalar product for the space $(H^{-1}(\Omega))^3$ defined according to (28). It is easy to check that the bilinear form $a_{\text{stab}} : ((H_0^1(\Omega))^3 \times L_0^2(\Omega))^2 \rightarrow \mathbb{R}$ which is defined by

$$(39) \quad a_{\text{stab}}(u, p; v, q) = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} p \nabla \cdot v + \int_{\Omega} \nabla \cdot u q + \gamma(-\Delta u + \nabla p, \nabla q)_{-1},$$

and which corresponds to such a formulation, is continuous. Moreover it is possible to prove that it is coercive for suitable choices of the constant γ . More precisely the following lemma holds [2].

LEMMA 1. – *There exists a constant γ_0 (depending on the domain Ω) such that if γ satisfies $0 < \gamma < \gamma_0$, then the bilinear form a_{stab} is coercive.*

Given any finite dimensional subspaces $U_h \subset (H_0^1(\Omega))^3$ and $Q_h \subset L_0^2(\Omega)$ we can then consider the following discrete problem:

Discrete Stabilised Problem. Find $u_h \in U_h$ and $p_h \in Q_h$ such that $\forall v_h \in U_h$ and $q_h \in Q_h$ we have

$$(40) \quad \int_{\Omega} (\nabla u_h \cdot \nabla v_h - p_h \nabla \cdot v_h) = \int_{\Omega} f \cdot v_h,$$

$$(41) \quad \int_{\Omega} \nabla u_h q_h + \gamma(-\Delta u_h + \nabla p_h, \nabla q_h)_{-1} = \gamma(f, \nabla q_h)_{-1}.$$

for which, using the standard theory for the Galerkin discretisation of coercive operators we immediately obtain the following error estimate.

PROPOSITION 1. – *Let (u, p) be the solution of problem (31) and (u_h, p_h) the solution of problem (40) Then the following error estimate holds:*

$$(42) \quad \|u - u_h\|_{1, \Omega} + \|p - p_h\|_{0, \Omega} \lesssim \left(\inf_{v_h \in U_h} \|u - v_h\|_{1, \Omega} + \inf_{q_h \in Q_h} \|p - q_h\|_{0, \Omega} \right).$$

The use of the stabilised formulation gives then rise to an optimal discretisation of the Stokes problem, for which the choice of the approximation spaces is not subject to limitations on the coupling of the discretisations for velocity and pressure.

Many problems share the same characteristics as the Stokes problem, and

can therefore benefit from an analogous approach. An abstract result can be found [2] [4]. This has been applied in the domain decomposition framework (see [8], [5]), as well as to the Lagrange multiplier formulation of Dirichlet problems [6]. The use of wavelets to realize negative and/or fractionary scalar product has also been applied in the framework of the *Least Squares* method (see [11], [20]). An analogous technique, based on the wavelet evaluation of a $-1/2$ scalar product, has also been applied to *convection-diffusion* problems with dominating convection [7].

4. – Nonlinear approximation.

One of the most significant development that took place thanks to the theory of wavelets is the new insight on the concept of *nonlinear approximation* [24]. It is well known that, in order to approximate a function with singularities in an efficient way, the approximation space must be tailored to the function itself, by concentrating the degrees of freedom near the singularities. This process takes a particularly simple form when considering wavelet approximations.

Assume that we want to approximate in $H^r(\Omega)$ a given function f ,

$$(43) \quad f = \sum_{\lambda \in \Lambda} f_{\lambda} \psi_{\lambda},$$

as a sum of $O(N)$ wavelets.

The first, more classical, possibility is to take all the terms of the sum (43) below a given scale J , with $J \sim \log_2 N/n$, (we recall that $\Omega \subseteq \mathbb{R}^n$), that is to look for an approximation v belonging to the linear space V_J . Clearly, the best $H^r(\Omega)$ approximation to f in V_J is its $H^r(\Omega)$ orthogonal projection on V_J . It is not difficult to see that

$$\inf_{v \in V_J} \|f - v\|_{H^r(\Omega)} \leq \|f - P_J f\|_{H^r(\Omega)} \leq \left\| \sum_{j \geq J} \sum_{\lambda \in \Lambda_j} f_{\lambda} \psi_{\lambda} \right\|_{H^r(\Omega)}.$$

The following well known theorem holds.

