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Sofia Giuffrè, Giovanna Idone

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Global Regularity for Solutions to Dirichlet Problem for Discontinuous Elliptic Systems with Nonlinearity $q > 1$ and with Natural Growth

SOFIA GIUFFRÈ - GIOVANNA IDONE

Sunto. – *In questo lavoro studiamo la regolarità fino alla frontiera di soluzioni di un problema di Dirichlet non omogeneo per sistemi ellittici discontinui del secondo ordine con non linearità $q > 1$ e con andamenti naturali. Scopo del lavoro è illustrare che le soluzioni del suddetto problema sono sempre globalmente hölderiane nel caso di dimensione $n = q$ senza alcun tipo di condizione di regolarità sui coefficienti. Come conseguenza di questo risultato, gli insiemi singolari $\Omega_0 \subset \Omega$, $\Sigma_0 \subset \partial\Omega$ sono sempre vuoti per $n = q$. Inoltre dimostriamo che anche per $1 < q < 2$, ma q sufficientemente vicino a 2, le soluzioni sono globalmente hölderiane per $n = 2$.*

Summary. – *In this paper we deal with the Hölder regularity up to the boundary of the solutions to a nonhomogeneous Dirichlet problem for second order discontinuous elliptic systems with nonlinearity $q > 1$ and with natural growth. The aim of the paper is to clarify that the solutions of the above problem are always global Hölder continuous in the case of the dimension $n = q$ without any kind of regularity assumptions on the coefficients. As a consequence of this sharp result, the singular sets $\Omega_0 \subset \Omega$, $\Sigma_0 \subset \partial\Omega$ are always empty for $n = q$. Moreover we show that also for $1 < q < 2$, but q close enough to 2, the solutions are global Hölder continuous for $n = 2$.*

1. – Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with boundary $\partial\Omega$ of class C^2; let q be a real number > 1 and $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$. If $u : \Omega \to \mathbb{R}^N$, we set $Du = (D_1 u, \dots, D_n u)$ and we denote by $p = (p^1, \dots, p^n)$, with $p^i \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$ and by $(u|v)$, with $u, v \in \mathbb{R}^N$, the inner product in $\mathbb{R}^N$.

The aim of this paper is to study the global Hölder continuity in $\overline{\Omega}$ of a solution $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$ to the following Dirichlet problem

$$(1) \qquad \begin{cases} u - g \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega) \\ \displaystyle\sum_{i=1}^n D_i a^i(x, u, Du) = -B(x, u, Du) \quad \text{in } \Omega \end{cases}$$

where $a^i(x,u,p)$, $i=1,2,\dots,n$, are vectors of $\mathbb{R}^N$, defined on $\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$, such that $a^i(x,u,p)$ are measurable in x and continuous in u, p, and $a^i(x,u,0)=0$ for a.a.$x\in\Omega$, $\forall u\in\mathbb{R}^N$.

For solution u to (1) we mean that $u=g+w$, where $w\in H_0^{1,q}(\Omega)\cap L^\infty(\Omega)$ is such that, $\forall\varphi\in H_0^{1,q}(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega\sum_{i=1}^n\big(a^i(x,w+g,Dw+Dg)|D_i\varphi\big)dx=\int_\Omega(B(x,w+g,Dw+Dg)|\varphi)dx. \tag{2}$$

We do not require regularity assumptions on the coefficients, but setting

$$V(p)=\Big(1+\|p\|^2\Big)^{\frac12}\quad\text{and}\quad W(p)=V^{\frac{q-2}{2}}(p)\,p,\quad\forall p\in\mathbb{R}^K,\ K\ge1, \tag{3}$$

we only assume that there exist two positive constants M,υ such that for a.a. $x\in\Omega$, $\forall\ u\in\mathbb{R}^N$, $p\in\mathbb{R}^{nN}$, it results

$$\|a^i(x,u,p)\|\le M\,V^{q-2}(p)\|p\|,\qquad i=1,\dots,n \tag{4}$$

$$\sum_{i=1}^n\big(a^i(x,u,p)|p^i\big)\ge\upsilon\,V^{q-2}(p)\,\|p\|^2. \tag{5}$$

Let us note that an example of operator satisfying conditions (4), (5) is given by $\sum_i D_i[(1+\|Du\|^2)^{\frac{q-2}{2}}D_iu]$ and that in the case $1<q<2$ conditions (4), (5) appear as degenerate conditions.

