Bollettino
Unione Matematica Italiana

Tommaso Ruggeri

The entropy principle: from continuum mechanics to hyperbolic systems of balance laws

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **8-B** (2005), n.1, p. 1–20.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2005_8_8B_1_1_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2005.

Bollettino U. M. I.
(8) 8-B (2005), 1-20

The Entropy Principle: from Continuum Mechanics to Hyperbolic Systems of Balance Laws.

TOMMASO RUGGERI (*)

Sunto. – *Si presenta una breve rassegna dei diversi ruoli che ha il principio di entropia nella moderna termodinamica. Nell'ambito della termodinamica razionale il principio di entropia diventa un criterio di selezione per le equazioni costitutive ammissibili mentre nel caso di soluzioni deboli di sistemi iperbolici non lineari diventa un criterio di selezione dei processi fisicamente ammissibili. Inoltre tutti i sistemi iperbolici di leggi di bilancio che sono compatibili con un principio di entropia convessa sono simmetrici ed è possibile riconoscere teorie a nido mediante l'introduzione dei sottosistemi principali. Particolare attenzione è dedicata all'analisi qualitativa dimostrando che in presenza di dissipazione il problema di Cauchy è ben posto in senso globale ed esistono, per dati iniziali sufficientemente piccoli, soluzioni regolari per tutti i tempi che tendono a stati costanti di equilibrio. Infine vengono applicati questi risultati alla teoria della Termodinamica Estesa che governa i processi dei gas rarefatti.*

Summary. – *We discuss the different roles of the entropy principle in modern thermodynamics. We start with the approach of rational thermodynamics in which the entropy principle becomes a selection rule for physical constitutive equations. Then we discuss the entropy principle for selecting admissible discontinuous weak solutions and to symmetrize general systems of hyperbolic balance laws. A particular attention is given on the local and global well-posedness of the relative Cauchy problem for smooth solutions. At the end we give some recent results on closure procedure for the moments theory associated to the Boltzmann equation (Extended Thermodynamics).*

1. – Entropy principle in continuum mechanics.

The concept and the name of entropy originated in the early 1850's in the work of Rudolph Julius Emmanuel Clausius (1822-1888) with the famous statement of the second principle of thermodynamics: «*heat cannot pass by itself from a cold to a hot body*».

From then on many researchers worked on thermodynamics in the two

(*) Conferenza tenuta a Milano l'8 settembre 2003 in occasione del XVII Congresso U.M.I.

complementary aspects: the macroscopic view formulated originally by Carnot, Clausius, Gibbs, Planck, and Caratheodory and the microscopic approach associated with Boltzmann and Maxwell.

The thermostatics – the thermodynamics of equilibrium – is fully accepted. This cannot be said instead for the thermodynamics in the proper sense, i.e. the theory of non equilibrium processes.

The entropy principle characterizes the irreversibility of the processes and at the beginning was thought only as an arrow in the time direction. A different important point of view was proposed in the '60s in the context of Rational Thermodynamics. In order to explain this, it is necessary to recall the structure of Continuum theories.

The Physical laws in continuum theories are *balance laws*:

$$\frac{d}{dt}\int_{\Omega} \boldsymbol{\Psi} d\Omega = -\int_{\Sigma} \boldsymbol{\Phi}^i n_i \, d\Sigma + \int_{\Omega} \boldsymbol{f} \, d\Omega \,, \tag{1}$$

where $\boldsymbol{\Psi}(\boldsymbol{x}, t)$; $\boldsymbol{x} \in \Omega$, $t \in R^+$, is a generic density. The first integral on the r.h.s. represents the flux of some quantities $\boldsymbol{\Phi}^i$ trough the surface Σ of unit normal $\vec{n} \equiv (n_i)$ and velocity $\vec{v} \equiv (v_i)$, while the last integral represents the productions (source terms).

Under regularity assumptions the system can be put in the local form:

$$\frac{\partial \boldsymbol{\Psi}}{\partial t} + \frac{\partial}{\partial x^i}(\boldsymbol{\Psi} v^i + \boldsymbol{\Phi}^i) = \boldsymbol{f} \,. \tag{2}$$

For example, in the case of fluids by the identifications:

$$\boldsymbol{\Psi} \equiv \begin{bmatrix} \varrho \\ \varrho v_j \\ \dfrac{\varrho v^2}{2} + \varrho e \end{bmatrix}, \qquad \boldsymbol{\Phi}^i \equiv \begin{bmatrix} 0 \\ -t_{ij} \\ q_i - t_{ij} v_j \end{bmatrix}, \qquad \boldsymbol{f} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3}$$

we have:

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} + \dfrac{\partial \varrho v_i}{\partial x^i} = 0 & \text{(conservation of mass)} \\[2ex] \dfrac{\partial (\varrho v_j)}{\partial t} + \dfrac{\partial}{\partial x^i}(\varrho v_i v_j - t_{ij}) = 0 & \text{(balance of momentum)} \\[2ex] \dfrac{\partial}{\partial t}\left(\dfrac{\varrho v^2}{2} + \varrho e\right) + \dfrac{\partial}{\partial x^i}\left\{\left(\dfrac{\varrho v^2}{2} + \varrho e\right) v_i + q_i - t_{ij} v_j\right\} = 0 & \text{(energy conservation)} \end{cases} \tag{4}$$

where ϱ, $\boldsymbol{t} \equiv (t_{ij})$, $\boldsymbol{q} \equiv (q_i)$, e are respectively the mass density, the stress tensor, the heat flux and the internal energy.

Of course the balance law systems are not closed, having more unknowns than equations and we need the so called *constitutive equations* in order to close them. In the modern constitutive theory all the constitutive equations must obey the two principles:

- The objectivity principle: the proper constitutive equations are independent of the Observer;

- The second principle of thermodynamics that in the Rational Thermodynamics requires that any solutions of the full system satisfies the inequality of Clausius-Duhem (Coleman-Noll 1963) [1]:

$$\frac{\partial \varrho S}{\partial t} + \frac{\partial}{\partial x^i}\left(\varrho S v^i + \frac{q^i}{T}\right) \geqslant 0 \quad \text{for all processes}\,. \tag{5}$$

S is the entropy density that is given by a constitutive relation to be determined by the compatibility between (4) and (5) while T denotes the absolute temperature. The requirement that all the solutions of the balance law system satisfies also the new balance law (5) is so strong that several restrictions arise for admissible costitutive equations. For instance in the case of a classical approach for fluids with Fourier Navier-Stokes assumptions

$$q_i = -\chi \frac{\partial T}{\partial x^i}\,, \qquad \sigma_{\langle ij\rangle} = \mu \frac{\partial v_{\langle i}}{\partial x^{j\rangle}}\,, \qquad \sigma_{ll} = \nu \,\mathrm{div}\,\boldsymbol{v}\,,$$

the constitutive equations compatible with (5) must satisfy:

$$TdS = de - \frac{p}{\varrho^2} d\varrho \quad \text{(Gibbs relation)} \tag{6}$$

$$\chi,\ \ \mu,\ \ \nu \geqslant 0\,,$$

($\boldsymbol{t} = -p\boldsymbol{I} + \boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ is the shear stress, $\boldsymbol{\sigma}_{\langle ij\rangle}$ denotes the deviatoric part, χ the heat conductivity, μ the shear viscosity and ν the bulk viscosity).

