Bollettino Unione Matematica Italiana

L. M. Ezquerro, M. Gómez-Fernández, X. Soler-Escrivà

On lattice properties of S-permutably embedded subgroups of finite soluble groups

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **8-B** (2005), n.2, p. 505–517.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2005_8_8B_2_505_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2005.

Bollettino U. M. I.
(8) 8-B (2005), 505-517

On Lattice Properties of S-Permutably Embedded Subgroups of Finite Soluble Groups.

L. M. Ezquerro - M. Gómez-Fernández - X. Soler-Escrivà

Sunto. – *In questo lavoro proviamo i seguenti risultati. Sia* π *un insieme di numeri primi e* G *un* π*-gruppo risolubile. Consideriamo* $U, V \leq G$ *e* $H \in \mathrm{Hall}_\pi(G)$ *tali che* $H \cap V \in \mathrm{Hall}_\pi(V)$ *e* $1 \neq H \cap U \in \mathrm{Hall}_\pi(U)$*. Supponiamo anche che* $H \cap U$ *sia un* π*-sottogruppo di Hall di un sottogruppo S-permutabile di G. Allora* $H \cap U \cap V \in \mathrm{Hall}_\pi(U \cap V)$ *e* $\langle H \cap U, H \cap V \rangle \in \mathrm{Hall}_\pi(\langle U, V \rangle)$.
Oltre a ciò, l'insieme di tutti i sottogruppi S-permutabilmente immersi di un gruppo risolubile G in cui un dato sistema di Hall Σ *si riduce è un sottoreticolo del reticolo di tutti i sottogruppi* Σ*-permutabili di G. Si verifica anche che due qualsiasi sottogruppi di questo reticolo di ordini primi fra loro permutano.*

Summary. – *In this paper we prove the following results. Let* π *be a set of prime numbers and* G *a finite* π*-soluble group. Consider* $U, V \leq G$ *and* $H \in \mathrm{Hall}_\pi(G)$ *such that* $H \cap V \in \mathrm{Hall}_\pi(V)$ *and* $1 \neq H \cap U \in \mathrm{Hall}_\pi(U)$*. Suppose also* $H \cap U$ *is a Hall* π*-subgroup of some S-permutable subgroup of G. Then* $H \cap U \cap V \in \mathrm{Hall}_\pi(U \cap V)$ *and* $\langle H \cap U, H \cap V \rangle \in \mathrm{Hall}_\pi(\langle U, V \rangle)$.
Therefore, the set of all S-permutably embedded subgroups of a soluble group G into which a given Hall system Σ *reduces is a sublattice of the lattice of all* Σ*-permutable subgroups of G. Moreover any two subgroups of this sublattice of coprime orders permute.*

Introduction and statement of results.

All groups considered are finite.

In the last years, many papers have been interested in finding out some sufficient conditions to obtain affirmative answers of these questions

Questions. – *Let* π *be a set of prime numbers. Suppose that* H *is a Hall* π*-subgroup of a* π*-soluble group* G*, and* U *and* V *are two subgroups of* G *such that* H *contains Hall* π*-subgroups of* U *and* V.

i) *Does* H *contain a Hall* π*-subgroup of* $U \cap V$*?*

ii) *Does* H *contain a Hall* π*-subgroup of* $\langle U, V \rangle$*?*

It is well-known that if U and V permute, i. e. if UV is a subgroup of G, then the answer to both questions is affirmative (see [8; I,4 Th. 22(b) and Exercise 5]). However, in general the answer is negative in both cases (see [7; Beispiel 1]). In most papers (see [7], [3], [2] and [5]) we observe that some embedding properties like normal embedding or permutable embedding of U and V in G play an important role to ensure positive answers to our questions.

On the other hand, **S-permutable** (or **S-quasinormal**) subgroups of a group, or subgroups which permute with every Sylow subgroup of the group were introduced by Kegel in [12] and have been analyzed quite extensively by some authors (see [6] and [13]). S-permutable subgroups form a sublattice of the lattice of subnormal subgroups ([12; Satz 2]). Moreover the quotient group of an S-permutable subgroup over its core in the group is a nilpotent group (see [6; Th. 1]) and [13; Prop. A]).

Our main interest here is to prove the following.

THEOREM A. – *Let π be a set of prime numbers and G a π-soluble group. Suppose that U is a subgroup of G such that every Hall π-subgroup of U is a Hall π-subgroup of some subnormal subgroup of G. Assume that V is a subgroup of G and H is a Hall π-subgroup of G such that $H \cap U \in \mathrm{Hall}_\pi(U)$ and $H \cap V \in \mathrm{Hall}_\pi(V)$.*

Then $H \cap U \cap V \in \mathrm{Hall}_\pi(U \cap V)$.

THEOREM B. – *Let π be a set of prime numbers and G a π-soluble group. Suppose that U is a subgroup of G whose Hall π-subgroups are non-trivial and such that every Hall π-subgroup of U is a Hall π-subgroup of some S-permutable subgroup of G. Assume that V is a subgroup of G and H is a Hall π-subgroup of G such that $1 \neq H \cap U \in \mathrm{Hall}_\pi(U)$ and $H \cap V \in \mathrm{Hall}_\pi(V)$.*

Then $\langle H \cap U, H \cap V\rangle \in \mathrm{Hall}_\pi(\langle U, V\rangle)$.

