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Uniform Exponential Stability for Linear Discrete Time Systems with Stochastic Perturbations in Hilbert Spaces.

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Sunto. – *In questo lavoro è trattato il problema della stabilità esponenziale e della stabilità esponenziale uniforme per i sistemi discreti variabili in tempo, perturbati con le variabili aleatorie independenti. Ci sono date due rappresentazioni delle soluzioni dei sistemi discussi e si è stabilito il legame tra esse. Ognuna delle due rappresentazioni conduce a stabilire delle condizioni necessarie e sufficienti per ottenere i due tipi di stabilità. C'è dato un teorema di caratterizzazione della stabilità esponenziale uniforme usando le ecuazioni Lyapunov. Nel caso stazionario, i due tipi di stabilità sono equipollenti.*

Summary. – *In this paper we study the exponential and uniform exponential stability problem for linear discrete time-varying systems with independent stochastic perturbations. We give two representations of the solutions of the discussed systems and we use them to obtain necessary and sufficient conditions for the two types of stability. A deterministic characterization of the uniform exponential stability, in terms of Lyapunov equations are given.*

1. – Introduction.

The main object of this paper is to discuss the problem of the exponential and uniform exponential stability of time-varying systems described by linear difference equations in infinite dimensional Hilbert spaces. We give two representations of the solutions of these systems and we establish a relation between them. These representations are very important in order to obtain the characterizations of the two types of stability.

One of these two representations of solutions allows us to reduce the stability problem in the stochastic case to the same one in the deterministic case (see Theorem 9). So, the characterization of the uniform exponential stability of the stochastic systems can be obtained as a consequence of the results of [6]. The other representation (see Theorem 6) leads us to obtain similar results (see Theorem 13) as those formulated in [7], where it is treated the case of linear discrete-time systems with Markov perturbations in finite dimensional spaces. The Theorem 12 establish a relation between the two representations.

A characterization of the uniform exponential stability is given by using the discrete-time Lyapunov equations. This result is similar to those obtained in [6], for the deterministic case and in [9], for the stochastic time-invariant case. Finally, we treated as an application the time-invariant case. We obtained some equivalent characterizations of the uniform exponential stability property of the solutions of the discussed systems and we solved the algebraic Lyapunov equations associated with these systems.

2. – Preliminaries.

Let H be a real separable Hilbert space and $L(H)$ be the Banach space of all bounded linear operators transforming H into H. We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. We denote by $a \otimes b$, $a, b \in H$ the bounded linear operator of $L(H)$ given by $a \otimes b(h) = \langle h, b \rangle a$ for all $h \in H$.

Nuclear operators. The operator $A \in L(H)$ is said to be nonnegative, and we write $A \geqslant 0$, if A is self adjoint and $\langle Ax, x \rangle \geqslant 0$ for all $x \in H$. For $A, B \in L(H)$, $A \geqslant 0$ we denote by $A^{1/2}$ the square root of A (see [3]) and by $|B|$ the operator $(B^* B)^{1/2}$. Let $A \in L(H)$, $A \geqslant 0$ and $\{e_n\}_{n \in \mathbf{N}^*}$ be an orthonormal basis in H. We define $Tr(A)$ by $Tr(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$. It is not difficult to see that $Tr(A)$ is a well defined number independent of the choice of the orthonormal basis $\{e_n\}_{n \in \mathbf{N}^*}$.

If $A \in L(H)$ we put $\|A\|_1 = Tr(|A|) \leqslant \infty$ and we denote by $C_1(H)$ the set $\{A \in L(H) / \|A\|_1 < \infty\}$. The elements of $C_1(H)$ are called nuclear operators. Using the polar decomposition of $A \in L(H)$, it can be proved that $\|A\|_1 = \sup \left\{ \sum_{n=1}^{\infty} |\langle A\xi_n, \eta_n \rangle|, \xi_n, \eta_n \text{ orthonormal systems in } H \right\}$ and by theorems T.9, T.7′ pp. 54-55 in [4] it follows that the definition of the nuclear operator introduced above is equivalent with that given in [4].

It is known (see [4]) that $C_1(H)$ (the operators' trace class) is a Banach space endowed with the norm $\|.\|_1$ and for all $A \in L(H)$ and $B \in C_1(H)$ we have $AB, BA \in C_1(H)$.