We observe that, within this new approach, the Gibbs relation (6) – that give a differential link between S, e and ϱ – comes as a consequence of the entropy principle and is not assumed a priori as in the thermodynamics of irreversible process (TIP) (*local equilibrium assumption*).

The same condition (6) hold also in the particular case of hyperbolic Euler fluids in which χ, μ and ν are zero.

Therefore in the modern Rational Thermodynamics the entropy principle becomes a constraint for the acceptable constitutive equations.

On the other hand the principle is also supported by the kinetic theory of gases. The kinetic theory describes the state of a rarefied gas through the phase density $f(\boldsymbol{x}, t, \boldsymbol{c})$, where $f(\boldsymbol{x}, t, \boldsymbol{c})\,d\boldsymbol{c}$ is the number density of atoms at point $\boldsymbol{x}$ and time t that have velocities between $\boldsymbol{c}$ and $\boldsymbol{c} + d\boldsymbol{c}$. The phase density obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + c^i \frac{\partial f}{\partial x^i} = Q \tag{7}$$

where Q represents the collisional terms. Introducing as moments (k is the Boltzmann constant):

$$\varrho S = \int (-k \log f)\, f\, d\boldsymbol{c}, \qquad \phi^i = \int (-k \log f)\, f c^i\, d\boldsymbol{c}\ , \tag{8}$$

for the properties of Q we have the so called H-theorem:

$$\frac{\partial \varrho S}{\partial t} + \frac{\partial}{\partial x^i}(\varrho S v^i + \phi^i) \geqslant 0\ . \tag{9}$$

But it is interesting to observe that the non convective entropy flux ϕ^i, given by $(8)_2$ is in general different from q^i/T. In fact, in kinetic theory $q^i = \frac{1}{2}\int f c^2 c^i\, d\boldsymbol{c}$. Starting from this observation Ingo Müller (1967) [2], proposed for a generic continuum model the inequality (9) as extension of the entropy principle of Coleman and Noll: not only the entropy density S but also the non convective entropy flux ϕ^i are not a priori prescribed but are constitutive equations to be determined by the compatibility between the system of balance laws and the entropy law.

Another conceptual advantage of the Müller approach was that it does not require a priori the definition of the temperature as in the previous Clausius-Duhem approach. For example in the case of fluids the temperature is not a primitive concept in the balance laws (4) nor in the new entropy principle (9) but it's a consequence of their compatibility. In fact appears as integral factor of the Gibbs relation (6).

Today the general form (9) is universally accepted in the continuum community and all the constitutive equations in new models are tested by the entropy principle. For a review concerning these arguments the interested reader can refer to the two chapters by Müller that are included in a very recent book dedicated to the entropy [3].

2. – The Riemann Problem and the non uniqueness of weak solutions.

In a complete different context the entropy principle plays a fundamental role. It is well known that weak solutions are not unique for hyperbolic systems of conservation laws. The trivial classroom example of a Riemann problems is:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0; \qquad u(x,0) = \begin{cases} u_0 & \text{for } x > 0 \\ u_1 & \text{for } x < 0 \end{cases} \tag{10}$$

that admits the two solutions:

$$u(x,t) = \begin{cases} u_0 & \text{for } x > st \\ u_1 & \text{for } x < st \end{cases} \qquad s = \frac{1}{2}(u_0 + u_1) \qquad \text{Shock Wave} \tag{11}$$

$$u(x,t) = \begin{cases} u_0 & \text{for } x > u_0 t \\ \dfrac{x}{t} & \text{for } u_1 t \leqslant x \leqslant u_0 t \\ u_1 & \text{for } x < u_1 t\,. \end{cases} \qquad \text{Rarefaction Wave} \tag{12}$$

To restore the uniqueness we observe the two different behavior in Figure 1 of the characteristic curves in space time and we have the so called Lax conditions [4]:

The shock is «stable» and the solution of (10) *is* (11) *if*

$$\lambda(u_0) < s < \lambda(u_1)\,.$$

Viceversa if $\lambda(u_1) < \lambda(u_0)$, *the solutions is the rarefaction* (12). λ is the characteristic velocity that in the present case is $\lambda = u$.

What is the physical meaning of the Lax conditions? In this trivial example

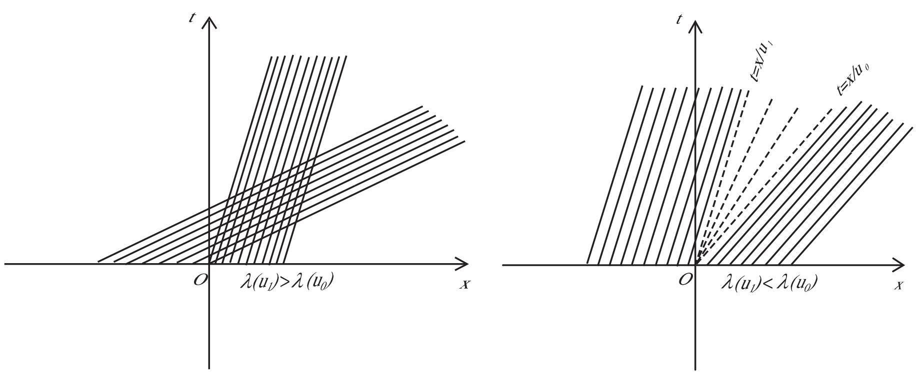

Fig. 1. – $u_0 < u_1$: Shock Wave; $u_0 > u_1$: Rarefaction Wave.

if we take as entropy law the equation

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x \leqslant 0\,, \tag{13}$$

it is a simple matter to verify that all the classical solutions of (10) are solution of (13) with equality, while the requirement that across the shock the entropy growth corresponds (for genuinely non linear waves) to the Lax conditions. This question can be generalized for a generic quasi-linear system of conservation laws compatible with an entropy principle and endowed with a convex entropy density (Friedrichs and Lax (1971) [5]):

$$\partial_t \boldsymbol{u} + \partial_i \boldsymbol{F}^i(\boldsymbol{u}) = 0 \qquad \partial_t h(\boldsymbol{u}) + \partial_i h^i(\boldsymbol{u}) = \Sigma \leqslant 0 \tag{14}$$

($\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, h, h^i and Σ corresponds to the physical entropy density, the entropy flux and the entropy production except by a change of sign).