Theorem A improves the main result of [7] and Theorem 5 of [2]. Theorem 5 of [5] is improved by Theorem B.

Observe that the hypotheses in Theorem B are stronger than in Theorem A. We will see some examples after the proof of Theorem B to show that these extra conditions are necessary.

Motivated by these satisfactory results, we focus our attention to subgroups whose Sylow subgroups are also Sylow subgroups of some S-permutable subgroups.

DEFINITION. (See [4]). – *Let G be a group. A subgroup V of a group G is said to be an* ***S-permutably embedded subgroup****, or an* ***S-quasinormally embedded subgroup****, of G if for each prime p dividing the order of V, a Sylow p-subgroup of V is also a Sylow p-subgroup of some S-permutable subgroup of G.*

Normally embedded subgroups and permutably embedded subgroups (see [2]) are trivial examples of S-permutably embedded subgroups. Notice that any non-normal subgroup of a p-group is an example of a non-pronormal S-permutably embedded subgroup. In [1] and [4], it becomes clear the influence in the structure of the group of the fact that some relevant subgroups are S-permutably embedded in the whole group.

Recall that if G is soluble and Σ is a Hall system of G, then the set of all subgroups U of G such that U permutes with all members of Σ, i. e. the set $\mathcal{P}(\Sigma)$ of all Σ-permutable subgroups of G, is a lattice (see [8; I,4.29]). The set $\mathcal{N}(\Sigma)$ of all normally embedded subgroups U of G such that Σ reduces into U is a sublattice of $\mathcal{P}(\Sigma)$ and any two subgroups of $\mathcal{N}(\Sigma)$ permute (see [8; I,7.10]). We prove the following.

THEOREM C. -- *Let G be a soluble group and Σ a Hall system of G.*

The set $\mathcal{SP}(\Sigma)$ of all S-permutably embedded subgroups of G into which Σ reduces is a sublattice of $\mathcal{P}(\Sigma)$, the lattice of all Σ-permutable subgroups of G.

Moreover, if $U, V \in \mathcal{SP}(\Sigma)$ and U and V have coprime orders, then $UV = VU$.

For permutably embedded subgroups, the reference is the elegant paper [2] by A. Ballester-Bolinches. In this paper the author remarks that if U and V are two permutably embedded subgroups of a soluble group G, then $U \cap V$ is not a permutably embedded subgroup in general. An example due to N. Itô, which appears in [12] and in [6], shows a p-group, for p a prime, and two permutable subgroups whose intersection is not permutable in the group. Hence, in general, the set of all permutably embedded subgroups into which Σ reduces is not a lattice.

Furthermore, in [2; Th. 1] it is proved that if U and V are permutably embedded subgroups of a soluble group G such that there exists a Hall system Σ of G which reduces into U and V, then U always permutes with V. The corresponding result for S-permutably embedded subgroups is not true in general. All subgroups of a p-group G, p a prime, are trivially S-permutable in G and they do not permute in general.

The following Lemma will be very useful in inductive arguments. Its proof is a routine checking.

LEMMA 1. (see [4; Lemma 1]). -- *Let G be a group, two subgroups V and M of G and a normal subgroup K of G. Then we have*

i) *if V is S-permutably embedded in G and $V \leq M$, then V is S-permutably embedded in M;*

ii) *if V is S-permutably embedded in G, then VK is S-permutably embedded in G and VK/K is S-permutably embedded in G/K;*

iii) *if* $K \leq V$ *and* V/K *is S-permutably embedded in* G/K*, then* V *is S-permutably embedded in* G.

The proofs.

Before starting the proofs we make with some previous considerations and calculations. Let N be a normal subgroup of G and consider the quotient group $\overline{G} = G/N$. Denote with a bar the epimorphism $G \to \overline{G} = G/N$, i. e. if X is a subgroup of G then $\overline{X} = XN/N$. Clearly $\overline{H} \in \mathrm{Hall}_\pi(\overline{G})$. Suppose that T is a subnormal subgroup of G such that $H \cap U = H \cap T \in \mathrm{Hall}_\pi(T)$.

Observe that if X is any subgroup of G such that $H \cap X \in \mathrm{Hall}_\pi(X)$, then $\overline{H \cap X} \in \mathrm{Hall}_\pi(\overline{X})$, and therefore $\overline{H} \cap \overline{X} = \overline{H \cap X}$. In particular $\overline{H} \cap \overline{U} = \overline{H \cap U} \in \mathrm{Hall}_\pi(\overline{U})$ and $\overline{H} \cap \overline{V} = \overline{H \cap V} \in \mathrm{Hall}_\pi(\overline{V})$. Analogously $\overline{T}$ is a subnormal subgroup of $\overline{G}$ and $\overline{H} \cap \overline{T} = \overline{H \cap T} \in \mathrm{Hall}_\pi(\overline{T})$.

