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A Sharp Weighted Wirtinger Inequality.

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Sunto. – *Si ottiene una stima ottimale per la migliore costante* $C > 0$ *nella disuguaglianza di tipo Wirtinger*

$$\int_0^{2\pi} \gamma^p w^2 \leq C \int_0^{2\pi} \gamma^q w'^2$$

dove γ *è limitata superiormente e dotata di estremo inferiore positivo,* w *è periodica di periodo* 2π *e tale che* $\int_0^{2\pi} \gamma^p w = 0$, *e* $p + q \geq 0$. *Tale risultato generalizza una disuguaglianza di Piccinini e Spagnolo.*

Summary. – *We obtain a sharp estimate for the best constant* $C > 0$ *in the Wirtinger type inequality*

$$\int_0^{2\pi} \gamma^p w^2 \leq C \int_0^{2\pi} \gamma^q w'^2$$

where γ *is bounded above and below away from zero,* w *is* 2π*-periodic and such that* $\int_0^{2\pi} \gamma^p w = 0$, *and* $p + q \geq 0$. *Our result generalizes an inequality of Piccinini and Spagnolo.*

Let $C(a, b) > 0$ denote the best constant in the following weighted Wirtinger type inequality:

$$\int_0^{2\pi} aw^2 \leq C(a, b) \int_0^{2\pi} bw'^2, \tag{1}$$

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where $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ is 2π-periodic and satisfies the constraint

$$\int_0^{2\pi} aw = 0\,, \tag{2}$$

and $a, b \in \mathcal{B}$ with

$$\mathcal{B} = \{a \in L^\infty(\mathbb{R}) : a \text{ is } 2\pi\text{-periodic and } \inf a > 0\}\,.$$

Here and in what follows, for every measurable function a we denote by $\inf a$ and $\sup a$ the essential lower bound and the essential upper bound of a, respectively. For every $L > 1$, we denote

$$\mathcal{B}(L) = \left\{a \in L^\infty(0, 2\pi) : a \text{ is } 2\pi\text{-periodic, } \inf \mathrm{a} = 1 \text{ and } \sup a = L\right\}.$$

Our aim in this note is to prove:

THEOREM 1. – *Suppose $a = \gamma^p$ and $b = \gamma^q$ for some $\gamma \in \mathcal{B}(M)$, $M > 1$, and for some $p, q \in \mathbb{R}$ such that $p + q \geq 0$. Then*

$$C(\gamma^p, \gamma^q) \leq \left(\frac{\frac{1}{2\pi} \int_0^{2\pi} \gamma^{(p-q)/2}}{\frac{4}{\pi} \arctan\left(M^{-(p+q)/4}\right)} \right)^2. \tag{3}$$

If $p + q > 0$, then equality holds in (3) *if and only if $\gamma(\theta) = \overline{\gamma}_{p,q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$, where*

$$\overline{\gamma}_{p,q}(\theta) = \begin{cases} 1, & \text{if } 0 \leq \theta < c_{p,q}\dfrac{\pi}{2}, \quad \pi \leq \theta < \pi + c_{p,q}\dfrac{\pi}{2} \\ M, & \text{if } c_{p,q}\dfrac{\pi}{2} \leq \theta < \pi, \quad \pi + c_{p,q}\dfrac{\pi}{2} \leq \theta < 2\pi \end{cases},$$

with

$$c_{p,q} = \frac{2}{1 + M^{-(p-q)/2}}\,.$$

Furthermore, equality holds in (1)-(2) *with $a(\theta) = \overline{\gamma}^p_{p,q}(\theta + \varphi)$ and $b(\theta) =$*

$\overline{\gamma}^q_{p,q}(\theta+\varphi)$ *if and only if* $w(\theta)=\overline{w}_{p,q}(\theta+\varphi)$ *where*

