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Unitary Groups Acting on Hyperbolic Substructures (*) (**).

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Sunto. – *Data un'estensione quadratica L/K di campi e uno spazio λ-Hermitiano regolare (V, h) di dimensione finita su L, si studiano le orbite del gruppo delle isometrie di (V, h) nell'insieme delle K-sottostrutture iperboliche di V.*

Summary. – *Given a quadratic extension L/K of fields and a regular λ-Hermitian space (V, h) of finite dimension over L, we study the orbits of the group of isometries of (V, h) in the set of hyperbolic K-substructures of V.*

Let K be a field, $L = K(\eta)$ be a quadratic extension of K and σ be the involutory automorphism of L associated with the extension. Also let $\lambda \in L$ with $\lambda\lambda^\sigma = 1$ and (V, h) be a regular λ-Hermitian space of finite dimension over L, i.e. a L-vector space V equipped with a non-degenerate σ-sesquilinear form $h : V \times V \longrightarrow L$ such that $h(x, y) = \lambda h(y, x)^\sigma$.

A *K-substructure* W of V is a K-subspace generated by vectors which are linearly independent over L, i.e. the L-linear map $W \otimes_K L \longrightarrow V$ induced by the inclusion $W \subseteq V$ is injective. According to [3] we shall term the K-substructure W *hyperbolic* if it is the direct sum of two (maximal) totally isotropic K-substructures and there is no nonzero vector in W orthogonal to the whole W. This means that $\dim_K W = 2m$, where, of course, m is not greater than the Witt index of (V, h).

The aim of this paper is to study the natural action of the group $\mathcal{G}(V, h)$ of isometries of (V, h) on the set $\mathcal{X}_{2m}$ of all hyperbolic K-substructures of V of dimension $2m$ over K.

Let $W \in \mathcal{X}_{2m}$ and let

$$W = W_1 \oplus W_2$$

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be a decomposition of W into the direct sum of two totally isotropic K-substructures. The restriction of h to vectors in W yields a K-bilinear mapping $h_{|W} : W \times W \longrightarrow L$ which can be represented in the usual way through the matrix with entries $h(e_i, e_j)$, if we refer to a fixed basis $\mathcal{B} = \{e_1, \dots, e_{2m}\}$ of W.

If $\mathcal{B}$ is obtained by putting together a basis $\mathcal{B}_1$ of W_1 and a basis $\mathcal{B}_2$ of W_2, we have for $h_{|W}$ a representation of the shape

$$H = \begin{pmatrix} \mathbf{0} & P \\ \lambda\, {}^{\mathrm{t}}P^{\sigma} & \mathbf{0} \end{pmatrix}$$

with $P \in GL_m(L)$. Let (P_1, P_2) and (λ_1, λ_2) be the components of P and λ over K, then we have

$$H = \begin{pmatrix} \mathbf{0} & P_1 \\ \lambda_1 {}^{\mathrm{t}}P_1 - \eta^2 \lambda_2 {}^{\mathrm{t}}P_2 & \mathbf{0} \end{pmatrix} + \eta \begin{pmatrix} \mathbf{0} & P_2 \\ \lambda_2 {}^{\mathrm{t}}P_1 - \lambda_1 {}^{\mathrm{t}}P_2 & \mathbf{0} \end{pmatrix}.$$

Let h_i, $i = 1, 2$, be the components of h over K, i.e.

$$h(x, y) = h_1(x, y) + \eta h_2(x, y), \quad \forall\, x, y \in V.$$

Then (h_1, h_2) is a pair of K-bilinear forms on V and we can associate with W the *Kronecker module* $\Phi_W = [W, W^*; \phi_1, \phi_2]$, consisting of the pair of K-linear mappings

$$\phi_i(x) : y \longmapsto h_i(x, y) \quad (x, y \in W,\ i = 1, 2),$$

from W into the dual W^* of W. The pair

$$\left[\begin{pmatrix} \mathbf{0} & P_1 \\ \lambda_1 {}^{\mathrm{t}}P_1 - \eta^2 \lambda_2 {}^{\mathrm{t}}P_2 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & P_2 \\ \lambda_2 {}^{\mathrm{t}}P_1 - \lambda_1 {}^{\mathrm{t}}P_2 & \mathbf{0} \end{pmatrix} \right] \tag{1}$$

is a representation of Φ_W in terms of matrices, if we give coordinates with respect to $\mathcal{B}$ in W and the dual basis $\mathcal{B}^*$ of $\mathcal{B}$ in W^*.