For the intersection, notice that

$$\overline{H} \cap \overline{U} \cap \overline{V} = \overline{H \cap U} \cap \overline{H \cap V} = \overline{H \cap U \cap (H \cap V)N}$$

and $\overline{U} \cap \overline{V} = \overline{U \cap VN}$, by the Dedekind law. Moreover

$$(1) \qquad |\overline{U} \cap \overline{V} : \overline{H} \cap \overline{U} \cap \overline{V}| = |\overline{U \cap VN} : \overline{H \cap U \cap (H \cap V)N}|$$

$$= |(U \cap VN)N : (H \cap U \cap (H \cap V)N)N| = \frac{|U \cap VN : H \cap U \cap (H \cap V)N|}{|U \cap N : H \cap U \cap N|}.$$

Concerning the join, we observe that $\langle \overline{H} \cap \overline{U}, \overline{H} \cap \overline{V} \rangle = \overline{\langle H \cap U, H \cap V \rangle}$ and $\langle \overline{U}, \overline{V} \rangle = \overline{\langle U, V \rangle}$. Moreover

$$(2) \qquad |\langle \overline{U}, \overline{V} \rangle : \langle \overline{H} \cap \overline{U}, \overline{H} \cap \overline{V} \rangle| = |\overline{\langle U, V \rangle} : \overline{\langle H \cap U, H \cap V \rangle}|$$

$$|\langle U, V \rangle N : \langle H \cap U, H \cap V \rangle N| = \frac{|\langle U, V \rangle : \langle H \cap U, H \cap V \rangle|}{|\langle U, V \rangle \cap N : \langle H \cap U, H \cap V \rangle \cap N|}.$$

Proof of Theorem A.

Assume that the result is false and let G be a counterexample of minimal order. Then there exist a Hall π-subgroup H of G and an ordered pair of subgroups (A, B) such that every Hall π-subgroup of A is a Hall π-subgroup of some subnormal subgroup of G, $H \cap A \in \mathrm{Hall}_\pi(A)$ and $H \cap B \in \mathrm{Hall}_\pi(B)$ but $H \cap A \cap B \notin \mathrm{Hall}_\pi(A \cap B)$. Among all such pairs of subgroups, we choose (U, V) such that the pair $(|G : U \cap V|, |U| + |V|)$ is minimal with respect to the lexicographical order.

Let N be any minimal normal subgroup of G. Since G is π-soluble, then N is either a π-group or a π'-group. All hypotheses hold in $\overline{G}$. By minimality of G, the

Theorem is true for the quotient group $\overline{G} = G/N$. Therefore $|\overline{U} \cap \overline{V} : \overline{H} \cap \overline{U} \cap \overline{V}|$ is a π'-number.

If N is a π'-group, then $H \cap V \in \mathrm{Hall}_\pi(V) \subseteq \mathrm{Hall}_\pi(VN)$ and $H \cap V = H \cap (H \cap V)N$. Now, in (1) we have that $|U \cap VN : H \cap U \cap (H \cap V)N| = |U \cap VN : H \cap U \cap V|$ is a π'-number and so is $|U \cap V : H \cap U \cap V|$; in other words $H \cap U \cap V \in \mathrm{Hall}_\pi(U \cap V)$. This is a contradiction. Therefore $O_{\pi'}(G) = 1$. Thus, N is a π-group. Suppose that $N \leq U$. Then $N \leq H \cap U$. In this case in (1) we have that

$$|\overline{U} \cap \overline{V} : \overline{H} \cap \overline{U} \cap \overline{V}| = |U \cap VN : (H \cap U) \cap (H \cap V)N|$$
$$= |(U \cap V)N : (H \cap U \cap V)N|$$

is a π'-number. A simple calculation shows that $|(U \cap V)N : (H \cap U \cap V)N| = |U \cap V : H \cap U \cap V|$ and then $H \cap U \cap V \in \mathrm{Hall}_\pi(U \cap V)$. This is a contradiction. Hence U is a core-free subgroup of G.

Suppose now that $U \cap V = UN \cap V$. It is easy to see that the pair (UN, VN) satisfies the hypotheses of the Theorem. Observe that since U is core-free in G, we have that $U \cap V$ is a proper subgroup of $N(U \cap V)$. Hence $U \cap V < N(U \cap V) = N(UN \cap V) = UN \cap VN$. By the minimal election of the pair (U, V), we have that $H \cap UN \cap VN = H \cap N(U \cap V) = N(H \cap U \cap V)$ is a Hall π-subgroup of $UN \cap VN = N(U \cap V)$. Since $N \leq H$, this implies that $H \cap U \cap V$ is a Hall π-subgroup of $U \cap V$, a contradiction.

Therefore $U \cap V$ is a proper subgroup of $UN \cap V$. Now, we notice that the pair (UN, V) satisfies the hypotheses of the Theorem and by the minimal election of the pair (U, V), we have that $H \cap UN \cap V \in \mathrm{Hall}_\pi(UN \cap V)$. This implies that the pair $(U, UN \cap V)$ satisfies the hypotheses of the Theorem. If $UN \cap V$ is a proper subgroup of V, then, by minimality of (U, V), we have that $H \cap U \cap UN \cap V = H \cap U \cap V$ is a Hall π-subgroup of $U \cap UN \cap V = U \cap V$, a contradiction. Therefore $UN \cap V = V$, or, in other words $V \leq UN$.

Notice that $H \cap UN = (H \cap U)N \in \mathrm{Hall}_\pi(UN)$. If UN were a proper subgroup of G, then, by minimality of G, we would reach a contradiction. Therefore $G = UN$, with N a soluble minimal normal subgroup of G. Since U is core-free in G, we have indeed that U is maximal in G and G is a primitive group with abelian socle. Then, the minimal normal subgroup N is self-centralizing in G.