$$\overline{w}_{p,q}(\theta)=\begin{cases} \sin\left[\sqrt{\mu}\left(c_{p,q}^{-1}\theta-\dfrac{\pi}{4}\right)\right], & \text{if } 0\leqslant\theta<c_{p,q}\dfrac{\pi}{2}\\ M^{-(p+q)/4}\cos\left[\sqrt{\mu}\left(\dfrac{\pi}{2}+c_{p,q}^{-1}M^{(p-q)/2}\left(\theta-c_{p,q}\dfrac{\pi}{2}\right))-\dfrac{3\pi}{4}\right)\right], & \text{if } c_{p,q}\dfrac{\pi}{2}\leqslant\theta<\pi\\ -\sin\left[\sqrt{\mu}\left(\pi+c_{p,q}^{-1}(\theta-\pi)-\dfrac{5\pi}{4}\right)\right], & \text{if } \pi\leqslant\theta<\pi+\mathrm{c}_{\mathrm{p,q}}\dfrac{\pi}{2}\\ -M^{-(p+q)/4}\cos\left[\sqrt{\mu}\left(\dfrac{3\pi}{2}+c_{p,q}^{-1}M^{(p-q)/2}\left(\theta-\pi-c_{p,q}\dfrac{\pi}{2}\right)-\dfrac{7\pi}{4}\right)\right], & \text{if } \pi+c_{p,q}\dfrac{\pi}{2}\leqslant\theta<2\pi \end{cases},$$

and $\mu=\left((4/\pi)\arctan M^{-(p+q)}\right)^2$.

If $p+q=0$, *then* (3) *is an equality for any weight function* γ. *Equality is attained in* (1)-(2) *with* $a=\gamma^p$ *and* $b=\gamma^{-p}$ *if and only if*

$$w(\theta)=C\cos\left(\frac{2\pi}{\int_0^{2\pi}\gamma^p}\int_0^\theta\gamma^p+\varphi\right),$$

for some $C\neq 0$ *and* $\varphi\in\mathbb{R}$.

Note that when $p=q=0$, Theorem 1 yields $C(1,1)=1$ according to the classical Wirtinger inequality. When $p=q\neq 0$, the estimate (3) reduces to the estimate obtained by Piccinini and Spagnolo in [4]. More related results may be found in [1,2,3] and in the references therein. We begin by recalling in the following lemma the Wirtinger inequality of Piccinini and Spagnolo [4].

LEMMA 1 ([4]). – *Suppose* $b=a\in\mathcal{B}(L)$. *Then,*

$$C(a,a)\leqslant\left(\frac{4}{\pi}\arctan L^{-1/2}\right)^{-2}. \tag{4}$$

Equality holds in (4) *if and only if* $a(\theta) = \overline{a}(\theta + \varphi)$ *for some* $\varphi \in \mathbb{R}$, *where* $\overline{a}$ *is defined by*

$$\overline{a}(\theta) = \begin{cases} 1, & \text{if } 0 \leqslant \theta < \dfrac{\pi}{2}, \quad \pi \leqslant \theta < \dfrac{3\pi}{2} \\ L, & \text{if } \dfrac{\pi}{2} \leqslant \theta < \pi, \quad \dfrac{3\pi}{2} \leqslant \theta < 2\pi \end{cases} \tag{5}$$

and equality holds in (1)-(2) *with* $a(\theta) = b(\theta) = \overline{a}(\theta + \varphi)$ *if and only if* $w(\theta) = \overline{w}(\theta + \varphi)$, *where*

$$\overline{w}(\theta) = \begin{cases} \sin\left[\sqrt{\lambda}\left(\theta - \dfrac{\pi}{4}\right)\right], & \text{if } 0 \leqslant \theta < \dfrac{\pi}{2} \\ L^{-1/2}\cos\left[\sqrt{\lambda}\left(\theta - \dfrac{3\pi}{4}\right)\right], & \text{if } \dfrac{\pi}{2} \leqslant \theta < \pi \\ -\sin\left[\sqrt{\lambda}\left(\theta - \dfrac{5\pi}{4}\right)\right], & \text{if } \pi \leqslant \theta < \dfrac{3\pi}{2} \\ -L^{-1/2}\cos\left[\sqrt{\lambda}\left(\theta - \dfrac{7\pi}{4}\right)\right], & \text{if } \dfrac{3\pi}{2} \leqslant \theta < 2\pi \end{cases}, \tag{6}$$

where $\lambda = (4\pi^{-1} \arctan L^{-1/2})^2$.

In order to prove Theorem 1, we need the following lemma, which yields an estimate for $C(a, b)$ for *arbitrary* weight functions a, b.