THEOREM 1. – *Assume that V_J contains polynomials up to degree $M - 1$, and let $f \in H^{r+s}(\Omega)$, $0 \leq s \leq M - r$, then we have*

$$\inf_{v \in V_J} \|f - v\|_{H^r(\Omega)} \leq N^{-s/n} \|f\|_{H^{r+s}(\Omega)}.$$

where $N = 2^{Jn}$.

The second way in which we can approximate f is to choose which coefficients f_{λ} to retain and which to discard without fixing a priori a maximum level of resolution. Then, we rather look for an approximation v to f belonging to the

non linear space Σ_N

$$(44) \quad \Sigma_N = \left\{ v = \sum_{\lambda} v_{\lambda} \psi_{\lambda} : \vec{v} = \{v_{\lambda}\}_{\lambda \in A} \in \sigma_N \right\}$$

of functions of $L^2(\Omega)$ which can be expressed as a linear combination of at most N basis functions, where we denote by σ_N

$$\sigma_N = \{ \vec{v} \in \ell^2(A) : \#\{\lambda : v_{\lambda} \neq 0\} \leq N \},$$

the set of ℓ^2 sequences with at most N non zero elements.

The space Σ_N is clearly non linear. The sum of two elements of Σ_N will be, in general, an element of Σ_{2N} rather than an element of Σ_N .

In order to construct an approximant to any given function $f = \sum_{\lambda \in A} f_{\lambda} \psi_{\lambda}$, a non linear projector $P_N: H^r(\Omega) \rightarrow \Sigma_N$ can be defined as follows: we at first re-scale the basis functions and the coefficients in the wavelet decomposition (43) in such a way that the basis functions are uniformly bounded in $H^r(\Omega)$. More precisely, for $\lambda \in A_j$ we let

$$\check{\psi}_{\lambda} = 2^{-jr} \psi_{\lambda}, \quad \check{f}_{\lambda} = 2^{jr} f_{\lambda},$$

so that $f = \sum_{\lambda \in A} f_{\lambda} \psi_{\lambda} = \sum_{\lambda \in A} \check{f}_{\lambda} \check{\psi}_{\lambda}$. We next introduce a decreasing rearrangement $\{ |\check{f}_{\lambda(n)}| \}_{n \in \mathbb{N}}$

$$|\check{f}_{\lambda(1)}| \geq |\check{f}_{\lambda(2)}| \geq |\check{f}_{\lambda(3)}| \geq \dots \geq |\check{f}_{\lambda(n)}| \geq |\check{f}_{\lambda(n+1)}| \geq \dots$$

of the sequence $\{ |\check{f}_{\lambda}| \}_{\lambda \in A}$, where the function $n \in \mathbb{N} \rightarrow \lambda(n) \in A$ is any bijective function such that

$$n < m \rightarrow |\check{f}_{\lambda(n)}| \geq |\check{f}_{\lambda(m)}|;$$

$P_N(f)$ is then defined by:

$$P_N(f) = \sum_{n=1}^N \check{f}_{\lambda(n)} \psi_{\lambda(n)},$$

that is only the N greatest (in rescaled absolute value) coefficients of f are retained. By abuse of notation we will also indicate by $P_N: \ell^2 \rightarrow \sigma_N$ the operator associating to the sequence \vec{f} the coefficients of the function $P_N(\sum_{\lambda} f_{\lambda} \psi_{\lambda})$:

$$\vec{w} = (w_{\lambda})_{\lambda \in A} = P_N(\vec{f}) \Leftrightarrow \sum_{\lambda \in A} w_{\lambda} \psi_{\lambda} = P_N\left(\sum_{\lambda} f_{\lambda} \psi_{\lambda}\right).$$

Following [24, 25] it is possible to prove the following estimate on the approximation error.