In the one dimensional case the Riemann problem for initial sufficiently small jump is solved as a «superposition» of shocks, characteristic shocks, rarefaction waves and constant states. The physical shocks are admissible (see e.g. [6]):

$$\begin{array}{lllll} \text{if} & \nabla\lambda\cdot\boldsymbol{r} \neq 0, & \lambda(\boldsymbol{u}_0) < s < \lambda(\boldsymbol{u}_1) & \Leftrightarrow\ \eta > 0 & \text{Shock} \\ \text{if} & \nabla\lambda\cdot\boldsymbol{r} \equiv 0, & \lambda(\boldsymbol{u}_0) = s = \lambda(\boldsymbol{u}_1). & \Leftrightarrow\ \eta = 0 & \text{Characteristic Shock,} \end{array}$$

while if $\lambda(\boldsymbol{u}_1) < \lambda(\boldsymbol{u}_0)$ we have a rarefaction wave ($\eta = -s[h] + [h^1]$ is the entropy production across the shock wave and the square bracket denotes the jump across the wave front with velocity s, $\boldsymbol{r}$ is the right characteristic eigenvector associated with the eigenvalue λ and $\nabla = \partial/\partial\boldsymbol{u}$).

The problem fails in the special case of local exceptionality $\nabla\lambda\cdot\boldsymbol{r} = 0$ for some $\boldsymbol{u}$. In this case the stability of the shock must be satisfy the Liu conditions [7], [8] that implies the generalized Lax condition $\lambda(\boldsymbol{u}_0) \leqslant s \leqslant \lambda(\boldsymbol{u}_1)$ but the entropy growth alone is not sufficient for the admissibility. In fact it is necessary to add a new *superposition principle* for the shocks (Liu and Ruggeri (2003) [9]).

Therefore the entropy principle becomes a selection rules for constitutive equations for classical solutions and a selection rules for physical processes for weak solutions.

The entropy principle for hyperbolic systems plays an important role not only in the field of the uniqueness of weak solutions but in recent years we found also several new properties for the systems that are compatible with this principle. The main are:

1. The systems endowed with an entropy principle with a convex entropy density can be written as a symmetric system if we choose a privileged

field variables «*the main field*» and the local Cauchy problem is well-posed;

2. The main field induced the possibility to find nesting theories trough the definition of *principal subsystem*. In particular there exists an *equilibrium manifold* in which the entropy reach the maximum value;

3. If the hyperbolic system of balance laws have a dissipative character in the sense of Kawashima, then there exist global smooth solutions and the equilibrium manifold is attractive provided the initial data are sufficiently small.

The first two results are valid in three-dimensional space while the last results are proved only in a one-dimensional case. In the following I will give some details on this points and then we apply these results to the Extended Thermodynamics theory.

3. – Balance laws systems, entropy and generators.

We rewrite the system and the entropy principle (14) in the most general case of an hyperbolic system of N balance laws ($\alpha = 0, 1, 2, 3$; $x^0 = t$; $\partial_\alpha = \partial / \partial x^\alpha$):

$$\partial_\alpha \boldsymbol{F}^\alpha(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}) \tag{15}$$

$$\partial_\alpha h^\alpha(\boldsymbol{u}) = \Sigma(\boldsymbol{u}) \leqslant 0\,. \tag{16}$$

The compatibility between (15) and (16) implies the existence of a *main field* $\boldsymbol{u}'$ such that [5], [10]:

$$\partial_\alpha h^\alpha - \Sigma \equiv \boldsymbol{u}' \cdot (\partial_\alpha \boldsymbol{F}^\alpha - \boldsymbol{F})\,. \tag{17}$$

As a consequence of the above identity, we have

$$dh^\alpha = \boldsymbol{u}' \cdot d\boldsymbol{F}^\alpha, \qquad \Sigma = \boldsymbol{u}' \cdot \boldsymbol{F} \leqslant 0\,. \tag{18}$$

Boillat [11] in the non relativistic case and Ruggeri and Strumia [10] in a covariant formulation had the idea to use as filed the main field $\boldsymbol{u}'$ (that, for convexity arguments, is global univalent to the field of the densities in any convex domain) and to introduce four potentials h'^α (*generators*):

$$h'^\alpha = \boldsymbol{u}' \cdot \boldsymbol{F}^\alpha - h^\alpha, \tag{19}$$

such that from $(18)_1$

$$\boldsymbol{F}^\alpha = \frac{\partial h'^\alpha}{\partial \boldsymbol{u}'}\,. \tag{20}$$

It follows that, upon selecting the main field as the field variables, the original system (15) can be written with Hessian matrices in the symmetric form

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial \boldsymbol{u}'} \right) = \boldsymbol{F} \quad \Leftrightarrow \quad \frac{\partial^2 h'^\alpha}{\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'} \partial_\alpha \boldsymbol{u}' = \boldsymbol{F} \tag{21}$$

provided that $h = h^0$ is a convex function of $\boldsymbol{u} \equiv \boldsymbol{F}^0$ (or equivalently the Legendre transform h'^0 is a convex function of the dual field $\boldsymbol{u}'$). Euler equations was already written in this form by Godunov [12].

4. – Principal subsystems.