LEMMA 2. – *Let* $a, b \in \mathcal{B}$. *The following estimate holds:*

$$C(a, b) \leqslant \left(\frac{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sqrt{ab^{-1}}}{\dfrac{4}{\pi}\arctan\left(\dfrac{\inf ab}{\sup ab}\right)^{1/4}} \right)^2. \tag{7}$$

If $\sqrt{ab} \in \mathcal{B}(L)$, $L > 1$, *then*

$$\frac{C(a, b)}{\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\sqrt{ab^{-1}}\right)^2} = \sup_{\sqrt{a'b'} \in \mathcal{B}(L)} \frac{C(a', b')}{\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\sqrt{a'b'^{-1}}\right)^2} = \left(\frac{4}{\pi}\arctan L^{-1/2}\right)^{-2} \tag{8}$$

if and only if the following equation is satisfied:

$$a(\theta(\tau))\, b(\theta(\tau)) = \overline{a}^2(\tau + \varphi) \quad \text{a.e.} \quad \tau \in (0, 2\pi), \quad \text{for some } \varphi \in \mathbb{R}, \tag{9}$$

where $\theta(\tau)$ *is the homeomorphism of* $\mathbb{R}$ *defined by*

(10) $$\tau(\theta) = \frac{1}{c}\int_0^{\theta}\sqrt{\frac{a(\tilde\theta)}{b(\tilde\theta)}}\,d\tilde\theta\,,$$

c is defined by

(11) $$c = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{a(\tilde\theta)}{b(\tilde\theta)}}\,d\tilde\theta\,,$$

and $\bar a$ *is the function defined in Lemma* 1.

If $b = a^{-1}$, *then* $C(a, a^{-1}) = \left((2\pi)^{-1}\int_0^{2\pi} a\right)^2$ *and equality is attained in* (1)-(2) *with* $b = a^{-1}$ *if and only if* $w(\theta) = C\cos\left(2\pi\left(\int_0^{2\pi} a\right)^{-1}\int_0^{\theta} a + \varphi\right)$ *for some* $C \neq 0$ *and* $\varphi \in \mathbb{R}$.

PROOF. – Under the change of variables $\theta = \theta(\tau)$ defined by (10)-(11), setting $\alpha(\tau) = a(\theta(\tau))$, $\beta(\tau) = b(\theta(\tau))$, $\xi(\tau) = w(\theta(\tau))$, we obtain

$$\alpha\theta' = c\sqrt{\alpha\beta}\,,\qquad \beta\theta'^{-1} = c^{-1}\sqrt{\alpha\beta},$$

and therefore:

$$\int_0^{2\pi} aw^2\,d\theta = \int_0^{2\pi}\alpha\theta'\,\xi^2\,d\tau = c\int\sqrt{\alpha\beta}\xi^2\,d\tau$$

$$\int_0^{2\pi} aw\,d\theta = \int_0^{2\pi}\alpha\theta'\,\xi\,d\tau = c\int\sqrt{\alpha\beta}\xi\,d\tau = 0$$

$$\int_0^{2\pi} bw'^2\,d\theta = \int_0^{2\pi}\beta\theta'^{-1}\,\xi'^2\,d\tau = c^{-1}\int\sqrt{\alpha\beta}\xi'^2\,d\tau\,.$$

Upon substitution, (1)-(2) takes the form:

(12) $$\int_0^{2\pi}\sqrt{\alpha\beta}\xi^2\,d\tau \leq \frac{C(a, b)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{ab^{-1}}\right)^2}\int_0^{2\pi}\sqrt{\alpha\beta}\xi'^2\,d\tau\,,$$

with constraint

(13) $$\int_0^{2\pi}\sqrt{\alpha\beta}\xi\,d\tau = 0\,.$$

If $\sqrt{ab} \in \mathcal{B}(L)$, in view of Lemma 1 we obtain

$$\text{(14)} \qquad \frac{C(a,\,b)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{ab^{-1}}\right)^2} = C(\sqrt{\alpha\beta},\,\sqrt{\alpha\beta}) \leqslant \left(\frac{4}{\pi}\arctan\sqrt{\frac{\inf\sqrt{\alpha\beta}}{\sup\sqrt{\alpha\beta}}}\right)^{-2}$$

$$= \left(\frac{4}{\pi}\arctan\left(\frac{\inf ab}{\sup ab}\right)^{1/4}\right)^{-2}.$$

This yields (7). Moreover, we have $C(\sqrt{\alpha\beta},\,\sqrt{\alpha\beta}) = \big((4/\pi)\arctan L^{-1/2}\big)^{-2}$ if and only if $\sqrt{\alpha(\tau)\,\beta(\tau)} = \overline{\alpha}(\tau+\varphi)$, for some $\varphi \in \mathbb{R}$. That is, (8) holds if and only if (9) holds.

