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Boundary Map and Overrings of Half-factorial Domains.

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Sunto. – *In questo articolo studiamo la fattorizzazione di elementi nei sopranelli di un dominio metà-fattoriale A in funzione del comportamento della funzione di bordo di A. A tale riguardo, troviamo che gioca un ruolo centrale una condizione sulle estensioni, che chiamiamo condizione* $\mathcal{C}^\star$. *Quindi studiamo quando questa condizione* $\mathcal{C}C^\star$ *è verificata. Infine, applichiamo i risultati ottenuti al caso speciale degli anelli di polinomi.*

Summary. – *We investigate factorization of elements in overrings of a half-factorial domain A in relation with the behaviour of the boundary map of A. It turns out that a condition, called* $\mathcal{C}^\star$, *on the extension plays a central role in this study. We finally apply our results to the special case of* $A + XB[X]$ *polynomial rings.*

In 1960, Carlitz [4] proved that the class number of an algebraic number ring is less or equal to 2 if and only if each nonzero nonunit x factors as a product of irreducible elements so that the number of such irreducible factors only depends on the element x. Then, we say that a domain R is *atomic* if each nonzero nonunit of R factors as a product of irreducible elements, and that an atomic domain is a *half-factorial domain* (or HFD) if each equality

$$\pi_1 \ldots \pi_r = \tau_1 \ldots \tau_s$$

with the π_i, τ_j's irreducible in R, implies $r = s$.

The study of the properties of HFDs has been a fruitful topic these last past years (see [5] for a survey). In particular, since HFDs generalize UFDs, we aim to know which of the properties of UFDs are still true for HFDs. For instance, a domain R may be a HFD whereas the polynomial ring $R[X]$ is not – more precisely, Coykendall proved in [8] that, if $R[X]$ is a HFD, then R is integrally closed whereas there are non-integrally closed HFDs (for instance $\mathbb{Z}[\sqrt{-3}]$). Another question in the same vein was to know if a localization of a HFD is a HFD, this question has been studied by D.F. Anderson, Chapman and Smith in [1] and by D.F. Anderson and Park in [2] for the case of Dedekind domains. More generally, we can ask if an overring of a HFD is a HFD. Of course, it is false in general (for instance, if R is not one-dimensional, then it admits non-discrete valuation overrings, whence non-atomic overrings) but we aim to characterize which overrings are HFDs. In particular, a natural conjecture then turns out: if R is a HFD, is its integral closure $\overline{R}$ also a HFD? In 1983, Halter-Koch gave a positive response for the case of orders in quadratic algebraic number rings [14], which was generalized to the general case of algebraic number rings by Coykendall in 1999 [9] who nevertheless proved in [11] that this conjecture fails in general. Anyway, in [9], Coykendall introduced a new tool, the boundary map of a HFD, which allows us better investigations of factorization properties in the overrings of a HFD.

The aim of this paper is, given a half-factorial domain A, to study the behaviour of the boundary map of A on its overrings and then, to derive conditions for these overrings to be half-factorial.

If R is an integral domain, then $\mathcal{U}(R)$ will denote its group of units and R^* its set of nonzero elements. We will often use the word *atom* for an irreducible element of an integral domain. As usual, $\mathbb{Z}$ will denote the ring of integers and $\mathbb{N}$ the set of nonnegative integers. All rings are commutative with identity and integral domains.

1. – Integral characters of an integral domain.

DEFINITION 1.1. – Let A be an integral domain with quotient field K. We call *an integral character* on A, each function $\varphi : A \to \mathbb{Z}$ such that

$$\varphi(xy) = \varphi(x) + \varphi(y)$$

for all $x, y \in A$. If $\varphi(A) \neq \{0\}$, we say that φ is *non trivial* on A.

Then, for every $x, y \in A$, set:

$$\varphi\left(\frac{x}{y}\right) = \varphi(x) - \varphi(y).$$

That is, we extend the integral character φ to K, and we then say that φ is an *integral character* on K. If $\varphi(K) \neq \{0\}$, we say that φ is *non trivial* on K.