Let T be a subnormal subgroup of G such that $H \cap U \in \mathrm{Hall}_\pi(T)$. It is clear that $O_\pi(T) \leq H \cap U$. If p is the prime dividing $|N|$, then $N = O_p(G)$, and $N \cap T = O_p(T)$. But observe that $O_p(T) \leq N \cap O_\pi(T) \leq N \cap U = 1$. Hence $N \cap T = 1$. By a well-known theorem of Wielandt (see [15; Satz 12.8, p. 454] or [8; A,14.3]), the minimal normal subgroup N normalizes T. Hence $NT = N \times T$. This implies that $T \leq C_G(N) = N$. Hence $T = 1$. But then U is a π'-group and so is $U \cap V$. Trivially in this case $1 = H \cap U \cap V$ is a Hall π-subgroup of $U \cap V$. This is the final contradiction. ∎

Proof of Theorem B.

Assume that the result is false and let G be a counterexample of minimal order. Then there exist a Hall π-subgroup H of G, and an ordered pair of subgroups (A, B) such that every Hall π-subgroup of A is a Hall π-subgroup of some S-permutable subgroup of G, $1 \neq H \cap A \in \mathrm{Hall}_\pi(A)$ and $H \cap B \in \mathrm{Hall}_\pi(B)$ but $\langle H \cap A, H \cap B\rangle \notin \mathrm{Hall}_\pi(\langle A, B\rangle)$. Among all such pairs of subgroups, we choose (U, V) such that $|G : U| + |V|$ is minimal.

Let N be any minimal normal subgroup of G. Since G is π-soluble, then N is either a p-group, for some prime $p \in \pi$, or a π'-group. All hypotheses hold in $\overline{G} = G/N$. By minimality of G, the Theorem is true the for quotient group $\overline{G}$. Therefore $|\langle \overline{U}, \overline{V}\rangle : \langle \overline{H} \cap \overline{U}, \overline{H} \cap \overline{V}\rangle|$ is a π'-number.

If N is a π'-group, then $|\langle U, V\rangle \cap N : \langle H \cap U, H \cap V\rangle \cap N| = |\langle U, V\rangle \cap N|$ is a π'-number. In (2) we have that $|\langle U, V\rangle : \langle H \cap U, H \cap V\rangle|$ is a π'-number. Then $\langle H \cap U, H \cap V\rangle \in \mathrm{Hall}_\pi(\langle U, V\rangle)$. This is a contradiction. Therefore $O_{\pi'}(G) = 1$.

Thus N is a p-group, for some prime $p \in \pi$. Suppose that $N \leq U$. Then $N \leq H \cap U$. Now in (2) we have that

$$|\langle \overline{U}, \overline{V}\rangle : \langle \overline{H} \cap \overline{U}, \overline{H} \cap \overline{V}\rangle| = |\langle U, V\rangle : \langle H \cap U, H \cap V\rangle|$$

is a π'-number and $\langle H \cap U, H \cap V\rangle \in \mathrm{Hall}_\pi(\langle U, V\rangle)$. This is a contradiction. Hence U is a core-free subgroup of G. Suppose that T is an S-permutable subgroup of G such that $H \cap U = H \cap T \in \mathrm{Hall}_\pi(T)$.

If $N \leq T$, then $N \leq H \cap T = H \cap U \leq U$, a contradiction. Hence T is core-free in G and therefore T is nilpotent. Moreover, since T is subnormal and $O_{\pi'}(G) = 1$, we have that $O_{\pi'}(T) = 1$. Hence T is a nilpotent π-group. This is to say that $H \cap U = T$ is an S-permutable subgroup of G. In particular, $H \cap U$ is subnormal in G and is the normal Hall π-subgroup of U.

Consider now the subgroup $M = \langle U, V\rangle N$. Observe that

$$H^* = \langle H \cap U, H \cap V\rangle N$$

is a Hall π-subgroup of M, by the above-exposed calculations. Moreover $H^* \cap U = H \cap U$ and $H^* \cap V = H \cap V$. Furthermore, since $T = H \cap U \leq M$, then T is an S-permutable subgroup of M. If M is a proper subgroup of G, then, by minimality of G, we have that $\langle H \cap U, H \cap V\rangle \in \mathrm{Hall}_\pi(\langle U, V\rangle)$, a contradiction. Hence $G = \langle U, V\rangle N$ and $H^* = H$.

Suppose that $\langle U, V\rangle$ is a proper subgroup of G. Since, by the above arguments, $G = \langle U, V\rangle N$ for any minimal normal subgroup N of G, we have that $\langle U, V\rangle$ is a core-free maximal subgroup of G and G is a primitive group. Hence there exists a unique minimal normal subgroup N in G such that N is soluble and $\langle U, V\rangle \cap N = 1 = \langle H \cap U, H \cap V\rangle \cap N$. Therefore $|G : H| = |\langle U, V\rangle : \langle H \cap U, H \cap V\rangle|$ is a π'-number. This is a contradiction. Therefore $G = \langle U, V\rangle$.

Denote $U_\pi = U \cap H$ and $V_\pi = V \cap H$. Notice that the Hall π-subgroup H of G is factorized as $H = \langle U_\pi, V_\pi \rangle N$ for every minimal normal subgroup N of G. In particular $\langle U_\pi, V_\pi \rangle$ is a core-free subgroup of G.