Looking at (1) we see that the decomposition $W = W_1 \oplus W_2$ yields two further Kronecker modules:

$$\Psi_{(W_1, W_2)} = [W_1, W_2^*; \psi_1, \psi_2],$$

where $\psi_i(x) = \phi_i(x)$, $\forall x \in W_1$ $(i = 1, 2)$, and

$$\Psi^{\dagger}_{(W_1, W_2)} = [W_2, W_1^*; \psi_1^{\dagger}, \psi_2^{\dagger}],$$

where $\psi_1^{\dagger} = \lambda_1 {}^{\mathrm{t}}\psi_1 - \eta^2 \lambda_2 {}^{\mathrm{t}}\psi_2$ and $\psi_2^{\dagger} = \lambda_2 {}^{\mathrm{t}}\psi_1 - \lambda_1 {}^{\mathrm{t}}\psi_2$, with ${}^{\mathrm{t}}\psi_i$ the K-linear mapping which is obtained by transposing ψ_i. Of course,

$$\Phi_W = \Psi_{(W_1, W_2)} \oplus \Psi^{\dagger}_{(W_1, W_2)},$$

$\Psi^{\dagger}_{(W_1, W_2)} = \Psi_{(W_2, W_1)}$ and $\Psi^{\dagger\dagger}_{(W_1, W_2)} = \Psi_{(W_1, W_2)}$ because $\lambda\lambda^{\sigma} = 1$.

The pair $[P_1, P_2]$ of matrices in (1) gives, manifestly, a representation of $\Psi_{(W_1,W_2)}$ (with respect to $\mathcal{B}_1$ and $\mathcal{B}_2^*$), while the pair

$$[P_1^\dagger = \lambda_1\,{}^{\mathrm{t}}P_1 - \eta^2\lambda_2\,{}^{\mathrm{t}}P_2,\ P_2^\dagger = \lambda_2\,{}^{\mathrm{t}}P_1 - \lambda_1\,{}^{\mathrm{t}}P_2] \tag{2}$$

gives a representation of $\Psi^\dagger_{(W_1,W_2)}$ (with respect to $\mathcal{B}_2$ and $\mathcal{B}_1^*$).

Assume now $\Psi_{(W_1,W_2)}$ decomposes into the direct sum $\Psi_{(W_1,W_2)} = \Psi' \oplus \Psi''$ of two nontrivial Kronecker modules $\Psi' = [W_1', W_2'^*; \psi_1, \psi_2]$ and $\Psi'' = [W_1'', W_2''^*; \psi_1, \psi_2]$ with $W_1 = W_1' \oplus W_1''$ and $W_2 = W_2' \oplus W_2''$, (hence $\psi_i(W_1') \subseteq W_2'^*$ and $\psi_i(W_1'') \subseteq W_2''^*$). In terms of matrices this means that $\Psi_{(W_1,W_2)}$ has a representation

$$\left[\begin{pmatrix} P_1' & \mathbf{0} \\ \mathbf{0} & P_1'' \end{pmatrix}, \begin{pmatrix} P_2' & \mathbf{0} \\ \mathbf{0} & P_2'' \end{pmatrix}\right], \tag{3}$$

with $P_i' \in \mathbf{Mat}_{t\times t}(K)$ and $P_i'' \in \mathbf{Mat}_{(m-t)\times(m-t)}(K), 0 < t < m$. Since $\Psi^\dagger_{(W_1,W_2)}$ belongs to the pencil of K-linear mappings generated by ${}^{\mathrm{t}}\psi_1$ and ${}^{\mathrm{t}}\psi_2$ (see [2], Chap. XII), the above decomposition of $\Psi_{(W_1,W_2)}$ yields a decomposition of $\Psi^\dagger_{(W_1,W_2)}$

$$\Psi^\dagger_{(W_1,W_2)} = \Psi'^\dagger \oplus \Psi''^\dagger,$$

with corresponding representation

$$\left[\begin{pmatrix} P_1'^\dagger & \mathbf{0} \\ \mathbf{0} & P_1''^\dagger \end{pmatrix}, \begin{pmatrix} P_2'^\dagger & \mathbf{0} \\ \mathbf{0} & P_2''^\dagger \end{pmatrix}\right], \tag{4}$$

where $P_i'^\dagger$ and $P_i''^\dagger$ are defined analogously to (2). Assume we obtain (3) and (4) through a basis $\{\varepsilon_1, \ldots, \varepsilon_m\}$ of W_1 and a basis $\{e_1, \ldots, e_m\}$ of W_2 with

$$W_1' = \langle \varepsilon_1, \ldots, \varepsilon_t\rangle,\ W_1'' = \langle \varepsilon_{t+1}, \ldots, \varepsilon_m\rangle,\ W_2' = \langle e_1, \ldots, e_t\rangle,\ W_2'' = \langle e_{t+1}, \ldots, e_m\rangle.$$