We denote by $\mathcal{H}$ and $\mathcal{N}$ the subspaces of $L(H)$ and $C_1(H)$ formed by all self-adjoint operators and by $\mathcal{K}$ (respectively $\mathcal{K}_1$) the cones of all nonnegative operators of $\mathcal{H}$ (respectively $\mathcal{N}$). $\mathcal{H}$ is a Banach space and since $\mathcal{N}$ is closed in $C_1(H)$ with respect to $\|.\|_1$ we deduce that it is a Banach space, too. In this paper we need some well-known results of operators' theory, which we resume below (see [1], [5], [4], [8]).

THEOREM 1 (see [1]). – *If* $A \in \mathcal{H}$ *is a compact operator then there exists an orthonormal basis* $\{e_n\}_{n \in N^*} \subset H$ *and a sequence* $\{\lambda_n\}_{n \in N^*} \subset \boldsymbol{R}$, $\lambda_n \xrightarrow[n \to \infty]{} 0$ *such that* $Ae_n = \lambda_n e_n$ *for all* $n \in \boldsymbol{N}^*$, *that is*

$$A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n, \tag{1}$$

where the convergence is in norm. By convenience we will say that the relation (1) *is a Hilbert-Schmidt decomposition of* A.

PROPOSITION 2 [8]. – *Let A belongs to* $\mathcal{N}$. *Then it is compact and from the above theorem we have* (1) *and* $\|A\|_1 = \sum_{n=1}^{\infty} |\lambda_n|$.

Using Theorem 1 it is easy to establish (see [4]) the following corollary:

COROLLARY 3. – *If* $A \in \mathcal{N}$ *and* $A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ *is the Hilbert-Schmidt decomposition of* A (*Theorem* 1), *where the series is norm convergent, then* $A = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ *is* $\|.\|_1$ *convergent.*

Covariance operators. Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ be a real (or H) valued random variable on Ω. We write $E(\xi)$ for his mean value (expectation). We denote by $L^2 = L^2(\Omega, \mathcal{F}, P, H)$ the space of all equivalence class of H-valued random variables ξ such that $E\|\xi\|^2 < \infty$ (with respect to the equivalence relation $\xi \sim \eta \Leftrightarrow E(\|\xi - \eta\|^2) = 0$).

It is useful to recall (see [2]) that if ξ is a H valued random variable such as $E\|\xi\|^2 < \infty$, then we have $\langle E(\xi), u \rangle = E\langle \xi, u \rangle$ for all $u \in H$.

If $\xi \in L^2$, we define the operator $E(\xi \otimes \xi): H \to H$, $E(\xi \otimes \xi)(u) = E(\langle u, \xi \rangle \xi)$ for all $u \in H$.

It is easy to see that $E(\xi \otimes \xi)$, which is called the covariance operator of ξ, is a linear, bounded and nonnegative operator. Let $\{e_n\}_{n \in N^*}$ be an orthonormal basis in H. Using the Monotone Convergence Theorem and the possibility to commute the inner product and the expectation we have $TrE(\xi \otimes \xi) = \sum_{n=1}^{\infty} E\|\langle \xi, e_n \rangle\|^2 = E \sum_{n=1}^{\infty} \|\langle \xi, e_n \rangle\|^2 = E\|\xi\|^2 < \infty$. Thus $E(\xi \otimes \xi)$ is nuclear and

$$\|E(\xi \otimes \xi)\|_1 = E\|\xi\|^2. \tag{2}$$

3. – Representations of the solutions of linear discrete-time systems.

Let us consider the stochastic system

$$x_{n+1} = A_n x_n + \xi_n B_n x_n, \tag{3}$$

where $A_n, B_n \in L(H)$ and ξ_n are real independent random variables, which satisfy the conditions $E(\xi_n) = 0$ and $E|\xi_n|^2 = b_n < \infty$ for all $n \in \boldsymbol{N}$.

We denote by $X(n, k)$, $n \geq k \geq 0$ the random evolution operator associated with the linear system (3) i.e $X(k, k) = I$ and $X(n, k) = (A_{n-1} + \xi_{n-1} B_{n-1}) \ldots (A_k + \xi_k B_k)$ for all $n > k$.

If $x_n = x_n(k, x)$ is the solution of the system (3) with the initial condition

$$x_k = x, \tag{4}$$

then it is unique and $x_n(k, x) = X(n, k)\, x$.