Note that $\varphi(K)$ is a subgroup of $\mathbb{Z}$. Thus we can always assume that $\varphi(K) = \mathbb{Z}$. From now on, we will always make this assumption. The following example will be the main interest of this paper.

EXAMPLE 1.2. – Let us consider an atomic domain A with quotient field K and a pseudo-length function $\ell : A \to \mathbb{N}$ on A that is [13]:

(*i*) $\ell(xy) = \ell(x) + \ell(y)$ for all x, y in A^*

(*ii*) $\ell(x) > 0$ for each nonprime irreducible element x of A.

We can then extend this function to K^* by setting:

$$\partial_{A,\ell}\left(\frac{a}{b}\right) = \ell(a) - \ell(b)$$

for each a, $b \in A^*$. The function $\partial_{A,\ell}$ is called the *boundary map related to* A *and* ℓ.

In the particular case of a HFD A, there exists a (pseudo-)length function ℓ on A such that $\ell(x) = 1$ if and only if x is an atom [17] (in particular, $\ell(x) = 0$ if and only if x is a unit of A). Then the associated *boundary map* is defined by $\partial_A(x) = r - s$ where $x = \frac{\pi_1 \dots \pi_r}{\tau_1 \dots \tau_s}$ with the π_i, τ_j's irreducible in A [9].

If A is an integral domain with quotient field K and φ is an integral character on K, then we will often say that φ is an integral character on A.

DEFINITION 1.3. – Let A be an integral domain with quotient field K and let φ be a non trivial integral character on K. Then φ is said to be *positive* on A if $\varphi(x) \geqslant 0$ for all $x \in A$. If moreover, $\varphi(x) > 0$ for all nonunit $x \in A$, then φ is said to be *strictly positive* on A.

Respectively, we say that φ is *negative on* A if $\varphi(x) \leqslant 0$ for all $x \in A$, and that φ is *strictly negative on* A if moreover $\varphi(x) < 0$ for all nonunit $x \in A$.

EXAMPLE 1.4. – Let us consider an atomic domain A with quotient field K and a pseudo-length function ℓ on A. Then the boundary map associated to ℓ is positive on A.

EXAMPLE 1.5. – Let us consider a (rank-one) discrete valuation ring V with quotient field K and let us denote v the valuation. Then v is a non-trivial integral character on K^* which is strictly positive on V.

We first give a consequence of the positiveness of an integral character.

LEMMA 1.6. – *If* φ *is positive on* A, *then* $\varphi(u) = 0$ *for each unit* u *of* A.

PROOF. – We have $\varphi(u) + \varphi(u^{-1}) = \varphi(1) = 0$ and the result follows as φ is positive. ■

Note that it may occur that $\varphi(u) = 0$ for each unit u of A but φ takes both positive and negative values on A. Indeed, consider the integral character φ defined on $K[X, Y]$ by $\varphi = v_X - v_Y$, where v_X and v_Y respectively denote the X-adic and the Y-adic valuations on $K[X, Y]$. Then $\varphi(u) = 0$ for each unit u of $K[X, Y]$ nevertheless $\varphi(X) = 1$ and $\varphi(Y) = -1$.

PROPOSITION 1.7. – *If A is not a field and φ is an integral character on A, then the following are equivalent*:

(*i*) *φ is either strictly positive or strictly negative on A*
(*ii*) *$\varphi(x) \neq 0$ for each nonunit $x \in A$.*

PROOF. – The fact that (*i*) implies (*ii*) is clear. Conversely, assume that $\varphi(x) \neq 0$ for each nonunit $x \in A$, it suffices to show that φ is either positive or negative. Assume, by way of contradiction, that there exist nonunits x, y in A with $\varphi(x) = m > 0$ and $\varphi(y) = -n < 0$, then we have:

$$\varphi(x^n y^m) = 0 \,.$$

It follows that the element $x^n y^m$ is invertible in A, whence x and y are both invertible in A. This contradicts the choice of x and y. ■

The next proposition gives an interesting example of a strictly positive integral character which will be useful in the remainder of this paper.