We split the main field $\boldsymbol{u}' \in R^N$ into two parts $\boldsymbol{u}' \equiv (\boldsymbol{v}', \boldsymbol{w}')$, $\boldsymbol{v}' \in R^M$, $\boldsymbol{w}' \in R^{N-M}$, $(0 < M < N)$ and the system (21) with $\boldsymbol{F} \equiv (\boldsymbol{f}, \boldsymbol{g})$, reads:

$$\partial_\alpha \left(\frac{\partial h'^\alpha(\boldsymbol{v}', \boldsymbol{w}')}{\partial \boldsymbol{v}'} \right) = \boldsymbol{f}(\boldsymbol{v}', \boldsymbol{w}'), \tag{22}$$

$$\partial_\alpha \left(\frac{\partial h'^\alpha(\boldsymbol{v}', \boldsymbol{w}')}{\partial \boldsymbol{w}'} \right) = \boldsymbol{g}(\boldsymbol{v}', \boldsymbol{w}'). \tag{23}$$

Given some assigned constant value $\boldsymbol{w}'_*$ of $\boldsymbol{w}'$, we call principal subsystem of (21) the system (1):

$$\partial_\alpha \left(\frac{\partial h'^\alpha(\boldsymbol{v}', \boldsymbol{w}'_*)}{\partial \boldsymbol{v}'} \right) = \boldsymbol{f}(\boldsymbol{v}', \boldsymbol{w}'_*). \tag{24}$$

In other words a principal subsystem (there are $2^N - 2$ of such subsystems) coincide with the first block of the system putting $\boldsymbol{w}' = \boldsymbol{w}'_*$.

The principal subsystems have two important properties: they admit also a convex subentropy law and the spectrum of the characteristic velocities is contained in the one of the full system (subcharacteristic conditions). In fact it is possible to prove the following simple theorems [13].

THEOREM 1 (Subentropy Law). – *The solutions of a principal subsystem satisfy also a supplementary law* (*subentropy law*):

$$\partial_\alpha \overline{h}^\alpha = \overline{\Sigma} \tag{25}$$

where the entropy four-vector $\overline{h}^\alpha(\boldsymbol{v}', \boldsymbol{w}'_*)$ *and the entropy production* $\overline{\Sigma}$ *are related to the restrictions of the entropy four-vector* $h^\alpha(\boldsymbol{v}', \boldsymbol{w}'_*)$ *and of the en-*

(1) The definition and the properties remain valid for prescribed values of $\boldsymbol{w}'_*$ depending in arbitrary manner on x^α. In this case the principal subsystem is not autonomous [13].

tropy production $\Sigma(\boldsymbol{v}', \boldsymbol{w}'_*)$ of the full system through:

$$\overline{h}^{\alpha}(\boldsymbol{v}', \boldsymbol{w}'_*) = h^{\alpha}(\boldsymbol{v}', \boldsymbol{w}'_*) - \boldsymbol{w}'_* \cdot \left(\frac{\partial h'^{\alpha}}{\partial \boldsymbol{w}'} \right)_{\boldsymbol{w}' \equiv \boldsymbol{w}'_*}$$

$$\overline{\Sigma} = \Sigma(\boldsymbol{v}', \boldsymbol{w}'_*) - \boldsymbol{w}'_* \cdot \boldsymbol{g}(\boldsymbol{v}', \boldsymbol{w}'_*) .$$

The subentropy is convex and therefore every principal subsystem are also symmetric hyperbolic.

Let $\lambda^{(k)}(\boldsymbol{v}', \boldsymbol{w}', \vec{n})$ and $\overline{\lambda}^{(\overline{k})}(\boldsymbol{v}', \boldsymbol{w}'_*, \vec{n})$ the characteristic velocities of the total system and of the subsystem respectively ($\vec{n}$ is the unit normal to the wave front). In general the solutions of the subsystem are not particular solutions of the system (for $\boldsymbol{w}' = \boldsymbol{w}'_*$) and the spectrum of the $\overline{\lambda}$'s is not part of the spectrum of the λ's. However let define

$$\lambda_{\max} = \max_{k = 1, 2, \ldots, N} \lambda^{(k)}, \qquad \overline{\lambda}_{\max} = \max_{\overline{k} = 1, 2, \ldots, M} \overline{\lambda}^{(\overline{k})}$$

and similarly for the minima. Then

THEOREM 2 (Subcharacteristic conditions). – *Under the assumption that h^o is a convex function, the following subcharacteristic conditions hold for every principal subsystem*:

$$\begin{cases} \lambda_{\max}(\boldsymbol{v}', \boldsymbol{w}'_*, \vec{n}) \geqslant \overline{\lambda}_{\max}(\boldsymbol{v}', \boldsymbol{w}'_*, \vec{n}); \\ \lambda_{\min}(\boldsymbol{v}', \boldsymbol{w}'_*, \vec{n}) \leqslant \overline{\lambda}_{\min}(\boldsymbol{v}', \boldsymbol{w}'_*, \vec{n}), \end{cases} \tag{26}$$

$\forall \boldsymbol{v}' \in R^M$ *and* $\forall \vec{n} \in R^3 \colon \|\vec{n}\| = 1$.

The proof of the theorems are in [13].

5. – Equilibrium subsystem.

A particular case of (22), (23) is when the first M equations are conservation laws, i.e. $\boldsymbol{f} \equiv 0$. In this case it is possible to define, as usual in thermodynamics, the equilibrium state:

DEFINITION 1. – *An equilibrium state is a state for which the entropy production $-\Sigma|_E$ vanishes and hence attains its minimum value.*

It is possible to prove the following theorem [14], [13]:

THEOREM 3 (Equilibrium manifold). – *In an equilibrium state, under the assumption of dissipative productions i.e. if*

$$\boldsymbol{D} = \frac{1}{2}\left\{ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{w}'} + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{w}'} \right)^T \right\}\Bigg|_E \quad \textit{is negative definite}, \tag{27}$$

the production vanishes and the main field components vanish except for the first M ones. Thus

$$\boldsymbol{g}\,|_E = 0, \qquad \boldsymbol{w}'\,|_E = 0\,. \tag{28}$$

Therefore, in the main field components the equilibrium manifold is the hyperplane $\boldsymbol{w}' = 0$ and this confirm once again the importance of the main field.

Now we have another important characteristic property of the equilibrium state [15], [16]:

THEOREM 4 (Maximum of entropy). – *At equilibrium the entropy density $-h$ is maximal, i.e.*

$$h > h\,|_E \quad \forall \boldsymbol{u} \neq \boldsymbol{u}\,|_E, \qquad \textit{where } h\,|_E = h(\boldsymbol{v}, \boldsymbol{w}\,|_E(\boldsymbol{v}))\,.$$

Hence we find also at this general level the well known thermodynamical statement of maximum of entropy in equilibrium.