It is easy to see that if $n > k$, then there exists a continuous function $F : R^{n-k} \to H$ (F is dependent of n, k, x) such that $x_n(\omega) = F(\xi_k(\omega), \ldots, \xi_{n-1}(\omega))$, $\omega \in \Omega$. Thus, it follows that x_n is a H valued random variable.

From the independence of ξ_m, $m = 0, 1, 2, \ldots$ and by using the properties of the independent random variables it results that x_n and ξ_n are independent, too. In the case $n = k$ the last statement is obviously true.

Using the induction, we can prove that $x_n \in L^2$ for all $n \in \boldsymbol{N}$, $n \geq k$.

Since $x_n \in L^2$ and (2) holds we deduce that $E(x_n \otimes x_n)$ is a nuclear, nonnegative operator and

$$\|E(x_n \otimes x_n)\|_1 = E\|x_n\|^2. \tag{5}$$

We consider the linear operator $\overline{A}_n : \mathcal{N} \to \mathcal{N}$, $\overline{A}_n(Y) = A_n Y A_n^*$, which is well-defined because $\mathcal{N}$ is a (left and right) ideal of the space $L(H)$. Since $\|\overline{A}_n(Y)\|_1 \leq \|A_n\|^2 \|Y\|_1$ we deduce that $\overline{A}_n \in L(\mathcal{N})$.

By analogy, we deduce that $\overline{B}_n : \mathcal{N} \to \mathcal{N}$, $\overline{B}_n(Y) = B_n Y B_n^*$ is an element of $L(\mathcal{N})$. We associate to (3) the deterministic system defined on $\mathcal{N}$:

$$y_{n+1} = \overline{A}_n y_n + b_n \overline{B}_n y_n, \tag{6}$$

where $\overline{A}_n$, $\overline{B}_n$, $n \in \boldsymbol{N}$ are the linear operators defined as above.

We consider the bounded linear operator

$$U_n : \mathcal{N} \to \mathcal{N},\ U_n(Y) = \overline{A}_n(Y) + b_n \overline{B}_n(Y). \tag{7}$$

If $Y(n, k)$ is the evolution operator associated with the system (6) then $Y(n, k) = U_{n-1} U_{n-2} \ldots U_k$ if $n - 1 \geq k$ and $Y(k, k) = I$, where I is the identity operator on $\mathcal{N}$. Since, $U_n \in L(\mathcal{N})$ it follows that $Y(n, k) \in L(\mathcal{N})$ for all $n \geq k \geq 0$. Let us denote by $y_n = y_n(k, R)$ the solution of (6) with $y_k = R \in \mathcal{N}$; it is clear that it is unique and $y_n(k, R) = Y(n, k)(R)$ for all $n, k \in \boldsymbol{N}$, $n \geq k$, $R \in \mathcal{N}$.

REMARK 4. – *It is a simple exercise to verify that* $U_n(\mathcal{K}_1) \subseteq \mathcal{K}_1$ *and* $Y(n, k)(\mathcal{K}_1) \subseteq \mathcal{K}_1$ *for all* $n \geq k$, $n, k \in \boldsymbol{N}$.

The following theorem gives a representation of the covariance operator associated to the solution of (3) by using the evolution operator $Y(n, k)$.

THEOREM 5. – *If* $x_n = x_n(k, x)$ *is the solution of* (3), (4), *then* $E(x_n \otimes x_n)$ *is the solution of the system* (6) *with the initial condition* $y_k = x \otimes x$.

PROOF. – Since $x_n \in L^2$ and $\{x_n, \xi_n\}$ are independent random variables for all $n \geq k \geq 0$, we have successively:

$$\langle E(x_n \otimes x_n)\, u, v\rangle = E(\langle u, x_n\rangle\langle x_n, v\rangle) =$$

$$E(\langle u, A_{n-1}x_{n-1} + \xi_{n-1}B_{n-1}x_{n-1}\rangle\langle A_{n-1}x_{n-1} + \xi_{n-1}B_{n-1}x_{n-1}, v\rangle) =$$

$$E(\langle u, A_{n-1}x_{n-1}\rangle\langle A_{n-1}x_{n-1}, v\rangle + \xi_{n-1}\langle u, A_{n-1}x_{n-1}\rangle\langle B_{n-1}x_{n-1}, v\rangle +$$

$$\xi_{n-1}\langle u, B_{n-1}x_{n-1}\rangle\langle A_{n-1}x_{n-1}, v\rangle + \xi_{n-1}^2\langle u, B_{n-1}x_{n-1}\rangle\langle B_{n-1}x_{n-1}, v\rangle).$$