If $b = a^{-1}$, then (12)-(13) takes the form

$$\int_0^{2\pi}\xi^2\,\mathrm{d}\tau \leqslant \frac{C(a,\,a^{-1})}{\left(\frac{1}{2\pi}\int_0^{2\pi}a\right)^2}\int_0^{2\pi}\xi'^2\,\mathrm{d}\tau$$

with constraint

$$\int_0^{2\pi}\xi\,\mathrm{d}\tau = 0\,.$$

Therefore, by the classical Wirtinger inequality,

$$C(a,\,a^{-1}) = \left(\frac{1}{2\pi}\int_0^{2\pi}a\right)^2$$

and equality holds in (1)-(2) with $b = a^{-1}$ if and only if $\xi(\tau) = C\cos(\tau+\varphi)$ for some $C \neq 0$ and $\varphi \in \mathbb{R}$, that is, if and only if $w(\theta) = C\cos\left(2\pi\left(\int_0^{2\pi}a\right)^{-1}\int_0^{\theta}a + \varphi\right)$, as asserted. ■

Lemma 3. – *Suppose a, b satisfy $\sqrt{ab} \in \mathcal{B}(L)$, $L > 1$, and* (9), *where $\theta(\tau)$ is defined in* (10) *and c is defined by* (11). *Suppose*

$$a = \gamma^p, \qquad b = \gamma^q$$

(15)

for some $\gamma \in \mathcal{B}(M)$, with $M = L^{2/(p+q)}$, and for some p, $q \in \mathbb{R}$ such that $p + q > 0$. Then $\gamma(\theta) = \overline{\gamma}_{p,\,q}(\theta+\varphi)$ for some $\varphi \in \mathbb{R}$, where $\overline{\gamma}_{p,\,q}$ is the function defined in Theorem 1.

PROOF. – When $p+q>0$, we have $\gamma^{(p+q)/2} \in \mathcal{B}(L)$. In view of (9) and (15) we have

$$\gamma(\theta(\tau)) = \overline{a}^{2/(p+q)}(\tau+\psi), \qquad \forall \tau \in \mathbb{R}$$

for some $\psi \in \mathbb{R}$. It follows that

$$\theta(\tau) = c\int_0^\tau \sqrt{\frac{b(\theta(\overline{\tau}))}{a(\theta(\overline{\tau}))}}\, d\overline{\tau} = c\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau}+\psi)\, d\overline{\tau} \tag{16}$$

and, in view of the 2π-periodicity of a and b,

$$c = \left(\frac{1}{2\pi}\int_0^{2\pi} \sqrt{\frac{b(\theta(\overline{\tau}))}{a(\theta(\overline{\tau}))}}\, d\overline{\tau}\right)^{-1} = \left(\frac{1}{2\pi}\int_0^{2\pi} \overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\, d\overline{\tau}\right)^{-1}.$$

Setting

$$h_{p,q}(\tau) = c\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\, d\overline{\tau}\,,$$

we have $\theta(\tau-\psi) = h_{p,q}(\tau) - h_{p,q}(\psi)$ for every $\tau \in \mathbb{R}$, and consequently $\tau(\theta) = h_{p,q}^{-1}(\theta + h_{p,q}(\psi)) - \psi$. In view of the definition of $\overline{a}$ with $L = M^{(p+q)/2}$, we have:

$$\int_0^\tau \overline{a}^{-(p-q)/(p+q)}(\overline{\tau})\, d\overline{\tau} =$$

$$\begin{cases} \tau\,, & \text{if } 0 \leqslant \tau < \dfrac{\pi}{2} \\[2ex] \dfrac{\pi}{2} + M^{-(p-q)/2}\left(\tau - \dfrac{\pi}{2}\right), & \text{if } \dfrac{\pi}{2} \leqslant \tau < \pi \\[2ex] \dfrac{\pi}{2}(1+M^{-(p-q)/2}) + \tau - \pi\,, & \text{if } \pi \leqslant \tau < \dfrac{3\pi}{2} \\[2ex] \dfrac{\pi}{2}(2+M^{-(p-q)/2}) + M^{-(p-q)/2}\left(\tau - \dfrac{3\pi}{2}\right), & \text{if } \dfrac{3\pi}{2} \leqslant \tau < 2\pi \end{cases}.$$