In the present case, when we limit our attention to the case of one dimensional space, the system (22), (23) assume the form:

$$\begin{cases} \boldsymbol{v}_t + (k'_{\boldsymbol{v}'})_x = 0 \\ \boldsymbol{w}_t + (k'_{\boldsymbol{w}'})_x = -\boldsymbol{G}(\boldsymbol{v}', \boldsymbol{w}')\,\boldsymbol{w}' \end{cases} \tag{29}$$

with $\boldsymbol{v} = h'_{\boldsymbol{v}'}$, $\boldsymbol{w} = h'_{\boldsymbol{w}'}$ and $\boldsymbol{G}(\boldsymbol{v}', 0)$ is a definite positive $(N-M)\times(N-M)$ matrix.

6. – Qualitative analysis.

In this section we discuss the importance of the entropy principle on the Cauchy problem.

6.1. *Local well posedness.*

In the general theory of hyperbolic conservation laws and hyperbolic-parabolic conservation laws, the existence of a strictly convex entropy function is a basic condition for the well-posedness. In fact, if the flux's $\boldsymbol{F}^i$ and the production $\boldsymbol{F}$ are smooth enough, in a suitable convex open set $D \in R^n$, it is well known that system (15) has a unique local (in time) smooth solution for smooth initial data [5], [17], [18].

However, in the general case, and even for arbitrarily small and smooth initial data, there is no global continuation for these smooth solutions, which may develop singularities, shocks or blowup, in finite time, see for instance [19], [6].

On the other hand, in many physical examples, thanks to the interplay between the source term and the hyperbolicity there exist global smooth solutions for a suitable set of initial data. This is the case, for example, of the isentropic Euler system with damping. Roughly speaking, for such a system the relaxation term induces a dissipative effect. This effect then competes with the hyperbolicity. If the dissipation is sufficiently strong to dominate the hyperbolicity, the system is *dissipative*, and we aspect that the smooth solution exists for all time and converges to a constant state. Otherwise, the dissipation and the hyperbolicity are equally important. Then we expect that only part of the perturbation diffuses. In the latter case the system is called *of composite type* by Zeng [20].

6.2. *The Kawashima condition.*

In general, there are several ways to identify whether a hyperbolic system with relaxation is dissipative or of composite type. One way is completely parallel to the case of the hyperbolic-parabolic system, which was discussed first by Kawashima [17] and for this reason is now called *the Kawashima condition* [21] or *genuine coupling* [16]:

In the equilibrium manifold any characteristic eigenvector is not in the null space of $\nabla \boldsymbol{F}$.

It is possible to verify that the Kawashima condition is equivalent in our notation to the following requirement (see [21]):

For every $\lambda \in R$ *and every* $\boldsymbol{X} \in R^M \setminus \{0\}$, *the vector* $\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0} \end{pmatrix} \in R^N$ *is not in the null space of* $-\lambda \boldsymbol{A}_0'^0 + \boldsymbol{A}_0'^1$, *where*

$$\boldsymbol{A}'^0 = \frac{\partial^2 h'}{\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'}\,, \qquad \boldsymbol{A}'^1 = \frac{\partial^2 k'}{\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'}$$

and the index 0 *denotes the equilibrium state.*

If we denote with $\boldsymbol{r}'$ the right eigenvectors of the symmetric system

$$(-\lambda \boldsymbol{A}'^0 + \boldsymbol{A}'^1)\, \boldsymbol{r}' = 0\,,$$

the K-condition is satisfied if every eigenvectors

$$\boldsymbol{r}_0' \neq \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0} \end{pmatrix}. \tag{30}$$

We observe that $\boldsymbol{r}' = \boldsymbol{A}'^0 \boldsymbol{r}$.

6.3. *Global Existence and stability of constant state.*

For dissipative one dimensional systems (29) satisfying the K-condition it is possible to prove the following global existence theorem due to Hanouzet and Natalini [21]:

THEOREM 5 (Global existence). – *Assume that the system* (29) *is strictly dissipative and the K-condition is satisfied. Then there exists* $\delta > 0$, *such that, if* $\|\boldsymbol{u}'(x, 0)\|_2 \leq \delta$, *there is a unique global smooth solution, which verifies*

$$\boldsymbol{u}' \in C^0([0, \infty); H^2(R) \cap C^1([0, \infty); H^1(R)).$$

Moreover Ruggeri and Serre [16] have proved that the constant state are stable:

THEOREM 6 (Stability). – *Under natural hypotheses of strongly convex entropy, strict dissipativeness, genuine coupling and «zero mass» initial for the perturbation of the equilibrium variables the constant solution stabilizes*

$$\|\boldsymbol{u}(t))\|_2 = 0(t^{-1/2}).$$

In [21] the authors report several examples of dissipative systems satisfying the K-condition: the p-system with damping, the Suliciu model for the isothermal viscoelasticity, the Kerr-Debye model in non linear electromagnetism and the Jin-Xin relaxation model.

6.4. *A counterexample of global existence without K-condition.*

Zeng [20] have considered a toy model of vibrational non equilibrium gas in Lagrangian variables, proving that also if the system is of composite type the global existence holds. Therefore the K-condition is only a sufficient condition for the global existence of smooth solutions.

An intriguing open problem is if it exists a weaker K-condition that is also necessary to ensure global solutions. And if it exists such condition, the question is if it has a physical meaning such that it is possible to consider it as a possible new principle of Extended Thermodynamics adding to the convexity of entropy.

Now we apply the previous results to the case of Extended Thermodynamics [22].

7. – The extended thermodynamics.

In the context of rarefied gas, as it is well known, most macroscopic thermodynamic quantities are identified as moments of the phase density

$$F_{k_1 k_2 \dots k_j} = \int f c_{k_1} c_{k_2} \dots c_{k_j} \, dc \,, \tag{31}$$

and due to the Boltzmann equation (7), the moments satisfy an infinity hierarchy of balance laws in which the flux in one equation becomes the density in the next one:

$$\begin{aligned}
&\partial_t F + \partial_i F_i = 0 \\
&\qquad \swarrow \\
&\partial_t F_{k_1} + \partial_i F_{ik_1} = 0 \\
&\qquad \swarrow \\
&\partial_t F_{k_1 k_2} + \partial_i F_{ik_1 k_2} = P_{k_1 k_2} \\
&\qquad \swarrow \\
&\partial_t F_{k_1 k_2 k_3} + \partial_i F_{ik_1 k_2 k_3} = P_{k_1 k_2 k_3} \\
&\vdots \\
&\partial_t F_{k_1 k_2 \dots k_n} + \partial_i F_{ik_1 k_2 \dots k_n} = P_{k_1 k_2 \dots k_n} \\
&\vdots
\end{aligned}$$

Taking into account that $P_{kk} = 0$, the first five equations are conservation laws and coincides (using different symbols) with (4), while the remaining ones are balance laws.