THEOREM 2. – *Let $f \in B_{\tau}^{r+s, \tau}(\Omega)$ with τ such that $1/\tau = s/n + 1/2$, then*

$$\inf_{w \in \Sigma_N} \|f - w\|_{H^r(\Omega)} \leq \|f - P_N f\|_{H^r(\Omega)} \leq N^{-r/n} \|f\|_{B_{\tau, \tau}^{r+s}(\Omega)},$$

where the implicit constants in the bounds are independent of N .

Remark that, for the space $B_{\tau}^{r+s, \tau}(\Omega)$ we have the norm equivalence

$$\|f\|_{B_{\tau}^{r+s, \tau}(\Omega)} \simeq \|f\|_{r+s, \tau, \tau} \simeq \left(\sum_j 2^{j\tau} \sum_{\lambda \in A_j} |\langle f, \tilde{\psi}_{\lambda} \rangle|^{\tau} \right)^{1/\tau} = \|\vec{f}\|_{\ell^{\tau}},$$

that is $B_{\tau}^{r+s, \tau}(\Omega)$ is the space of functions such that the sequence of rescaled coefficients is in ℓ^{τ} . For $r=0$ we obtain exactly the space $B_{\tau}^{s, \tau}(\Omega)$ mentioned in section 2.3.

REMARK 3. – *Actually the rate of convergence of the nonlinear approximation of a function is directly related to a weaker norm of the sequence of its (rescaled) wavelet coefficients. Let us in fact consider the weak- ℓ^{τ} space ℓ_w^{τ} which can be defined as follows: a sequence $\vec{v} = \{v_{\lambda}\}_{\lambda}$ belongs to the space ℓ_w^{τ} if and only if*

$$(45) \quad \#\{\lambda : |v_{\lambda}| \geq \varepsilon\} \leq C\varepsilon^{-\tau},$$

the norm $\|\vec{v}\|_{\ell_w^{\tau}}$ of the given sequence being defined as the smallest constant C which verifies relation (45). The main result in [25] is that $\|f - \mathbb{P}(f)\|_{H^r(\Omega)} \leq N^{-s/n}$ if and only if $\{\check{f}_{\lambda}\}_{\lambda \in A}$ belongs to ℓ_w^{τ} , with τ such that $1/\tau = s/n + 1/2$. Since $\ell^{\tau} \subset \ell_w^{\tau}$, Theorem 2 easily follows.

REMARK 4. – *The analog result for linear approximation, namely Theorem 1, tells us that, when we measure the approximation error in $H^r(\Omega)$, the same error rate $\mathcal{O}(N^{-s/d})$ is achieved by linear approximation only for functions belonging to the Sobolev space $H^{r+s}(\Omega)$, which is much smaller than the space that allow to obtain the same convergence rate for non linear approximation, namely the Besov space $B_{\tau, \tau}^{r+s}(\Omega)$. In particular, functions with isolated singularities may belong to $B_{\tau, \tau}^{r+s}(\Omega)$ for quite large values of r , while not belonging to $H^{s+r}(\Omega)$. On such functions one can then hope to obtain a good approximation rate through non linear approximation, while linear method would give a poor result.*

5. – Nonlinear methods for the solution of PDE's.

The perhaps most interesting new idea that has recently been put forward in the fields of wavelet methods for the solution of PDE's is related to the concept of nonlinear approximation described in the previous section 4.

To fix the ideas, let us consider a simple model problem: the *reaction-diffusion* equation. Given g find u such that

$$(46) \quad -\Delta u + u = g \quad \text{in } \Omega,$$

and assume that, rather than using the standard form of adaptive method (computing an approximate solution, evaluate an error indicator, refine/derefine the approximation space, iterate until convergence) one would like to desi-

gn an algorithm that looks for the solution in the space Σ_N previously defined.