PROPOSITION 1.8. – *Let φ be a non-trivial integral character on A and consider the multiplicatively closed set $S = \{x \in A;\ \varphi(x) = 0\}$.*

(*i*) *If φ is positive on A, then $S^{-1}A \neq K$ and φ is strictly positive on $S^{-1}A$.*
(*ii*) *If φ takes both positive and negative values, then $S^{-1}A = K$.*

PROOF. – (*i*) Let x be a nonzero element of $S^{-1}A$ and write $x = \frac{a}{s}$ with $a \in A^*$ and $s \in S$. Then $\varphi(x) = \varphi(a) - \varphi(s) = \varphi(a) \geqslant 0$ since φ is positive on A. Therefore φ is positive on $S^{-1}A$. Moreover, if x is a nonunit, then $a \notin S$, that is, $\varphi(a) > 0$. Thus $\varphi(x) > 0$, that is, φ is strictly positive on $S^{-1}A$. Lastly, assume that $S^{-1}A = K$, it follows from Lemma 1.6 that φ is trivial on $S^{-1}A$ thus on A, we reach a contradiction.

(*ii*) Let us consider an element $x \in A$ such that $\varphi(x) \neq 0$, say $\varphi(x) = m > 0$. Then there exists $y \in A$ with $\varphi(y) = -n < 0$. We have $\varphi(x^n y^m) = 0$ that is $x^n y^m$ is invertible in $S^{-1}A$, hence so is x. Since each nonzero element of A is invertible in $S^{-1}A$, $S^{-1}A$ is a field and $S^{-1}A = K$. ■

Now, we investigate some consequences of the notion of strictly positive integral character.

PROPOSITION 1.9. – *If φ is strictly positive integral character on an integral domain A, then A is a bounded factorization domain* (*BFD*). *In particular, A is an atomic domain.*

PROOF. – Let us consider an ascending chain $Ax_0 \subset Ax_1 \subset Ax_2 \subset \dots$ of principal ideals of A. Then, for each $n \geq 0$, we can write $x_{n+1} = x_n y_n$ where y_n is a nonunit of A. Since φ is strictly positive on A, we thus have $\varphi(x_{n+1}) < \varphi(x_n)$. Hence the sequence $(\varphi(x_n))_{n \in \mathbb{N}}$ strictly decreases in $\mathbb{N}$ and it thus follows that A satisfies the ascending chain condition on principal ideals. Therefore A is atomic.

Now, consider a nonzero nonunit x of A and a factorization $x = \xi_1 \dots \xi_n$ as a product of irreducible factors. Then

$$\varphi(x) = \varphi(\xi_1) + \dots + \varphi(\xi_n).$$

Since φ is strictly positive on A, the $\varphi(\xi_i)$'s are positive integers, thus n is bounded by $\varphi(x)$. ■

REMARK 1.10. – If φ is a strictly positive integral character on an integral domain A and if $\varphi(x) = 1$, then x is an atom. Indeed, write $x = ab$, then $\varphi(a) + \varphi(b) = \varphi(x) = 1$, whence $\varphi(a) = 0$ or $\varphi(b) = 0$ that is, a or b is a unit of A.

Note that the converse fails. Indeed, consider the X-adic valuation v_X on the integral domain $K[X^2, X^3]$, then X^2 is an atom but $v_X(X^2) = 2$.

In fact, if A is an atomic domain then, A is half-factorial if and only if there is a positive integral character on A which takes the value 1 exactly on the atoms (see [17]).

PROPOSITION 1.11. – *Let us consider two domains $A \subset B$ with the same quotient field K and an integral character φ on K. If φ is strictly positive on A and positive on B, then $\mathcal{U}(A) = \mathcal{U}(B) \cap A$.*

PROOF. – It is clear that the units of A are units of B. Conversely, if u is a unit of B which belongs to A then, from Lemma 1.6, $\varphi(u) = 0$. Since φ is strictly positive on A, it follows that u is a unit of A. ■

Note that it is not sufficent to assume φ strictly positive on B. For instance, the p-adic valuation is strictly positive on $\mathbb{Z}_{(p)}$ but not on $\mathbb{Z}$.