Then $U_1 = \langle \varepsilon_1, \ldots, \varepsilon_t, e_1, \ldots, e_t\rangle$ and $U_2 = \langle \varepsilon_{t+1}, \ldots, \varepsilon_m, e_{t+1}, \ldots, e_m\rangle$ give an orthogonal decomposition of W. In other words, we have for $h_{|W}$ the representation

$$\left(\begin{array}{cc|cc} \mathbf{0} & P_1' + \eta P_2' & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ P_1'^\dagger + \eta P_2'^\dagger & \mathbf{0} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & \mathbf{0} & P_1'' + \eta P_2'' \\ & & P_1''^\dagger + \eta P_2''^\dagger & \mathbf{0} \end{array}\right)$$

with respect to the basis $\{\varepsilon_1, \ldots, \varepsilon_t, e_1, \ldots, e_t, \varepsilon_{t+1}, \ldots, \varepsilon_m, e_{t+1}, \ldots, e_m\}$ of W.

Of course, we can conversely go from such an orthogonal decomposition of W back to a decomposition of $\Psi_{(W_1,W_2)}$. So we have

PROPOSITION 1. – *Let $W = W_1 \oplus W_2$ be a decomposition of W into two totally isotropic K-substructures. Then W does not split into the orthogonal sum of two non-trivial hyperbolic K-substructures if and only if $\Psi_{(W_1,W_2)}$ does not decompose into the direct sum of two non-trivial Kronecker modules.* □

The celebrated Krull-Schmidt Theorem applies in this context. Thus, from now on, we may confine our attention to the case where W is *indecomposable*, i.e. does not split into the orthogonal sum of two proper hyperbolic K-substructures. In view of the above Proposition this is equivalent to assume that the Kronecker module $\Psi_{(W_1,W_2)}$ does not decompose into the direct sum of two non-trivial Kronecker modules, i.e. $\Psi_{(W_1,W_2)}$ and $\Psi^{\dagger}_{(W_1,W_2)}$ are the *indecomposable factors* of Φ_W. Hence $(\Psi_{(W_1,W_2)}, \Psi^{\dagger}_{(W_1,W_2)})$, as an unordered pair of Kronecker modules, is independent (up to isomorphisms of Kronecker modules [1] of the particular decomposition $W_1 \oplus W_2$ of W. This allows one to write Ψ_W instead of $\Psi_{(W_1,W_2)}$. Now we have

THEOREM 2. – *Let $W, U \in \mathcal{X}_{2m}$ be indecomposable. Then W and U are in the same orbit under the action of the isometry group $\mathcal{G}(V,h)$ just if $\Psi_W \simeq \Psi_U$ or $\Psi_W \simeq \Psi^{\dagger}_U$.*

PROOF. – If W and U are in the same orbit under the action of $\mathcal{G}(V,h)$, we have that there exists an isomorphism $\rho : W \longrightarrow U$ such that $h(\rho(x),\rho(y)) = h(x,y), \forall x, y \in W$. As $u^*_x = \phi_i \rho(x) \in U^*, \forall x \in W$, for all $y \in W$ we have that

$$\begin{aligned}[\,^{\mathrm{t}}\rho\phi_i\rho(x)](y) = [\,^{\mathrm{t}}\rho(u^*_x)](y) = u^*_x(\rho(y)) = [\phi_i\rho(x)](\rho(y)) = \\ = h_i(\rho(x),\rho(y)) = h_i(x,y) = [\phi_i(x)](y).\end{aligned}$$

Hence the diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi_i} & W^* \\ \big\downarrow{\scriptstyle\rho} & & \big\uparrow{\scriptstyle {}^{\mathrm{t}}\rho} \\ U & \xrightarrow{\phi_i} & U^* \end{array}$$