In particular, we derive

$$c = \frac{2}{1+M^{-(p-q)/2}} = c_{p,q}.$$

It follows that $h_{p,q}(\tau)$ is the piecewise linear homeomorphism of $\mathbb{R}$ defined in

$[0, 2\pi)$ by

$$h_{p,q}(\tau) = \begin{cases} c_{p,q}\tau\,, & \text{if } 0 \leqslant \tau < \dfrac{\pi}{2} \\ c_{p,q}\left[\dfrac{\pi}{2} + M^{-(p-q)/2}(\tau - \dfrac{\pi}{2})\right], & \text{if } \dfrac{\pi}{2} \leqslant \tau < \pi \\ c_{p,q}\left[\dfrac{\pi}{2}(1 + M^{-(p-q)/2}) + \tau - \pi\right], & \text{if } \pi \leqslant \tau < \dfrac{3\pi}{2} \\ c_{p,q}\left[\dfrac{\pi}{2}(2 + M^{-(p-q)/2}) + M^{-(p-q)/2}\left(\tau - \dfrac{3\pi}{2}\right)\right], & \text{if } \dfrac{3\pi}{2} \leqslant \tau < 2\pi \end{cases}$$

and by $h_{p,q}(\tau + 2\pi n) = 2\pi n + h_{p,q}(\tau)$, for any $\tau \in [0, 2\pi)$ and for any integer n. Inversion yields

$$h_{p,q}^{-1}(\theta) = \begin{cases} c_{p,q}^{-1}\theta\,, & \text{if } 0 \leqslant \theta < c_{p,q}\dfrac{\pi}{2} \\ \dfrac{\pi}{2} + c_{p,q}^{-1}M^{(p-q)/2}\left(\theta - c_{p,q}\dfrac{\pi}{2}\right), & \text{if } c_{p,q}\dfrac{\pi}{2} \leqslant \theta < \pi \\ \pi + c_{p,q}^{-1}(\theta - \pi)\,, & \text{if } \pi \leqslant \theta < \pi + c_{p,q}\dfrac{\pi}{2} \\ \dfrac{3\pi}{2} + c_{p,q}^{-1}M^{(p-q)/2}\left(\theta - \pi - c_{p,q}\dfrac{\pi}{2}\right), & \text{if } \pi + c_{p,q}\dfrac{\pi}{2} \leqslant \theta < 2\pi \end{cases},$$

for $\theta \in [0, 2\pi)$ and $h_{p,q}^{-1}(\theta + 2\pi n) = 2\pi n + h_{p,q}^{-1}(\theta)$ for any $\tau \in [0, 2\pi)$ and for any integer n. Substitution yields $\gamma(\theta) = \overline{a}^{2/(p+q)}\big(h_{p,q}^{-1}(\theta + h_{p,q}(\psi))\big) = \overline{a}^{2/(p+q)}\big(h_{p,q}^{-1}(\theta + \varphi)\big) = \overline{\gamma}_{p,q}(\theta + \varphi)$, with $\varphi = h_{p,q}(\psi)$. ■

Now we can prove Theorem 1.

PROOF OF THEOREM 1. – Estimate (7) with $a = \gamma^p$ and $b = \gamma^q$ yields (3). Suppose $p + q > 0$. In view of Lemma 2 and Lemma 3 we have

$$\frac{C(\gamma^p, \gamma^q)}{\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\gamma^{(p-q)/2}\right)^2} = \left(\frac{4}{\pi}\arctan M^{-(p+q)/4}\right)^{-2}$$

if and only if $\gamma(\theta) = \overline{\gamma}_{p,q}(\theta + \varphi)$ for some $\varphi \in \mathbb{R}$. Equality is attained in (1)-(2) with $a(\theta) = \overline{\gamma}_{p,q}^{p}(\theta + \varphi)$ and $b(\theta) = \overline{\gamma}_{p,q}^{q}(\theta + \varphi)$ if and only if $w(\theta) = \overline{w}_{p,q}(\theta + \varphi)$.

If $p + q = 0$, then the conclusion follows by Lemma 2 with $a = \gamma^p$ and $b = \gamma^{-p}$. ■

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