On the free term B we suppose that there exist two positive constants a,b, such that for a.a. $x\in\Omega$, $\forall\ u\in\mathbb{R}^N$, $p\in\mathbb{R}^{nN}$ it results

$$\|B(x,u,p)\|\le a+b\|W(p)\|^2, \tag{6}$$

and, if u is a solution to Problem (1), also the following smallness condition holds

$$2b\ \|u-g\|_{L^\infty(\Omega)}<\upsilon. \tag{7}$$

Condition (6) is called natural growth condition. Further we would like to point out that the so called smallness condition (7) is necessary, in a certain sense, in order to obtain the regularity result of the solutions, in virtue of the well known counter-examples, as, for example, the one provided by [9]; however it seems that the optimal smallness condition could be $b\ \|u-g\|_{L^\infty(\Omega)}<\upsilon$, even if in the literature it is used condition (7).

Taking into account the counter-examples provided in [7], [11], [14], [21] and the general form of the coefficients $a^i(x,u,Du)$ of problem (1), it is well known that it is not possible to expect the global Hölder continuity of solutions for $n>q$. For $n<q$ the desired regularity easily follows from Sobolev imbedding theorems, then our goal remains only in proving the regularity up to the boundary for $n=q$. For what concerns the case $q\ge2$ we achieve this result by means of the following theorem on higher global integrability of the gradient:

THEOREM 1.1. – *Assume that conditions* (4), (5), (6) *and* (7) *are fulfilled. Let* $\partial\Omega$ *be of class* C^2 *and* $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, *with* $s > q$. *If* $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q \geq 2$, *is a solution to Dirichlet Problem (1), then there exists a number* $r > 1$ *such that* $u \in H^{1,qr}(\Omega)$.

From Theorem 1.1 we immediately derive the following corollary.

COROLLARY 1.1. – *Under the same assumptions of Theorem* 1.1, *a solution* u *to Dirichlet Problem* (1), *for* $q = n$, *belongs to* $C^{0,a}(\overline{\Omega})$, *with* $a = 1 - \dfrac{1}{r}$.

Considering now the case of nonlinearity $1 < q < 2$, we obtain again a result of higher global integrability of the gradient. In fact we prove the following theorem.

THEOREM 1.2. – *Assume that conditions* (4), (5), (6) *and* (7) *are fulfilled. Let* $\partial\Omega$ *be of class* C^2 *and* $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, *with* $s > q$. *If* $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$, $1 < q < 2$, *is a solution to Dirichlet Problem* (1), *then there exists a number* $r > 1$ *such that* $u \in H^{1,qr}(\Omega)$.

Then we can reach the second main result of this paper.

COROLLARY 1.2. –*Under the same assumptions of Theorem* 1.2, *a solution* u *to Dirichlet Problem (1), for* $q \in \left(\dfrac{n}{r}, 2\right)$, *belongs to* $C^{0,a}(\overline{\Omega})$, *with* $a = 1 - \dfrac{n}{qr}$.

It is interesting to note that even in the case $1 < q < 2$, for $n = 2$, we obtain the Hölder continuity up to the boundary for $\dfrac{2}{r} < q < 2$.

In the general case, namely without requirements on the dimension n and with coefficients a^i depending on x, u, Du, the result we can expect if $q > n$ is only the so called «partial Hölder regularity», that is there exists a closed singular set Ω_0 such that u is Hölder continuous in $\Omega \setminus \Omega_0$ and, even if the trace of u on $\partial\Omega$ is smooth, there exists a closed singular set Σ_0 on $\partial\Omega$ such that u is Hölder continuous up to the boundary except for the points of Σ_0 (for nonlinearity $q > 2$ see [4]; for nonlinearity $q = 2$ see [1], [3], [6], [12], [15], [17], [18], [23]; it is worth mentioning that all these results are obtained under suitable regularity assumptions on the data). Moreover in particular cases it is possible to estimate, always under suitable regularity assumptions on the data, the Hausdorff dimension of both the singular sets Ω_0 and Σ_0 (see for example [4], [15], [17], [18]).

The general case with $1 < q < 2$ is less studied than the case $q \geq 2$. We mention that differentiability results for $1 < q < 2$ are obtained in [2], [20]. Moreover in [19] a homogeneous system with coefficients of the type $a^i(Du)$ is studied and global Hölder continuity results are obtained.

Also we note that in [22] the author obtains the global Hölder continuity up to the boundary for $n = q$ in the case of term B fulfilling a growth of the type $\|p\|^{q-1}$.

Moreover we would point out that the behaviour of weak solutions with respect to the Hölder continuity is analogous to the one we meet when we consider elliptic nonvariational systems, namely we obtain global Hölder continuity up to the boundary only for low values of n and partial Hölder continuity in the general case (see [8], [16]).