[1] An *isomorphism* $\xi : \Gamma \longrightarrow \Delta$ from the Kronecker module $\Gamma = [V', V''; \gamma_1, \gamma_2]$ onto the Kronecker module $\Delta = [W', W''; \delta_1, \delta_2]$ (with $\dim_K V' = \dim_K W'$ and $\dim_K V'' = \dim_K W''$) is a pair of bijective linear mappings $\xi = (\xi' : V' \to W';\ \xi'' : V'' \to W'')$ which make the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\gamma_i} & V'' \\ \big\downarrow{\scriptstyle\xi'} & & \big\downarrow{\scriptstyle\xi''} \\ W' & \xrightarrow{\delta_i} & W'' \end{array}$$

commutative $(i = 1, 2)$.

commutes, i.e. $\Phi_W \simeq \Phi_U$. Since Ψ_W and $\Psi_W^\dagger$, (resp. Ψ_U and $\Psi_U^\dagger$) are the indecomposable factors of Φ_W (resp. Φ_U), by the Krull-Schmidt Theorem we have that $\Psi_W \simeq \Psi_U$ or $\Psi_W \simeq \Psi_U^\dagger$.

Conversely, let $\Psi_W \simeq \Psi_U$ (resp. $\Psi_W \simeq \Psi_U^\dagger$) and let $[P_1, P_2]$ and $[Q_1, Q_2]$ be pairs of matrices representing Ψ_W and Ψ_U respectively. Then

$$\begin{pmatrix} \mathbf{0} & P_1 + \eta P_2 \\ P_1^\dagger + \eta P_2^\dagger & \mathbf{0} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0} & Q_1 + \eta Q_2 \\ Q_1^\dagger + \eta Q_2^\dagger & \mathbf{0} \end{pmatrix} \tag{5}$$

are representations of $h_{|W}$ and $h_{|U}$ and we have to show that these matrices are congruent via a matrix in $\mathbf{GL}_{2m}(K)$.

The fact that Ψ_W and Ψ_U (resp. Ψ_W and $\Psi_U^\dagger$) are isomorphic Kronecker modules implies that there exist matrices $M, N \in \mathbf{GL}_m(K)$ such that $MP_iN = Q_i$ (resp. $MP_iN = Q_i^\dagger$), $i = 1, 2$. Then

$$\begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & {}^{\mathrm{t}}N \end{pmatrix} \qquad \left(\text{resp. } \begin{pmatrix} \mathbf{0} & {}^{\mathrm{t}}N \\ M & \mathbf{0} \end{pmatrix}\right)$$

is a matrix in $\mathbf{GL}_{2m}(K)$ making the matrices (5) congruent. □

The last Theorem reduces matters to classify the indecomposable Kronecker module Ψ_W. Looking at [1], we see that Ψ_W has a representation

$$\Psi_W \equiv \left[\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{\mathbf{I}_{m-1}} & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & -a_m \\ & & & \vdots \\ & \boxed{\mathbf{I}_{m-1}} & & -a_2 \\ & & & -a_1 \end{pmatrix} \right]$$

or

$$\Psi_W \equiv \left[\begin{pmatrix} 0 & \cdots & 0 & -a_m \\ & & & \vdots \\ & \boxed{\mathbf{I}_{m-1}} & & -a_2 \\ & & & -a_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{\mathbf{I}_{m-1}} & \\ 0 & & & \end{pmatrix} \right]$$

with $p(x) = \sum_{i=0}^{m-1} a_{m-i} x^i + x^m$ a power of an irreducible polynomial over K.

Summing up, we have

THEOREM 3. – *Let W be an indecomposable hyperbolic K-substructure of V. Then there exists a basis of W with respect to which $h_{|W}$ has the representation*

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_m + \eta R \\ \lambda(\mathbf{I}_m - \eta\, {}^{\mathrm{t}}R) & \mathbf{0} \end{pmatrix} \; or \; \begin{pmatrix} \mathbf{0} & R + \eta \mathbf{I}_m \\ \lambda({}^{\mathrm{t}}R - \eta \mathbf{I}_m) & \mathbf{0} \end{pmatrix}$$

where

$$R = \begin{pmatrix} 0 & \cdots & 0 & -a_m \\ \multicolumn{3}{|c|}{} & \vdots \\ \multicolumn{3}{|c|}{\mathbf{I}_{m-1}} & -a_2 \\ \multicolumn{3}{|c|}{} & -a_1 \end{pmatrix}$$

with $\sum_{i=0}^{m-1} a_{m-i}x^i + x^m$ *a power of an irreducible polynomial over* K. $\square$

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