Let $U_{\pi'}$ be a Hall π-complement of U. Let $G_{\pi'}$ be a Hall π'-subgroup of G such that $U_{\pi'} \leq G_{\pi'}$. Since U_π is S-permutable in G, $U^* = U_\pi G_{\pi'}$ is a subgroup of G. Suppose that $U < U^*$. Observe that $H \cap U^* = U_\pi \in \mathrm{Hall}_\pi(U^*)$. Then by minimality of $|G : U| + |V|$, we have that $\langle H \cap U, H \cap V \rangle \in \mathrm{Hall}_\pi(G)$, a contradiction. Hence $U = U^* = U_\pi G_{\pi'}$. Observe that U_π is normalized by $G_{\pi'}$.

Suppose that $V_\pi < V$. By minimality of $|G : U| + |V|$, we have that $\langle U_\pi, V_\pi \rangle \in \mathrm{Hall}_\pi(\langle U, V_\pi \rangle)$. Write $\langle U, V_\pi \rangle = \langle U_\pi, V_\pi \rangle G_{\pi'} = X$ and observe that $XN = HG_{\pi'} = G$. If $G = X$, then $\langle U_\pi, V_\pi \rangle \in \mathrm{Hall}_\pi(G)$, a contradiction. Hence X is a maximal subgroup of G. This implies that $X \cap N = 1$. This is true for any minimal normal subgroup N of G. Hence X is core-free in G and G is primitive. Since U_π is a nontrivial subnormal π-subgroup of G, then $1 \neq U_\pi \leq X \cap O_\pi(G) = 1$, a contradiction. Hence $V = V_\pi$, i. e. V is a π-group.

Observe that $G = \langle U, V \rangle = \langle U_\pi, V, G_{\pi'} \rangle$. Since $G_{\pi'}$ normalizes U_π, the normal closure of U_π in G, $\langle U_\pi \rangle^G$, is a subgroup of $\langle U_\pi, V \rangle = \langle U_\pi, V_\pi \rangle$. Since $U_\pi \neq 1$, then $\langle U_\pi \rangle^G$ is a nontrivial normal subgroup. But $\langle U_\pi, V_\pi \rangle$ is core-free in G. This is the final contradiction. ■

Remarks and examples.

(1) Theorem B fails when U is a π'-group.

Consider the alternating group of 4 letters $G = Alt(4)$ and the subgroups $U = \langle (123) \rangle$, isomorphic to a cyclic group of order 3, and $V = \langle (12)(34) \rangle$, isomorphic to a group of order 2. Let P be any Sylow 2-subgroup of G. Then $1 = P \cap U \in \mathrm{Syl}_2(U)$ but all the other hypotheses of Theorem B are fulfilled. Observe that $G = \langle U, V \rangle$. Clearly $\langle P \cap U, P \cap V \rangle = V \notin \mathrm{Syl}_2(G)$.

(2) Theorem B does not hold if we assume that every Hall π-subgroup of U is a Hall π-subgroup of some subnormal subgroup of G as in Theorem A. The following example shows that we cannot deduce even the weaker conclusion $H \cap \langle U, V \rangle \in \mathrm{Hall}_\pi(\langle U, V \rangle)$.

Consider the alternating group A on 4 letters. Then A is generated by two elements $a = (12)(34)$ and $c = (123)$. If we write $b = a^c = (23)(14)$, the subgroup $Q = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle$ is the unique minimal normal subgroup of A.There exists an irreducible and faithful A-module W over GF(3) (see [8; B,10.7]). Construct the semidirect product $S = [W]A$. If $C = \langle d : d^5 = 1 \rangle \cong C_5$ is a cyclic group of order 5, we consider the wreath product $G = C \wr S$ with respect to the regular action. Then $G = [C^\natural]S$ and $C^\natural$ is an $|S|$-dimensional S-module over GF(5) with the following action: there exists a basis $\{d_x : x \in S\}$, such that $d_x^y = d_{xy}$, for each $y \in S$.

Consider the subnormal subgroup $T = [C^\natural]W\langle a\rangle$. Clearly $U = \langle a\rangle \in \mathrm{Syl}_2(T)$. Therefore U is a Sylow 2-subgroup of a subnormal subgroup of G. Consider $V \in \mathrm{Syl}_3(S)$. It is clear that $S = \langle U, V\rangle$. Consider the element $g = d_1 d_a \in C^\natural$. Observe that $g^a = d_a d_{a^2} = d_a d_1 = g$, i. e. $g \in C_G(a)$. Then $Q^g = \langle a, b^g\rangle$ is a Sylow 2-subgroup of G such that $Q^g \cap Q = U$, since $b^g \notin Q$. We have that U is a Sylow 2-subgroup of a subnormal subgroup of G, $Q^g \cap U = U \in \mathrm{Syl}_2(U)$ and $Q^g \cap V = 1 \in \mathrm{Syl}_2(V)$. However $U = Q^g \cap S = Q^g \cap \langle U, V\rangle = \langle Q^g \cap U, Q^g \cap V\rangle \notin \mathrm{Syl}_2(\langle U, V\rangle)$.