The first step in order to do so (see [15], [10]), is to rewrite problem (46) as an equivalent infinite discrete problem, written in terms of the wavelet coefficients, with respect to a sufficiently regular (at least $H^1(\Omega)$) wavelet basis. Letting $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ denote the bilinear form

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + uv)$$

corresponding to the reaction-diffusion operator and expressing u in terms of the rescaled basis $\{\check{\psi}_{\lambda} = 2^{-j} \psi_{\lambda}\}_{\lambda}$, we can rewrite (46) in terms of the rescaled Fourier coefficients $\vec{u} = \{\check{u}_{\lambda}\}_{\lambda \in \Lambda} = \{2^j u_{\lambda}\}_{\lambda \in \Lambda}$ of the unknown solution $u = \sum_{\lambda} u_{\lambda} \psi_{\lambda}$, as an infinite dimensional system of linear equations:

$$(47) \quad \mathcal{A} \vec{u} = \vec{g}$$

where

$$\mathcal{A} = (a_{\mu, \lambda})_{\mu, \lambda \in \Lambda}, \quad a_{\mu, \lambda} = \langle -\Delta \check{\psi}_{\lambda}, \check{\psi}_{\mu} \rangle, \quad \vec{g} = \{g_{\mu}\}_{\mu \in \Lambda}, \quad g_{\mu} = \langle g, \check{\psi}_{\mu} \rangle,$$

are a bi-infinite matrix and an infinite array respectively. It is not difficult to check that $\mathcal{A} \in \mathcal{L}(\ell^2, \ell^2)$ ($\mathcal{L}(\ell^2, \ell^2)$ denoting the space of bounded linear operator from ℓ^2 to ℓ^2) and that it is boundedly invertible; that is:

$$\|\mathcal{A}\|_{\mathcal{L}(\ell^2, \ell^2)} < +\infty, \quad \|\mathcal{A}^{-1}\|_{\mathcal{L}(\ell^2, \ell^2)} \leq +\infty.$$

Moreover the basis $\{\check{\psi}_{\lambda}\}_{\lambda}$ can be chosen in such a way that for some $\tau_0 < 2$, $\mathcal{A} \in \mathcal{L}(\ell^{\tau_0}, \ell^{\tau_0})$ (this holds provided the wavelet basis is sufficiently well localised, which is true for a wide class of wavelet bases [15]).

Let us now recall that $\Sigma_N \subseteq H^1(\Omega)$, (see definition (44)) is defined as the non linear space of functions in $L^2(\Omega)$ which can be represented as the linear combination of at most N elements of the basis $\{\psi_{\lambda}\}_{\lambda}$ (or equivalently $\{\check{\psi}_{\lambda}\}_{\lambda}$). Let us also recall that, given a function $u = \sum_{\lambda \in \Lambda} u_{\lambda} \psi_{\lambda} \in H^1(\Omega)$, a quasi-best $H^1(\Omega)$ approximation $\mathbb{P}_N u \in \Sigma_N$ can be obtained as a truncate wavelet Fourier series by retaining, in the wavelet expansion, the components corresponding to the N biggest (in absolute value) rescaled coefficients \check{u}_{λ} , and discarding the remaining.

In order to find an approximation in Σ_N to the solution of (46), the idea is then to formally write a convergent iterative scheme for the solution of the infinite linear system (47), and then discretise each step of such a scheme by

- evaluating the infinite matrix-vector multiplication approximately in a finite number of operations
- force (by applying the projector \mathbb{P}_N) the iterates of the scheme to belong to Σ_N .

To fix the ideas let us consider the simplest iterative scheme, namely Richardson scheme:

$$(48) \quad \vec{u}^{n+1} = \vec{u}^n + \vartheta(\vec{g} - \mathcal{C}\vec{u}^n).$$

The convergence of such scheme is a consequence of a property of wavelets that is often referred at as *wavelet preconditioning*. The boundedness of the operator \mathcal{C} and of its inverse \mathcal{C}^{-1} (which are a consequence of the boundedness of the Laplace operator and of its inverse, and of the norm equivalences (22) and (23)) imply that

$$\text{cond}(\mathcal{C}) = \|\mathcal{C}\|_{\mathcal{L}(\ell^2, \ell^2)} \|\mathcal{C}^{-1}\|_{\mathcal{L}(\ell^2, \ell^2)} = \alpha < \infty,$$

and this implies that there exists $\vartheta_0 > 0$ (depending on α) such that, provided $\vartheta < \vartheta_0$ the iterate \vec{u}^n converges to the unique fixed point of the operator

$$v \rightarrow v + \vartheta(f - \mathcal{C}v),$$

that is the solution of (47).