We now focuse on the case of boundary maps. Let A be a half-factorial domain with quotient field K and B be an overring of A. Recall that the *boundary map* of A is the function $\partial_A \colon K^* \to \mathbb{Z}$ defined by $\partial_A(u) = 0$ for each $u \in \mathcal{U}(A)$ and

$$\partial_A\left(\frac{\pi_1 \dots \pi_r}{\tau_1 \dots \tau_s}\right) = r - s$$

for every irreducible elements π_i, τ_j of A. Since the boundary map ∂_A is clearly strictly positive on A, we obtain:

COROLLARY 1.12. – *If* ∂_A *is positive on* B, *then*:

(i) *For each unit* u *of* B, $\partial_A(u) = 0$.
(ii) $\mathcal{U}(A) = \mathcal{U}(B) \cap A$.

Then, from the previous corollary, Proposition 1.9 and Remark 1.10, we derive:

COROLLARY 1.13. – *If* ∂_A *is strictly positive on* B *then*:

(i) ∂_A *is positive on* B.
(ii) *For each unit* u *of* B, $\partial_A(u) = 0$.
(iii) B *is a BFD* (*in particular* B *is atomic*).
(iv) $\mathcal{U}(A) = \mathcal{U}(B) \cap A$.
(v) *Each atom of* A *is an atom of* B.

This result allows us to give an example of an atomic overring which admits a nonunit element of boundary zero (giving a negative answer to the last question of [11] or [6, Problem 27]): it is sufficient to find an irreducible element of A which does not remain irreducible in B.

EXAMPLE 1.14. – Set $A = \mathbb{Z} + X\mathbb{Z}[t][X]$ and $B = \mathbb{Z}[t, X]$. Then A is a HFD [12, Proposition 1.8] and B is a factorial overring of A. The element $f = X(t+X)$ is irreducible in A but not in B and $t + X = \frac{[X(t+X)]}{X}$ is a nonzero nonunit of B with boundary 0.

In this example, the element with boundary 0 is prime (since the top ring is a UFD). We can give another example with a boundary 0 element which is irreducible but not prime in B.

EXAMPLE 1.15. – Set $A = \mathbb{Z} + X\mathbb{Z}[t][X]$ and $B = \mathbb{Z}[t^2, t^3] + X\mathbb{Z}[t][X]$. Then A is a HFD and B is an overring of A which is not an HFD (since $\mathbb{Z}[t^2, t^3]$ is not an HFD). The element $f = Xt^2$ is irreducible in A but not in B and $t^2 = \frac{[Xt^2]}{X}$ is a nonzero nonunit of B with boundary 0 which is not prime in B.

It is easy to see that, in Corollary 1.13, (v) implies (iv) (but the converse fails). Moreover, (v) is an improvement of [9, Corollary 2.6]. Now, we ask:

QUESTION 1. – If B satisfies conditions (i), (iii) and (v), is ∂_A strictly positive on B?

In the following remark, we give a positive answer to the previous question in the case when the conductor $[A : B] = \{x \in B, xB \subseteq A\}$ contains a prime element of B.

REMARK 1.16. – Let us consider A an HFD, B an overring. We suppose that there exists a prime π of B such that $\pi B \subsetneq A$ (that is, $\pi \in [A : B]$). In this case, ∂_A is strictly positive on B if and only if each irreducible element of A remains irreducible in B and $\mathcal{U}(B) \cap A = \mathcal{U}(A)$.

Indeed, if the condition on units is satisfied π is an irreducible element of A. Let $b \in B$ such that $\partial_A(b) = 0$ then $\pi b \in A$ and $\partial_A(\pi b) = \partial_A(\pi) = 1$; thus $\pi b = \tau$ is an irreducible element of A. As each irreducible of A is irreducible in B, we conclude that b is a unit of B. The converse follows from corollary 1.13.

2. – Overrings of half-factorial domains.