Proof of Theorem C. (1) The set $\mathcal{SP}(\Sigma)$ of all S-permutably embedded subgroups of G into which Σ reduces is a lattice.

Let U and V be two S-permutably embedded subgroups of G into which Σ reduces. If p is a prime dividing $|G|$, denote by P the Sylow p-subgroup of G in Σ. Let T^p be an S-permutable subgroup of G such that $P \cap T^p = P \cap U$ and let S^p be an S-permutable subgroup of G such that $P \cap S^p = P \cap V$. Since S-permutable subgroups form a lattice, by [12; Satz 2], the subgroups $\langle S^p, T^p\rangle$ and $T^p \cap S^p$ are S-permutable in G.

By Theorem A, $P \cap U \cap V = P \cap T^p \cap S^p$ is a Sylow p-subgroup of $U \cap V$ and of $T^p \cap S^p$. This is true for any prime p and then the subgroup $U \cap V$ is S-permutably embedded in G.

If U and V are p'-groups, then $\langle U, V\rangle \leq G_{p'}$, where $G_{p'}$ is the Hall p-complement of G in Σ. Hence, if p divides the order of $\langle U, V\rangle$, then p divides the order of U or the order of V.

Thus, let p be a prime dividing the order of $\langle U, V\rangle$. Since either $P \cap U \neq 1$ or $P \cap V \neq 1$, we can apply Theorem B and deduce that $\langle P \cap U, P \cap V\rangle = \langle P \cap T^p, P \cap S^p\rangle$ is a Sylow p-subgroup of $\langle U, V\rangle$ and of $\langle T^p, S^p\rangle$. Hence the subgroup $\langle U, V\rangle$ is S-permutably embedded in G.

Therefore, the set $\mathcal{SP}(\Sigma)$ of all S-permutably embedded subgroups of G into which Σ reduces is a lattice.

(2) If U and V are subgroups in $\mathcal{SP}(\Sigma)$ with coprime orders, then U permutes with V.

Suppose that the result is not true and consider a soluble group G, minimal counterexample. Let $\mathcal{S}$ be the non-empty set composed by all pairs of subgroups $\{U, V\}$ of G such that

i) U and V are S-permutably embedded subgroups of G,
ii) $\gcd(|U|, |V|) = 1$,
iii) there exists a Hall system Σ of G which reduces into U and V,
iv) UV is not a subgroup of G.

We consider a pair $\{U, V\} \in \mathcal{S}$ such that $|U| + |V|$ is minimum.

Let N be any minimal normal subgroup of G. All hypotheses hold in the quotient group G/N. Therefore, by minimality of G, the subgroup UN/N

permutes with VN/N. That means that $(UN)(VN) = V(UN) = H$ is a subgroup of G. By Lemma 1(i) the subgroups U and V are S-permutably embedded in H. Since by ([8; I,4.22(b)]) Σ reduces into H, all hypotheses of the Theorem hold in H. Consequently, if H is a proper subgroup of G, the subgroups U and V permute, by minimality of G. But this cannot occur and this means that $G = H = V(UN)$.

Let q be a prime dividing $|V|$ and consider the Sylow q-subgroup V_q of V which is in $\Sigma \cap V$. If V_q is a proper subgroup of V, then by minimality of $|U| + |V|$, it follows that V_q permute with U. Therefore all Sylow subgroups of V permute with U and then V permute with U. Thus necessarily, V is a q-group.

Analogously, U is an r-group, for some prime r such that $r \neq q$.

Now we fix a minimal normal subgroup N of G and suppose that N is a p-group, for a prime p. Since U and V have coprime orders, we can assume that V is a p'-group, that is, $p \neq q$.

If $p \neq r$, then $U \in \mathrm{Syl}_r(G)$ and $V \in \mathrm{Syl}_q(G)$, since $G = V(UN)$. Moreover $U, V \in \Sigma$. Hence U and V permute, a contradiction. Therefore $r = p$ and U is a p-group. This implies that $UN \in \mathrm{Syl}_p(G)$ and $V \in \mathrm{Syl}_q(G)$. Since U is S-permutably embedded in G, there exists an S-permutable subgroup T of G such that $U \in \mathrm{Syl}_p(T)$. It follows that TV is a subgroup of G. Notice that

$$|TV : U| = |TV : T||T : U|$$

is a q-power. Therefore $U \in \mathrm{Syl}_p(TV)$. Moreover, $V \in \mathrm{Syl}_q(TV)$. Then $TV = UV$ is a subgroup of G. This is the final contradiction.

(3) If U is an S-permutably embedded subgroup of G such that Σ reduces into U, then U permutes with each member of Σ, i. e. U is Σ-permutable.

Let G_π and $G_{\pi'}$ denote the Hall π-subgroup and the Hall π'-subgroup respectively of G in Σ, for any set π of prime numbers. Then, by (2), we have that $\langle U, G_\pi \rangle = \langle U \cap G_{\pi'}, G_\pi \rangle = (U \cap G_{\pi'})G_\pi = UG_\pi$. Hence U permutes with all members of Σ, that is U is Σ-permutable. ■

Remarks and examples.

(1) In the proof of Theorem C we need permutability with at least Sylow subgroups. A similar result for subnormally embedded subgroups is not true.