In order to force the iterate of the numerical scheme to belong to the nonlinear space Σ_N we simply apply at each iteration the nonlinear projector P_N . We end then up with the following iterative scheme:

$$(49) \quad \vec{u}^{n+1} = P_N(\vec{u}^n + \vartheta(\vec{g} - \mathcal{C}\vec{u}^n)).$$

As already stated, this is still a formal algorithm, since it involves operations on infinite matrices and vectors. Such scheme has to be coupled with suitable compression steps applied both to the operator \mathcal{C} and to the right hand side \vec{g} , which will allow to actually implement it efficiently. Nevertheless it is interesting to consider such a scheme in order to analyze the influence of the nonlinear operator P_N . In particular it is possible to prove the following Theorem (see [9]):

THEOREM 3. – *Let $\mathcal{C} \in \mathcal{L}(\ell^{\tau_0}, \ell^{\tau_0}) \cap \mathcal{L}(\ell^2, \ell^2)$ for some $\tau_0 < 2$. Then there exists a $\tilde{\tau} < 2$ and a $\theta_0 > 0$ such that, for all θ , $0 < \theta < \theta_0$, it holds (the implicit constants in the inequalities depending on θ)*

(i) *stability: if $\vec{g} \in \ell^2$, we have*

$$(50) \quad \|\vec{u}^n\|_{\ell^2} \leq \|\vec{g}\|_{\ell^2}, \quad \forall n \in \mathbb{N};$$

(ii) *approximation error estimate: if $\vec{g} \in \ell_w^\tau$, $\tilde{\tau} < \tau \leq 2$ then for some $\mu < 1$:*

$$(51) \quad \|\vec{u} - \vec{u}^n\|_{\ell^2} \leq \mu^n \|\vec{u} - \vec{u}^0\|_{\ell^2} + \frac{1}{1-\mu} N^{-(\frac{1}{\tau} + \frac{1}{2})};$$

In other words, the method is stable and convergent. Remark that it is pos-

sible to prove that asking \vec{g} to belong to ℓ^2 is equivalent to asking that $g \in (H^1(\Omega))'$ (which is the natural space for the right hand side of equation (46)).

Clearly, this simple idea can be refined and generalised in many possible directions and applied to a wide class of PDEs (see [15], [18]). Other solvers than the Richardson method can be used (for example, the Conjugate Gradient Method); moreover it makes sense to vary the number N of degrees of freedom as the procedure goes on, in such a way that N is small (and therefore the computation is less expensive) during the first steps of the iterative method [15].

6. – Operations on infinite matrices and vectors.

For the applications considered we ended up with methods that implied the computation of either an infinite sum or of an infinite matrix/vector multiplication involving

- matrices \mathcal{A} expressing a differential operator with good properties (continuity, coercivity, ...);
- vectors \vec{u} of wavelet coefficients of a discrete function.

This is the case of both the computation of the equivalent scalar product $(\cdot, \cdot)_{-1}$ appearing in section 3 and of the infinite matrix/vector multiplication appearing in Section 5. These matrices and vector are not directly *maniable*. However, thanks to the properties of wavelets it is in general possible to approximate such infinite sums by finite sums with an arbitrary precision, or even to replace the infinite sum by a finite one without substantially changing the resulting method.

For the sake of simplicity let us concentrate on the case of Ω bounded domain, so that for any fixed level j the cardinality of Λ_j is finite. To fix the ideas, let us consider the equivalent scalar product $(\cdot, \cdot)_{-1}$ in Section 3. Heuristically, the argument that we have in mind is that *if a discrete function satisfies an inverse inequality (\sim it is «low frequency»), then the levels in the infinite sum corresponding to «high frequency» components will be superfluous and then the infinite sum in (28) can be truncated*. Just for this example we would like to show how this heuristics can be made rigorous. The aim is to replace the scalar product

$$(F, G)_{-1} = \sum_j \sum_{\lambda \in \Lambda_j} 2^{-2j} \langle F, \tilde{\psi}_\lambda \rangle \langle G, \tilde{\psi}_\lambda \rangle$$

in (37), with a bilinear form

$$(F, G)_{-1, J} = \sum_{j \leq J} \sum_{\lambda \in \tilde{\Lambda}_j} 2^{-2j} \langle F, \tilde{\psi}_\lambda \rangle \langle G, \tilde{\psi}_\lambda \rangle,$$

while retaining the properties (coercivity on the discrete space of the stabilized operator) of the resulting discrete method.