7.1. *The closure of extended thermodynamics.*

When we cut the hierarchy at the density with tensor of rank n, we have the problem of closure because the last flux end the production terms are not in the list of the densities. The idea of Rational Extended Thermodynamics (Müller and Ruggeri [22]) was to view the truncated system as a phenomenological system of continuum mechanics and then we consider the new quantities as constitutive functions:

$$\begin{aligned}
&F_{k_1 k_2 \dots k_n k_{n+1}} \equiv F_{k_1 k_2 \dots k_n k_{n+1}}(F, F_{k_1}, F_{k_1 k_2}, \dots F_{k_1 k_2 \dots k_n}) \\
&P_{k_1 k_2 \dots k_j} \equiv P_{k_1 k_2 \dots k_j}(F, F_{k_1}, F_{k_1 k_2}, \dots F_{k_1 k_2 \dots k_n}), \qquad 2 \leqslant j \leqslant n .
\end{aligned}$$

According with the continuum theory, the restrictions on the constitutive equations come only from *universal principles*, i.e.: *Entropy principle, Objectivity Principle* and *Causality and Stability* (convexity of the entropy).

The restrictions are so strong (in particular the entropy principle) that, at least, for processes not to far from the equilibrium the system is completely closed and in the case of 13 moments the results are in perfect agreement with the kinetic closure procedure proposed by Grad [23].

7.2. *Principal subsystems in ET.*

Now, that we have stated, that for any n we may use the closure of ET, the following question arises: What kind of relation do exist between two closure theories with different index, a theory S_n and a theory S_m with $n > m$, say? Boillat and Ruggeri [13] have proved, that

THEOREM 7 (Nesting theories). – *S_m is a principal subsystem of S_n obtained from S_n by setting $u'^{\alpha} = 0$, $(\alpha = m+1, \dots, n)$ and neglecting the corresponding equations for $\alpha = m+1, \dots, n$, i.e.*

$$S_n: \begin{cases} \dfrac{\partial u^a(u'^b, u'^\beta)}{\partial t} + \dfrac{\partial F_i^a(u'^b, u'^\beta)}{\partial x_i} = \Pi^a(u'^b, u'^\beta), \\[2ex] \dfrac{\partial u^\alpha(u'^b, u'^\beta)}{\partial t} + \dfrac{\partial F_i^\alpha(u'^b, u'^\beta)}{\partial x_i} = \Pi^\alpha(u'^b, u'^\beta) \end{cases} \tag{32}$$

$$a = 0, \dots, m; \quad \alpha = m+1, \dots, n\,.$$

$$S_m: \ \frac{\partial u^a(u'^b, 0)}{\partial t} + \frac{\partial F_i^a(u'^b, 0)}{\partial x_i} = \Pi^a(u'^b, 0). \tag{33}$$

In particular the Euler systems becomes the equilibrium subsystem of any ET theory.

In a work in progress, we have verified that the K-condition is satisfied and therefore for the previous theorems on the qualitative analysis if the initial data are sufficiently small, smooth solutions of ET exists for all time and converge to a constant state of the equilibrium Euler manifold!

7.3. *Examples of principal subsystems in E.T.: The* 13 *Moments Grad system*

As an example of the nesting theory of subsystems and the subcharacteristic properties we present the most simple case of Extended Thermodynamics:

the case of 13-moments know as Grad system [23]. In the usual symbols equations are:

$$\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x^i}(\varrho v_i) = 0\ ;$$

$$\frac{\partial}{\partial t}(\varrho v_j) + \frac{\partial}{\partial x^i}(\varrho v_i v_j + p\delta_{ij} - \sigma_{ij}) = 0\ ;$$

$$\frac{\partial}{\partial t}\left(\varrho e + \varrho\frac{v^2}{2}\right) + \frac{\partial}{\partial x^k}\left\{\left(\varrho e + \varrho\frac{v^2}{2} + p\right)v_k + q_k - \sigma_{kj}v_j\right\} = 0$$

$$\frac{\partial}{\partial t}\left\{\varrho\left(v_i v_j - \frac{v^2}{3}\delta_{ij}\right) - \sigma_{ij}\right\} + \frac{\partial F_{\langle ij\rangle k}}{\partial x_k} = \tau_0\sigma_{ij}$$

$$\frac{\partial}{\partial t}\{(\varrho v^2 + 5p)\,v_k + 2q_k - 2\sigma_{kj}v_j\} + \frac{\partial F_{ppik}}{\partial x_k} = 2\tau_0\sigma_{kj}v_j - \tau_1 q_k,$$

where

$$F_{\langle ij\rangle k} = F_{ijk} - \frac{1}{3}F_{hhk}\delta_{ij};$$

$$F_{ijk} = \varrho v_i v_j v_k + \left(pv_k + \frac{2}{5}q_k\right)\delta_{ij} + pv_i + \frac{2}{5}q_i\delta_{jk} + \left(pv_j + \frac{2}{5}q_j\right)\delta_{ik} - \sigma_{ij}v_k - \sigma_{ik}v_j - \sigma_{jk}v_i;$$

$$F_{ppij} = (\varrho v^2 + 7p)\,v_i v_j + (p\delta_{ij} - \sigma_{ij})\,v^2 - 2\sigma_{ik}v_k v_j - 2\sigma_{jk}v_k v_i + \frac{14}{5}(q_i v_j + q_j v_i) + \frac{4}{5}q_k v_k\delta_{ij} + \frac{p}{\varrho}(5p\delta_{ij} - 7\sigma_{ij}).$$

The first five equations are the usual conservation laws of mass, momentum and energy, while the remaining eight are the new evolution balance laws corresponding to the non-equilibrium variables q_i, heat flux, and σ_{ij}, shear stress (symmetric and traceless tensor). The equations $(34)_4$ and $(34)_5$ when relaxation times are small reduces – via the Maxwellian iteration procedure [22] – to the Navier-Stokes and Fourier equations respectively.