In $G = Alt(4)$, the alternating group of degree 4, if Σ is a Hall system of G, then the Sylow 3-subgroup of G in Σ and any subgroup of order 2 are Sylow subgroups of subnormal subgroups of G into which Σ reduces. However, G possesses no subgroups of order 6.

(2) In the general non-soluble case we have that a Sylow 5-subgroup and a Sylow 2-subgroup of the alternating group of degree 5 are S-permutably embedded subgroups of $Alt(5)$ of coprime orders and they do not permute.

Some final considerations and remarks

The fact that two S-permutably embedded subgroups of coprime orders such that the same Hall system reduces into both permute, by Theorem C, allows us to describe all S-permutably embedded subgroups of a soluble group. In [8; I,7. Exercise 4] appears a method, due to Fischer, to describe the set of all normally embedded subgroups of a soluble group. Later, an alternative description is given in [9]. For permutably embedded subgroups, Ballester-Bolinches gives in [2; Th. 2] a similar characterization. We can obtain an analogous result for S-permutably embedded subgroups of finite soluble groups.

PROPOSITION 1. – *Let G be a soluble group.*

A subgroup V is S-permutably embedded in G if and only if for every Hall system $\Sigma = \{G_\pi \ / \ \pi \subseteq \pi(G)\}$ such that Σ reduces into V, there exists a family of S-permutable subgroups $\{T^p \ / \ p \in \pi(G)\}$ of G such that

$$V = \prod_{p \| G|} (G_p \cap T^p).$$

The proof runs parallel to the one in [9] or in [2; Th. 2] taking into account the factorization of two S-permutably embedded subgroups of coprime orders proved in Theorem C. ■

We can also obtain a «dual» description. A description of normally embedded subgroups in a class $\mathcal{U}$ of non-necessarily finite groups appears in [14; Lemma 2.5]. The finite soluble groups are exactly all finite groups in the class $\mathcal{U}$. Therefore this Tomkinson description holds for finite soluble groups. We extend this result to S-permutably embedded subgroups of finite soluble groups.

PROPOSITION 2. – *Let G be a soluble group. A subgroup V is S-permutably embedded in G if and only if for every Hall system $\Sigma = \{G_\pi \ / \ \pi \subseteq \pi(G)\}$ such that Σ reduces into V, there exists a family of S-permutable subgroups $\{T^p \ / \ p \in \pi(G)\}$ of G such that*

$$V = \bigcap_{p \| G|} G_{p'} T^p.$$

Moreover, for every p, T^p is such that $V_p = V \cap G_p \in Syl_p(V) \cap Syl_p(T^p)$.

PROOF. – Let us suppose that V is S-permutably embedded in G and consider $\Sigma = \{G_\pi \ / \ \pi \subseteq \pi(G)\}$ be a Hall system of G such that Σ reduces into V. Write $V_p = V \cap G_p$ for every prime p dividing $|G|$.

For every $p \in \pi(G)$, we choose an S-permutable subgroup T^p of G such that $V_p \in Syl_p(V) \cap Syl_p(T^p)$. Since T^p is subnormal in G, every Hall system reduces into it and we have $V_p = V \cap G_p = T^p \cap G_p$.

On the other hand, since $G_{p'}$ is a product of Sylow subgroups of G, it follows that T^p permutes with $G_{p'}$ and Σ reduces into $T^pG_{p'}$. Thus, $T^pG_{p'} \cap G_p = T^p \cap G_p$ is a Sylow p-subgroup of $T^pG_{p'}$. Now, since Σ reduces into the p-group $T^pG_{p'} \cap G_p$, it follows that

$$T^pG_{p'} \cap G_p \leq G_{q'} \leq T^qG_{q'},$$

for any prime $q \neq p$. Therefore,

$$T^pG_{p'} \cap G_p \leq \bigcap_{q \neq p} T^qG_{q'} \qquad \text{and} \qquad T^pG_{p'} \cap G_p = \left(\bigcap_{q \| G|} T^qG_{q'} \right) \cap G_p.$$

Then, we have

$$\bigcap_{q \| G|} T^qG_{q'} = \prod_{p \| G|} \left[\left(\bigcap_{q \| G|} T^qG_{q'} \right) \cap G_p \right] = \prod_{p \| G|} (T^pG_{p'} \cap G_p) = \prod_{p \| G|} (T^p \cap G_p) = V.$$

Conversely, let $\Sigma = \{G_\pi \ / \ \pi \subseteq \pi(G)\}$ be a Hall system of G and $V = \bigcap_{q \| G|} G_{q'}T^q$, where $\{T^q \ / \ q \in \pi(G)\}$ is a family of S-permutable subgroups of G. Notice that Σ reduces into V. Then, arguing as before, we have

$$V \cap G_p = \left(\bigcap_{q \| G|} G_{q'}T^q \right) \cap G_p = G_{p'}T^p \cap G_p = T^p \cap G_p \in Syl_p(T^p) \cap Syl_p(V).$$

That is, V is S-permutably embedded in G. ■

FINAL REMARK. – We observe that the only properties of S-permutably embedded subgroups we have used in the proof of Theorem C are their definition, the basic embedding properties of Lemma 1 and the fact that S-permutable subgroups form a sublattice of the lattice of subnormal subgroups. This means that Theorem C remains true for some other types of subgroups satisfying the same properties.