Troughout this section, A is a half-factorial domain (HFD) with quotient field K and B is a proper overring of A, that is, $A \subset B \subset K$.

The purpose of the following is to investigate factorization in the overring B of A in relation with the behaviour of the boundary map ∂_A on B. The key fact of this section is that the boundary map is strictly positive on A.

PROPOSITION 2.1. – *Assume that the atoms of A are atoms of B and that B is a HFD. Then ∂_A is strictly positive on B.*

PROOF. – Let x be a nonzero element of B of boundary 0 and write

$$x = \frac{\pi_1 \dots \pi_r}{\tau_1 \dots \tau_r}$$

where the π_i, τ_j's are irreducible in A, then we obtain:

$$\tau_1 \dots \tau_r x = \pi_1 \dots \pi_r.$$

Since each atom of A is an atom of B and since B is a HFD, it follows that x is a unit of B. ■

Now, we recall a condition on extensions which is often used in factorization problems (see for instance [10], [12], [14], [15] and [16]).

DEFINITION 2.2. – We say that an extension of integral domains $R \subsetneq T$ satisfies the condition $\mathcal{C}^\star$ if for each element $t \in T$, there exists a unit u of T such that $ut \in R$.

REMARK 2.3. – Let A be an atomic domain and B be an overring of A such that the extension $A \subset B$ verifies $\mathcal{C}^\star$ and such that each atom of A is an atom of B. Thus $\mathcal{U}(B) \cap A = \mathcal{U}(A)$, B is also atomic and the atoms of B are of the form $u\pi$ where π is an atom of A.

Indeed, let x be a nonzero nonunit of B. Then, there exists a unit u of B

such that $ux \in A$. Since A is atomic and as ux is a nonunit of A, we can write $ux = \pi_1 \dots \pi_n$ where the π_i's are irreducible in A. That is $x = u^{-1}\pi_1 \dots \pi_n$, where u^{-1} is a unit of B and $\pi_1, \dots, \pi_n$ are atoms of A, whence of B. Therefore B is atomic.

Moreover, since the atoms of A are atoms of B, a product $u\pi$ (where u is a unit of B and π is an atom of A) is an atom of B. Conversely, let τ be an atom of B, then there exists a unit u of B with $u\tau \in A$. Write $u\tau = xy$ with x, y in A. As τ is an atom of B, x or y is a unit of B, say x. Then $x \in \mathcal{U}(B) \cap A$, that is $x \in \mathcal{U}(A)$. Therefore $u\tau$ is an atom of A.

Proposition 2.4. – *Assume that the extension $A \subset B$ satisfies $\mathcal{C}^\star$ and that $\partial_A(u) = 0$ for each unit u of B, then ∂_A is strictly positive on B and B is a HFD.*

Proof. – Let b be a nonzero nonunit of B. Then there exists a unit u of B such that ub is a nonzero nonunit of R, thus $\partial_A(ub) > 0$, therefore $\partial_A(b) = \partial_A(u) + \partial_A(b) > 0$.

It follows from Corollary 1.13 and the previous remark that B is atomic. Write $x_1 \dots x_m = y_1 \dots y_n$ with the x_i, y_j's irreducible in B. For each i, there is a unit u_i of B such that $x_i' = u_i x_i$ is an atom of A, and for each j, there is a unit v_j of B such that $y_j' = v_j y_j$ is an atom of A. Set $u = u_1 \dots u_m$ and $v = v_1 \dots v_n$, then $vx_1' \dots x_m' = uy_1' \dots y_n'$, thus:

$$\partial_A(v) + \partial_A(x_1') + \dots + \partial_A(x_m') = \partial_A(u) + \partial_A(y_1') + \dots + \partial_A(y_n')$$

Whence $m = n$. ■

Example 2.5. – [1] Let A be a Dedekind domain with class group $\mathbb{Z}_6$ and such that the set of nonzero ideal classes which contain prime ideal is $S_A = \{1, 2, 3\}$. Then A is HFD [7].