The components of the main field $\boldsymbol{u}'$ in the present case are:

$$\boldsymbol{u}' \equiv (\xi, \Lambda_j, \zeta, \Lambda_{\langle ij \rangle}, \Omega_k)$$

$$\xi = \frac{1}{T}\left\{\mu - \frac{v^2}{2} + \frac{1}{2p}\sigma_{ij} v_i v_j - \frac{\varrho}{5p^2} q_i v_i v^2\right\};$$

$$\Lambda_i = \frac{1}{T}\left\{v_i - \frac{1}{p}\sigma_{ij} v_j + \frac{\varrho}{5p^2}(v^2 q_i + 2 q_j v_j v_i)\right\};$$

$$\zeta = -\frac{1}{T}\left\{1 - \frac{2\varrho}{3p^2} q_k v_k\right\};$$

$$\Lambda_{\langle ij \rangle} = -\frac{1}{T}\left\{\frac{1}{2p}\sigma_{ij} + \frac{\varrho}{5p^2}\left(v_i q_j + v_j q_i - \frac{2}{3} v_k q_k \delta_{ij}\right)\right\};$$

$$\Omega_i = \frac{\varrho}{5Tp^2} q_i,$$

($\mu = e + p/\varrho - TS$ is the chemical potential).

For non degenerate mono-atomic ideal gas $p = (k/m)\varrho T$, $e = 3p/(2\varrho)$, the maximum characteristic velocity evaluated in an equilibrium state is

$$\lambda_{\max} = 1.65 c_S$$

where $c_S = \sqrt{\frac{5}{3}\frac{k}{m}T}$ is the sound velocity.

Taking into account the theorem 7 and (32), (33), the 10-moments system is a principal subsystem of the 13-moments, putting

$$\Omega_i = 0 \;\rightarrow\; q_i = 0$$

and neglecting the last equation of (34):

$$\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x^i}(\varrho v_i) = 0;$$

$$\frac{\partial}{\partial t}(\varrho v_j) + \frac{\partial}{\partial x^i}(\varrho v_i v_j + p\delta_{ij} - \sigma_{ij}) = 0;$$

$$\frac{\partial}{\partial t}\left(\varrho e + \varrho\frac{v^2}{2}\right) + \frac{\partial}{\partial x^k}\left\{\left(\varrho e + \varrho\frac{v^2}{2} + p\right) v_k - \sigma_{kj} v_j\right\} = 0$$

$$\frac{\partial}{\partial t}\left\{\varrho\left(v_i v_j - \frac{v^2}{3}\delta_{ij}\right) - \sigma_{ij}\right\} + \frac{\partial \overline{F}_{\langle ij \rangle k}}{\partial x_k} = \tau_0 \sigma_{ij}$$

($\overline{F}_{\langle ij \rangle k} = F_{\langle ij \rangle k}\,|_{q_m = 0}$). The maximum characteristic velocity is now smaller according with the subcharacteristic theorem:

$$\lambda_{\max} = 1.34 c_S .$$

While the equilibrium Euler system is a principal subsystem of the 13 and 10 moments system with

$$\Omega_i = 0, \quad \Lambda_{\langle i,j\rangle} = 0 \;\rightarrow\; q_i = 0, \quad \sigma_{ij} = 0$$

and neglecting the last equation of (35):

$$\begin{aligned}
&\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x^i}(\varrho v_i) = 0\,;\\
&\frac{\partial}{\partial t}(\varrho v_j) + \frac{\partial}{\partial x^i}(\varrho v_i v_j + p\delta_{ij}) = 0\,;\\
&\frac{\partial}{\partial t}\left(\varrho e + \varrho\frac{v^2}{2}\right) + \frac{\partial}{\partial x^k}\left\{\left(\varrho e + \varrho\frac{v^2}{2} + p\right) v_k\right\} = 0\,.
\end{aligned}$$

The maximum velocity is now

$$\lambda_{\max} = 1\, c_S.$$

7.4. *Characteristic velocities.*

Therefore for any n we have an ET theory with symmetric hyperbolic differential system and finite characteristic velocities.

Now we ask what happens if n become large. The special form of symmetric hyperbolic system permits (via Routh-Hurwitz inequalities) to deduce a lower bound for the maximum characteristic velocity in terms of the number n of truncation:

In the classical case, we have (Boillat and Ruggeri (1997) [24]):

$$\frac{\lambda_{\max}}{c_S} \geqslant \sqrt{\frac{6}{5}\left(n - \frac{1}{2}\right)} \tag{36}$$

where c_S is the sound velocity. Therefore, we have the surprising result that $\lambda_{\max}$ becomes unbounded when $n \rightarrow \infty$. While in the relativistic theory we obtain (c is the light velocity and K_n denotes Bessel functions of the second kind):

$$\frac{(2n-1)}{\gamma}\,\frac{K_{n+1}(\gamma)}{K_{n+2}(\gamma)} \leqslant \frac{\lambda^2_{\max}}{c^2} \leqslant 1, \quad \gamma = \frac{mc^2}{kT}\,.$$

and it is easy to prove that when the number of moments tends to infinity the maximum velocity in equilibrium tends to the light velocity (Boillat and Ruggeri (1999) [25], [26]; Brini and Ruggeri (1999) [27]).

7.5. *Applications of ET.*

The Extended Thermodynamics was applied in several physical problems with a good agreement with experimental data. In fact if the gas is very rarefied the Navier-Stokes Fourier theory gives bad results while the ET with many moments gives very satisfactory results.

Some of these new results concerns the dispersion relation of sound waves (Weiss 1990) [28]; Light Scattering (Weiss and Müller, 1995 [29]); Extended Thermodynamics of Radiation (Struchtrup, 1997 [30]); Evaluation of the heat conductivity and Bulk Viscosity in Reacting Gases (Kremer and Müller 1996 [31]); Hydrodynamical Models for Semiconductors (Anile et Coworkers see e.g. [32]); The nesting of theories of increasing order (Boillat and Ruggeri, 1996 [13]); The Extended Thermodynamics at kinetic level and Maximum Entropy Principle (Dreyer 1987 [33], Boillat and Ruggeri, 1996 [24]); Shock Structure problem (Ruggeri, 1993 [34], Weiss 1996 [35], Boillat and Ruggeri 1998 [14]).

8. – Conclusions.

We have seen that the entropy principle is not only an indication of an arrow in the time direction but plays an important role: to select the constitutive equations, to restore the uniqueness for weak solutions, to identify *symmetries* and *nesting* structure, to establish the well-posedness of the Cauchy problem and to close the system of the Extended Thermodynamics.