Since the aim of adding the stabilisation term to the original equation is to obtain control on the pressure p_h through a coercivity argument, that is one wants to obtain a bound of the form

$$a_{\text{stab}}(v_h, q_h; v_h, q_h) \geq \|v_h\|_{H^1(\Omega)}^2 + \|q_h\|_{L^2(\Omega)}^2,$$

and since the velocity is controlled through coercivity already for the original bilinear form a , the fundamental property of $(\cdot, \cdot)_{-1}$ that we want to keep, in this case, is that for arbitrary elements $q_h \in Q_h$ one has

$$(52) \quad (\nabla q_h, \nabla q_h)_{-1, J} = \sum_{j \leq J} \sum_{\lambda \in \tilde{\Lambda}_j} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2 \geq \|\nabla q_h\|_{H^{-1}(\Omega)}^2.$$

We will have to replace the heuristical concept q_h is «*low frequency*» by a suitable inverse inequality: more precisely we will assume that $\nabla Q_h \subset H^t(\Omega)$ for some t with $-1 < t$, and that for all $q_h \in Q_h$

$$\|\nabla q_h\|_t \leq h^{-t-1} \|\nabla q_h\|_{-1}.$$

Under this assumption it is actually possible to prove that there exists a $J = J(h)$ depending on the mesh-size parameter h such that (52) holds. The proof is simple and gives an idea of how these kind of argument works in general. For any given $J > 0$ we can write

$$\|\nabla q_h\|_{H^{-1}(\Omega)}^2 = \sum_{j \leq J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2 + \sum_{j > J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2.$$

Let us analyse the last term:

$$\begin{aligned} \sum_{j > J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2 &\leq 2^{-2(t+1)J} \sum_{j > J} \sum_{\lambda} 2^{2tj} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2 \leq \\ &2^{-2(t+1)J} \|\nabla q_h\|_{H^t(\Omega)}^2 \leq 2^{-2(t+1)J} h^{-2(t+1)} \|\nabla q_h\|_{H^{-1}(\Omega)}^2 \end{aligned}$$

Then we can write

$$\|\nabla q_h\|_{H^{-1}(\Omega)}^2 \leq \sum_{j \leq J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2 + C' 2^{-2(t+1)J} h^{-2(t+1)} \|\nabla q_h\|_{H^{-1}(\Omega)}^2$$

whence

$$(1 - C' 2^{-2(t+1)J} h^{-2(t+1)}) \|\nabla q_h\|_{H^{-1}(\Omega)}^2 \leq \sum_{j \leq J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_\lambda \rangle|^2$$

which, provided J is chosen in such a way that $(1 - C' 2^{-2(t+1)J} h^{-2(t+1)}) \leq 1/2$ yields

$$\|\nabla q_h\|_{H^{-1}(\Omega)}^2 \leq \sum_{j \leq J} \sum_{\lambda} 2^{-2j} |\langle \nabla q_h, \tilde{\psi}_{\lambda} \rangle|^2.$$

We will not go here into details on how this result allows to write down computable stabilised methods for the Stokes equation, and prove stability, convergence and optimal error estimates for such methods.

An analogous result holds for the evaluation of the infinite matrix/vector multiplication $\mathcal{C} \vec{u}$ that is found in (49). Taking advantage of the fact that, by construction, \vec{u} is the linear combination of a finite number of wavelets and that, thanks to the localisation property of wavelets, the infinite matrix \mathcal{C} can be sparsified (by retaining, per line, only a finite number of entries which are, in absolute value, greater than a suitable tolerance ε) one can compute the products $\mathcal{C} \vec{u}$ in a finite number of operations within a given tolerance [15].

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