Let $\mathfrak{p}$ be a prime ideal of A which lies in class 3. Then there exists an element $t \in \mathfrak{p}$ such that t is not in any prime of classes 1 and 2. Set $T = \{1, t, t^2, \dots\}$ and $B = T^{-1}D = D[1/t]$. The extension $A \subset B$ satisfies $\mathcal{C}^\star$ but ∂_A is not strictly positive on B. Indeed, there exist units in B with nonzero boundary. For example, as $\mathfrak{p}$ is a prime ideal which lies in class 3, there exists an irreducible element $\alpha \in A$ such that $A\alpha = \mathfrak{p}^2$.

Here is an example of a half-factorial polynomial overring of a HFD such that the extension satisfies the condition $\mathcal{C}^\star$.

Example 2.6. – Let $A = \mathbb{Z} + X\mathbb{Z}[t, X]$ and $B = \mathbb{Z}[t^2, t^3] + X\mathbb{Z}[t, X]$. We have seen, in Example 1, that there exist elements of B with boundary zero. Set $S = \{b \in B,\ \partial_A(b) = 0\}$, then:

$$A \subset B \subset \mathbb{Z}[t, X] \subset S^{-1}B$$

and $S^{-1}B \neq \mathbb{Q}(t, X)$. Indeed, let us suppose that $\frac{1}{X} = \frac{b}{s}$ with $b \in B$ and $s \in S$, that is $\partial_A(s) = 0$, thus $\partial_A(bX) = 0$. Since $bX \in A$, bX is a unit of A. We obtain a contradiction and then conclude, by Proposition 1.8, that, for each $b \in B$, $\partial_A(b) \geqslant 0$.

Each nonzero nonunit u of $S^{-1}B$ has a nonzero boundary. From Corollary 1.13, $S^{-1}B$ is atomic, $\mathcal{U}(S^{-1}B) \cap A = \mathcal{U}(A)$ and each irreducible element in A remains irreducible in $S^{-1}B$. We now prove that each irreducible element of $S^{-1}B$ is associated to an irreducible element of A, that is, the extension $A \subset S^{-1}B$ satisfies $\mathcal{C}^\star$.

Let g be an irreducible element of $S^{-1}B$, write $g = \frac{\alpha}{\beta}$ with $\alpha \in B$ and $\beta \in S \subset \mathcal{U}(S^{-1}B)$. So, up to a unit of $S^{-1}B$, we can assume that $g \in B$. If $g \in A$ then g is irreducible in A (by the condition on units), thus assume that $g \notin A$ and that g is not associated to any element of A. Consider the nonzero nonunit element gX of A and consider the following factorization $gX = f_1 \dots f_n$, where $f_1, \dots, f_n$ are atoms of A. Assume that $n = 1$ then $gX = f_1$. It follows that $\partial_A(g) = 0$ (indeed, $\partial_A(g) + \partial_A(X) = \partial_A(f_1)$) which contradicts the fact that g is irreducible in $S^{-1}B$. Thus $n \geqslant 2$. One of the f_i's is of order 1, say $f_1 = Xh$, where $h \in \mathbb{Z}[X, t] \subset S^{-1}B$. Thus we can write $g = h(f_2 \dots f_n)$. As g is irreducible in $S^{-1}B$ and $(f_2 \dots f_n)$ is a nonunit of A, we conclude that h is a unit of $S^{-1}B$. Consequently, g is associated to an element of A which contradicts our hypothesis. From Proposition 2.4, we then conclude that $S^{-1}B$ is HFD.

Recall that $\partial_A(\alpha) \geqslant 0$ whenever $\alpha \in K$ is almost integral over A [9, Lemma 2.3]. We first summarize some properties in this case.

PROPOSITION 2.7. – *If the extension $A \subset B$ is almost-integral, then:*

(*i*) *For each nonzero α in B, $\partial_A(\alpha) \geqslant 0$.*
(*ii*) *For each unit u of B, $\partial_A(u) = 0$.*
(*iii*) *$\mathcal{U}(A) = \mathcal{U}(B) \cap A$.*

Note that Coykendall gave, in [11], an example of an integral extension $A \subset B$ such that there exist nonunit elements with boundary 0, moreover in this case B is exactly the integral closure of the half-factorial domain A. It leads to the following question:

QUESTION 2. – Find an example of an integral extension $A \subset B$ such that B is atomic and there exist nonunit elements with boundary 0.