REFERENCES

[1] B. D. Coleman - W. Noll, *The thermomechanics of elastic materials with heat conduction and viscosity*, Arch. Rational Mech. Anal., **13** (1963), 167-178.

[2] I. Müller, *On the entropy inequality*, Arch. Rational Mech. Anal., **26** (1967), 118-141.

[3] A. Green - G. Keller - G. Warnecke Eds., *Entropy*, Series in Appl. Math. Princeton University Press, Princeton (2003).

[4] P. D. Lax, *Shock Waves and Entropy*, Contributions to Functional Analysis, 603-634, Ed. E. A. Zarantonello, New York, Academic Press (1971).

[5] K. O. Friedrichs - P. D. Lax, *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci. USA, **68** (1971), 1686-1688.

[6] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer Verlag, Berlin (2000).

[7] T.-P. Liu, *The Riemann Problem for General System of Conservation Laws*, J. Differential Equations, **18** (1975), 218-234.

[8] T.-P. LIU, *The Admissible Solutions of Hyperbolic Conservation Laws*, Memoir of AMS, **240** (1981), 78.

[9] T.-P. LIU - T. RUGGERI, *Entropy Production and Admissibility of Shocks*, Acta Math. Appl. Sin., Engl. Ser., **19**, No. 1 (2003), 1-12.

[10] T. RUGGERI - A. STRUMIA, *Main field and convex covariant density for quasi-linear hyperbolic systems. Relativistic fluid dynamics*, Ann. Inst. Henri Poincarè, **34-A** (1981), 65-84.

[11] G. BOILLAT, *Sur l'Existence et la Recherche d'Équations de Conservation Supplémentaires pour les Systèmes Hyperboliques*, C.R. Acad. Sc. Paris, **278-A** (1974), 909-912. Non Linear Fields and Waves. In CIME Course, *Recent Mathematical Methods in Nonlinear Wave Propagation*, Lecture Notes in Mathematics **1640**, T. Ruggeri Ed. Springer-Verlag (1995), 103-152.

[12] S. K. GODUNOV, *An interesting class of quasilinear systems*, Sov. Math., Dokl. 2 (1961), 947-949; translation from Dokl. Akad. Nauk SSSR, **139** (1961), 521-523.

[13] G. BOILLAT - T. RUGGERI, *Hyperbolic Principal Subsystems: Entropy Convexity and Sub characteristic Conditions*, Arch. Rat. Mech. Anal., **137** (1997), 305-320.

[14] G. BOILLAT - T. RUGGERI, *On the shock structure problem for hyperbolic system of balance laws and convex entropy*, Contin. Mech. Thermodyn., **10**, No. 5 (1998), 285-292.

[15] T. RUGGERI, *Maximum of Entropy Density in Equilibrium and Minimax Principle for an Hyperbolic System of Balance Laws*, Contributions to Continuum Theories, Anniversary Volume for Krzysztoff Wilmanski, B. Albers editor WIAS-Report No. 18 (2000).

[16] T. RUGGERI - D. SERRE, *Stability of constant equilibrium state for dissipative balance laws system with a convex entropy*, Quarterly of Applied Math. To appear (2003).

[17] S. KAWASHIMA, *Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. R. Soc. Edinb., Sect. A, **106** (1987), 169-194.

[18] A. E. FISCHER - J. E. MARSDEN, *The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system*, Commun. Math. Phys., **28** (1972), 1-38.

[19] A. MAJDA, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer Verlag, New York (1984).

[20] Y. ZENG, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Ration. Mech. Anal., **150**, No. 3 (1999), 225-279.

[21] B. HANOUZET - R. NATALINI, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Rat. Mech. Anal., **169** (2003), 89-117.

[22] I. MÜLLER - T. RUGGERI, *Rational Extended Thermodynamics*, 2nd ed., Springer Tracts in Natural Philosophy **37**, Springer-Verlag, New York (1998).

[23] H. GRAD, *On the kinetic theory of rarefied gases*, Comm. Appl. Math., **2** (1949), 331-407.

[24] G. BOILLAT - T. RUGGERI, *Moment equations in the kinetic theory of gases and wave velocities*, Contin. Mech. Thermodyn., **9**, No. 4 (1997), 205-212.

[25] G. BOILLAT - T. RUGGERI, *Maximum wave velocity in the moments' system of a relativistic gas*, Contin. Mech. Thermodyn., **11**, No. 2 (1999), 107-111.

[26] G. BOILLAT - T. RUGGERI, *Relativistic gas: Moment equations and maximum wave velocity*, J. Math. Phys., **40**, No. 12 (1999), 6399-6406.

[27] F. Brini - T. Ruggeri, *Maximum velocity for wave propagation in a relativistic rarefied gas*, Contin. Mech. Thermodyn., **11**, No. 5 (1999), 331-338.
[28] W. Weiss, *Die Berechnung von kontinuierlichen Stoßstrukturen in der Kinetischen Gastheorie*, Habilitation thesis TU Berlin (1997).
[29] W. Weiss - I. Müller, *Light scattering and extended thermodynamics*, Cont. Mech. Thermodyn., **7** (1995), 123-144.
[30] H. Struchtrup, *An extended moment method in radiative transfer: The matrices of mean absorption and scattering coefficients*, Annals of Physics, **257**, No. 2 (1997), 111-135.
[31] G. M. Kremer - I. Müller, *Thermal conductivity and dynamic pressure in extended thermodynamics of chemically reacting mixtures of gases*, Ann. Inst. Henri Poincaré, Phys. Théor., **69**, No. 3 (1998), 309-334.
[32] A. M. Anile - V. Romano - G. Russo, *Extended hydrodynamical model of carrier transport in semiconductors*, SIAM J. Appl. Math., **61**, No. 1 (2000), 74-101.
[33] W. Dreyer, *Maximisation of the Entropy in Non-Equilibrium*, J. Phys. A: Math. Gen., **20** (1987), 6505-6512.
[34] T. Ruggeri, *Breakdown of Shock Wave Structure Solutions*, Phys. Rev., **47-E**, (6) (1993), 4135-4140.
[35] W. Weiss, *Continuous shock structure in extended thermodynamics*, Phys. Review E, Part A, **52** (1995), 5760-5768.

Department of Mathematics and Research Center
of Applied Mathematics (CIRAM)
University of Bologna, Via Saragozza 8, 40123 Bologna, Italy
E-mail: ruggeri@ciram.unibo.it

Pervenuta in Redazione
il 6 dicembre 2003