2. – Sketch of the Proofs

We set $B(x^0,\sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$; and, if $x_n^0 = 0$, $B^+(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n > 0\}$, $\Gamma(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n = 0\}$. We will simply write $B(\sigma)$, $B^+(\sigma)$, $\Gamma(\sigma)$ and Γ instead of $B(0,\sigma)$, $B^+(0,\sigma)$, $\Gamma(0,\sigma)$ and $\Gamma(0,1)$, respectively. Moreover if $u \in L^1(\mathcal{B})$ and $\mathcal{B}$ is a measurable set with $\operatorname{meas}\mathcal{B} \neq 0$, then $u_{\mathcal{B}} = \fint_{\mathcal{B}} u(x)dx = \dfrac{1}{\operatorname{meas}\mathcal{B}}\displaystyle\int_{\mathcal{B}} u(x)dx.$

In both cases, $1 < q < 2$ and $q \geq 2$, a first step in order to obtain the global higher summability of the gradient is to achieve the interior higher summability of the gradient. To this end a crucial step is the following «Caccioppoli's type» inequality.

THEOREM 2.1. – *Assume that conditions* (4), (5), (6) *and* (7) *are fulfilled. Let* $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$, *and let* $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q > 1$, *be a solution of the strongly elliptic system* (2) $\forall \varphi \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$. *Then for every couples of concentric balls* $B(\sigma) \subset B(2\sigma) \subset \Omega$, *it results*

$$\int_{B(\sigma)} \|Dw\|^q dx \leq c\,\sigma^{-q} \int_{B(2\sigma)} \|w - w_{B(2\sigma)}\|^q dx + c_1 \int_{B(2\sigma)} (1 + \|Dg\|)^q dx \tag{8}$$

where c, c_1 *depend on* q, M, υ, a, b, $\|u - g\|_{L^\infty(\Omega)}$.

Using Poincaré inequality, Caccioppoli inequality (8) and Gehring-Giaquinta-Modica Lemma (see [1] p. 125, [10], [13]), we are in position to derive the interior higher summability of the gradient.

THEOREM 2.2. – *Assume that conditions* (4), (5), (6) *and* (7) *are fulfilled. Let* $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$, *and let* $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q > 1$, *be a solution of the strongly elliptic system* (2). *Then there exists a number* $\tilde{r} > 1$ *such that* $Du \in L_{loc}^{q\tilde{r}}(\Omega)$ *and* $\forall B(2\sigma) \subset \Omega$ *it results*

$$\left(\fint_{B(\sigma)} \|Dw\|^{q\tilde{r}} dx \right)^{\frac{1}{\tilde{r}}} \leq K \fint_{B(2\sigma)} \|Dw\|^q dx + K \left(\fint_{B(2\sigma)} (1 + \|Dg\|)^{q\tilde{r}} dx \right)^{\frac{1}{\tilde{r}}} \tag{9}$$

where the constant K *does not depend on* σ.

We remark that the proofs of Theorems 2.1, 2.2 are different in the case $1 < q < 2$ with respect to the case $q \geq 2$ in virtue of the effect of degeneration.

In the second step of the proof of the global higher summability, we prove a Caccioppoli type inequality near the boundary, from which it follows the higher summability up to the boundary for the gradient of a solution to Dirichlet Problem.

THEOREM 2.3. – *Assume that conditions* (4), (5), (6) *and* (7) *are fulfilled and* $g \in H^{1,s}(B^+(1)) \cap L^\infty(B^+(1))$, *with* $s > q > 1$. *If* $w \in H^{1,q}(B^+(1)) \cap L^\infty(B^+(1))$ *is a solution of the strongly elliptic problem*

$$(10)\qquad \begin{cases} \displaystyle\int_{B^+(1)} \sum_{i=1}^{n} \big(a^i(x, w+g, Dw+Dg)|D_i\varphi\big)dx = \\ \displaystyle\int_{B^+(1)} (B(x, w+g, Dw+Dg)|\varphi)dx, \quad \forall \varphi \in H_0^{1,q}(B^+(1)) \cap L^\infty(B^+(1)) \\ w(x) = 0 \quad \text{on } \Gamma, \end{cases}$$

then there exists a number $r' > 1$ *such that* $Dw \in L^{qr'}_{loc}(B^+(1))$ *and for all* $B^+(2\sigma) \subset B^+(1)$ *it results*

$$(11)\qquad \left(\fint_{B^+(\sigma)} \|Dw\|^{qr'} dx \right)^{\frac{1}{r'}} \leq K \fint_{B^+(2\sigma)} \|Dw\|^q dx + K \left(\fint_{B^+(2\sigma)} (1 + \|Dg\|^{qr'})\, dx \right)^{\frac{1}{r'}}$$

where K *is a positive constant which does not depend on* σ.

Finally, taking into account that $\partial\Omega$ is of class C^2, it is enough to use the usual covering procedure (see [5] Lemma 2.IV, 2.V and Section n.8 for details) in order to derive the global higher summability of the gradient as claimed in Theorems 1.1 and 1.2.

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D.I.M.E.T. Faculty of Engineering
University of Reggio Calabria
Loc. Feo di Vito, 89060 Reggio Calabria, Italy
sofia.giuffre@unirc.it, idone@ing.unirc.it

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