It seems that all known examples of integral extensions $A \subset B$ with A HFD and B atomic satisfy the condition $\mathcal{C}^\star$. This remark stresses the interest of the following result which can easily be deduced from Proposition 2.4.

THEOREM 2.8. – *Assume that the extension $A \subset B$ is almost-integral and satisfies $\mathcal{C}^\star$, then:*

(*i*) ∂_A *is strictly positive on B.*
(*ii*) *Each atom of A is an atom of B.*
(*iii*) *B is a HFD.*

Now, we give a sufficient condition for an extension to satisfy $\mathcal{C}^\star$.

PROPOSITION 2.9. – *Assume that there exists a prime element* π *of* B *such that* $\pi B \subseteq A$ *and that* $\mathcal{U}(A) = \mathcal{U}(B) \cap A$. *Then, for each atom* x *of* B *with* $\partial_A(x) \geqslant 1$, *there exists a unit* u *of* B *such that* $ux \in A$. *In particular, if* $\partial_A(x) = 1$, *then* ux *is an atom of* A *for some unit* u *of* B.

PROOF. – Since $\pi B \subseteq A$ and $\mathcal{U}(B) \cap A = \mathcal{U}(A)$, π is also an atom of A, thus $\partial_A(\pi) = 1$. Let x be an atom of B, set $\partial_A(x) = k \geqslant 1$, πx is in A and $\partial_A(\pi x) = k + 1 \geqslant 2$. Thus we can write $\pi x = \tau_1 \dots \tau_{k+1}$ where the τ_i's are irreducible in A. Since π is a prime element of B, one of the τ_i's, say τ_1, is in πB. Hence, there exists y in B such that $\tau_1 = \pi y$ and $x = y\tau_2 \dots \tau_{k+1}$. Since x is an atom of B, either y or one of the τ_i's for some $i \geqslant 2$ is a unit of B, whence a unit of A as $\mathcal{U}(B) \cap A = \mathcal{U}(A)$. Since the τ_i's are irreducible in A, they are nonunits, thus $u = y^{-1}$ is a unit of B such that $ux = \tau_2 \dots \tau_{k+1} \in A$.

In the case where $\partial_A(x) = 1$, we then obtain an element ux of A with $\partial_A(ux) = \partial_A(u) + \partial_A(x) = 1$, that is, ux is an atom of A. ∎

Then, from Proposition 2.4, we derive the following corollary which gives a partial positive answer to the conjecture stated in [11].

COROLLARY 2.10. – *Assume that there exists a prime element* π *of* B *such that* $\pi B \subseteq A$ *and that* ∂_A *is strictly positive on* B. *Then, the extension* $A \subset B$ *satisfies the condition* $\mathcal{C}^\star$. *In particular,* B *is a HFD.*

That is, the conjecture of [11] is true whenever the conductor of B in A contains a prime element of B.

3. – Application to polynomial rings.

In this section, we change the notations. Let $A \subset B$ be an extension of integral domains (not necessarily an overring). We set $R = A + XB[X]$ and study the factorization of elements in the overring $B[X]$ when R is a HFD.

So, we assume that $R = A + XB[X]$ is a HFD. Firstly, since the extension $R \subset B[X]$ is almost-integral, we have:

LEMMA 3.1. – *The boundary map* ∂_R *is positive on* $B[X]$. *In particular, we have* $\mathcal{U}(A) = \mathcal{U}(B) \cap A$.

Now, we investigate the boundary of the atoms of $B[X]$.