More precisely, given a group G, let $\mathcal{L}(G)$ be a sublattice of the lattice of all S-permutable subgroups of G satisfying that

i) if $T \in \mathcal{L}(G)$ and M is a subgroup of G, then $T \cap M \in \mathcal{L}(M)$,

ii) if $T \in \mathcal{L}(G)$ and N is a normal subgroup of G, then $TN/N \in \mathcal{L}(G/N)$.

Consider the set of all **locally $\mathcal{L}(G)$-subgroups**, i. e. the set of all subgroups V of G such that for each prime p dividing the order of V, a Sylow p-subgroup of V is also a Sylow p-subgroup of some subgroup of $\mathcal{L}(G)$.

Then we can deduce with the previous arguments that if G is soluble and Σ is a Hall system of G, then the set of all locally $\mathcal{L}(G)$-subgroups into which Σ reduces is a lattice.

Let us examine an example of such a lattice. Given any group G, if $\mathfrak{F}$ is a saturated formation, the set of all $\mathfrak{F}$-hypercentrally embedded subgroups in G, i. e. those subgroups T whose section $\langle T^G \rangle / Core_G(T)$ is $\mathfrak{F}$-hypercentral in G, is a lattice satisfying (i) and (ii), (see [10]). Hence if G is soluble, the set of all locally S-permutable $\mathfrak{F}$-hypercentrally embedded subgroups into which a fixed Hall system of G reduces, is a lattice.

The particular case of the class $\mathfrak{N}$ of nilpotent groups is especially interesting since all $\mathfrak{N}$-hypercentrally embedded subgroups (or simply hypercentrally embedded subgroups) are S-permut-able (see [13]).

THEOREM C$'$. *Let G be a soluble group and Σ a Hall system of G.*

The set $\mathcal{LZ}(\Sigma)$ of all locally hypercentrally embedded subgroups of G into which Σ reduces is a sublattice of the lattice of all Σ-permutable subgroups of G.

Moreover, if $U, V \in \mathcal{LZ}(\Sigma)$ and have coprime orders, then $UV = VU$.

Acknowledgment. This work was supported by *Proyecto BFM* 2001-1667-C03-01 of Ministerio de Ciencia y Tecnolog a of Spain.

REFERENCES

[1] M. ASAAD - A. A. HELIEL, *On S-quasinormally embedded subgroups of finite groups*, J. Pure App. Algebra, **165**, (2001), 129-135.

[2] A. BALLESTER-BOLINCHES, *Permutably embedded subgroups of finite soluble groups*, Arch. Math., **65** (1995), 1-7.

[3] A. BALLESTER-BOLINCHES - M. D. PÉREZ-RAMOS, *Permutability in finite soluble groups*, Math. Proc. Camb. Phil. Soc., **115** (1993), 393-396.

[4] A. BALLESTER-BOLINCHES - M. C. PEDRAZA-AGUILERA, *Sufficient conditions for supersolubility of finite groups*, J. Pure App. Algebra, **127** (1998), 118-134.

[5] A. BALLESTER-BOLINCHES - L. M. EZQUERRO, *On join properties of Hall π-subgroups of finite π-soluble groups*, J. Algebra, **204** (1998), 532-548.

[6] W. E. DESKINS, *On quasinormal subgroups of finite groups*, Math. Z., **82** (1963), 125-132.

[7] K. DOERK, *Eine Bemerkung ber das Reduziern von Hallgruppen in endlichen auflsbaren Gruppen*, Arch. Math., **60** (1993), 505-507.

[8] K. Doerk - T. HAWKES, *Finite soluble groups*, De Gruyter (Berlin, New York, 1992).

[9] L. M. EZQUERRO, *Una caracterización de los subgrupos inmersos normalmente en grupos finitos resolubles*, Actas XII Jor. Luso-Esp. Mat. Univ. Braga, **2** (1987), 68-71.

[10] L. M. EZQUERRO - X. SOLER-ESCRIV, *Some permutability properties of $\mathfrak{F}$-hypercentrally embedded subgroups of finite groups*. Preprint (to appear in J. Algebra), 2002.

[11] A. FELDMAN, *An intersection property of Sylow p-subgroups affecting p-length in finite p-solvable groups*, J. Algebra, **184** (1996), 251-254.

[12] O. H. KEGEL, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math. Z., **78** (1962), 205-221.

[13] P. SCHMID, *Subgroups permutable with all Sylow subgroups*, J. Algebra, **207** (1998), 285-293.

[14] M. J. TOMKINSON, *Prefrattini subgroups and cover-avoidance properties in U-groups*, Canad. J. Math., **27** (1975), 837-851.

[15] H. WIELANDT, *Subnormale Untergruppen endlicher Gruppen*, Mathematical Works, Vol. I, De Gruyter (Berlin, New York, 1994), 413-479.

L. M. Ezquerro: Departamento de Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía. 31006 Pamplona, Spain.
E-mail: ezquerro@unavarra.es

M. Gómez-Fernández: Departamento de Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía. 31006 Pamplona, Spain.
E-mail: marisol@unavarra.es

Xaro Soler Escrivà: Universitat d'Alacant Dpt. de Matemàtica Aplicada
Campus de Sant Vicent del Raspeig, Ap. Correus 99, E - 03080 Alacant, Spain.

Pervenuta in Redazione
l'1 luglio 2003 e in forma rivista il 27 aprile 2004