LEMMA 3.2. – *Let f be an irreducible element of $B[X]$, then either $\partial_R(f) = 0$ or $\partial_R(f) = 1$.*

PROOF. – Let f be an irreducible element of $B[X]$ such that f is in R then, as $\mathcal{U}(A) = \mathcal{U}(B) \cap A$, f is also irreducible in R and $\partial_R(f) = 1$. If f is associated to an element of R, there exists a unit u of $B[X]$ such that $uf \in R$. Hence uf is irreducible in R and $\partial_R(f) = \partial_R(uf) = 1$. So, assume that f is an irreducible of $B[X]$ which is not associated to any element of R. Then $fX \in R$ and fX is irreducible in R. Indeed, write $fX = gh$. We may assume that $h = Xh_1$ where $h_1 \in B[X]$. Then $f = gh_1$. As $h_1 \notin \mathcal{U}(B)$ (from the hypothesis), we have $g \in \mathcal{U}(B) \cap R$, that is, $g \in \mathcal{U}(R)$. Since fX is irreducible in R, one has $\partial_R(fX) = 1$. Whence $\partial_R(f) = 0$. ■

EXAMPLE 3.3. – Let $A \subset B$ be an extension such that $R = A + XB[X]$ is a HFD. Set $T = B[X]$ and $S = \{t \in T,\ \partial_R(T) = 0\}$. Then $R \subset S^{-1}T$ satisfies $\mathcal{C}^\star$. In particular, $S^{-1}T$ is HFD.

Indeed, from Proposition 1.8, we have $S^{-1}T \neq L(X)$ where L is the quotient field of B. Moreover, $S^{-1}T$ is atomic and $\mathcal{U}(S^{-1}T) \cap R = \mathcal{U}(R)$. Thus we just have to prove that each irreducible element of $S^{-1}T$ is associated to an (irreducible) element of R which is given by Proposition 2.9. From Proposition 2.4, we immediately have the last assertion.

Of course, it follows that when there are no boundary zero element in the overring $B[X]$, we obtain a positive answer to the following question:

QUESTION 3. – If $R = A + XB[X]$ is a HFD, is $B[X]$ a HFD?

In fact, we have a bit more than this partial answer:

THEOREM 3.4. – *Let $A \subset B$ an extension of integral domains such that the domain $R = A + XB[X]$ is a HFD and the domain $B[X]$ is atomic. Then the following two conditions are equivalent:*

(*i*) *The extension $A \subset B$ satisfies the condition $\mathcal{C}^\star$.*

(*ii*) *Each atom f of $B[X]$ verifies $\partial_R(f) = 1$.*

In particular, if the previous conditions are fulfilled, then $B[X]$ is a HFD.

PROOF. – Firstly, we assume that the extension $A \subset B$ satisfies $\mathcal{C}^\star$. It is clear that the extension $A + XB[X] \subset B[X]$ satisfies also $\mathcal{C}^\star$. Let f be an atom of $B[X]$, by Lemma 3.2, $\partial_R(f) = 0$ or $\partial_R(f) = 1$. There exists a unit u of B such that uf is an (irreducible) element of R and $\partial_R(f) = \partial_R(uf) = 1$.

Conversely, we conclude by using Proposition 2.9 (where the prime element is X). The last assertion follows from Proposition 2.4. ■

Note that the previous theorem improves one implication of [15, Theorem

13], namely it was proved that R is an HFD if and only if $B[X]$ is an HFD under the condition $\mathcal{C}^\star$ and another condition. Note that we can not improve the second implication in the same way, as attested by the next example [12, Example 2.8].

EXAMPLE 3.5. – We set $B = \mathbb{C}[\![t]\!]$ (the ring of power series with complexes coefficients) and $A = \mathbb{R} + t\mathbb{R} + t^2\mathbb{C}[\![t]\!]$. This ring has been proved to be atomic by Anderson and Park [3, theorem 2.1], and A is not a HFD since $\varrho(A) = 2$ [3, Theorem 3.2]. Thus $A + XB[X]$ is not a HFD, $B[X]$ is a HFD (in fact it is a UFD) and the extension $A \subset B$ satisfies $\mathcal{C}^\star$. Indeed, let f be a non zero element in B. We may write $f = t^r g$ where r is the order of f and g is a unit of B. Then $g^{-1}f = t^r$ is in A.

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