and

$$\langle E(x_n \otimes x_n)u, v\rangle =$$

$$E(\langle u, A_{n-1}x_{n-1}\rangle\langle A_{n-1}x_{n-1}, v\rangle) + b_{n-1}E(\langle u, B_{n-1}x_{n-1}\rangle\langle B_{n-1}x_{n-1}, v\rangle) =$$

$$E(\langle A_{n-1}^*u, x_{n-1}\rangle\langle x_{n-1}, A_{n-1}^*v\rangle) + b_{n-1}E(\langle B_{n-1}^*u, x_{n-1}\rangle\langle x_{n-1}, B_{n-1}^*v\rangle) =$$

$$E(\langle x_{n-1} \otimes x_{n-1}(A_{n-1}^*u), A_{n-1}^*v\rangle + b_{n-1}\langle x_{n-1} \otimes x_{n-1}(B_{n-1}^*u), B_{n-1}^*v\rangle) =$$

$$\langle(\overline{A}_{n-1}E(x_{n-1} \otimes x_{n-1}) + b_{n-1}\overline{B}_{n-1}E(x_{n-1} \otimes x_{n-1}))(u), v\rangle$$

for all $u, v \in H$. In order to obtain the last equality we have used the possibility to commute the inner product and the expectation. Thus we have $E(x_n \otimes x_n) = \overline{A}_{n-1}E(x_{n-1} \otimes x_{n-1}) + b_{n-1}\overline{B}_{n-1}E(x_{n-1} \otimes x_{n-1})$ and $E(x_k \otimes x_k) = x \otimes x$. The conclusion follows from the uniqueness of the solution of (6) with the initial condition $y_k = x \otimes x$. ■

From the above proposition it follows $E(x_n \otimes x_n) = Y(n, k)(x \otimes x)$. By (5), we have

$$E\|x_n(k, x)\|^2 = \|E(x_n \otimes x_n)\|_1 = \|y_n(k, x \otimes x)\|_1.$$

We get

$$E\|X(n, k)x\|^2 = \|Y(n, k)(x \otimes x)\|_1 \tag{8}$$

for all $n \geq k \geq 0$ and $x \in H$.

We consider the mapping $Q_n : \mathcal{H} \to \mathcal{H}$,

$$Q_n(S) = A_n^* S A_n + b_n B_n^* S B_n, \tag{9}$$

where A_n, B_n and $b_n = E|\xi_n|^2 < \infty$ are defined as above.

It is easy to see that Q_n is a linear and bounded operator.

Let us define the operator $T(n, k)$ by $T(n, k) = Q_k Q_{k+1} \dots Q_{n-1} \in L(\mathcal{H})$ for all $n - 1 \geq k$ and $T(k, k) = I$, where I is the identity operator on $\mathcal{H}$.

THEOREM 6. – *If $X(n, k)$ is the random evolution operator associated with the system* (3), *then we have*

$$\langle T(n, k)(S)\, x, y \rangle = E\langle SX(n, k)\, x, X(n, k)\, y \rangle \tag{10}$$

for all $n \geq k \geq 0$, $S \in \mathcal{H}$ and $x, y \in H$.

PROOF. – Let $S \in \mathcal{H}$ and $x, y \in H$. Since $x_{n-1} = X(n-1, k)\, x$ and ξ_{n-1} are independent random variables, we deduce that ξ_{n-1} and $\langle AX(n-1, k)\, x, BX(n-1, k)\, y \rangle$ (resp. ξ_{n-1}^2 and $\langle AX(n-1, k)\, x, BX(n-1, k)\, y \rangle$) are independent, too on $(\Omega, \mathcal{F}, P)$ for all $A, B \in L(H)$. Computing, we get

$$\begin{aligned} E\langle SX(n, k)\, x, X(n, k)\, y \rangle &= E(\langle SA_{n-1} X(n-1, k)\, x, A_{n-1} X(n-1, k)\, y \rangle \\ &\quad + \xi_{n-1} \langle SA_{n-1} X(n-1, k)\, x, B_{n-1} X(n-1, k)\, y \rangle \\ &\quad + \xi_{n-1} \langle SB_{n-1} X(n-1, k)\, x, A_{n-1} X(n-1, k)\, y \rangle \\ &\quad + \xi_{n-1}^2 \langle SB_{n-1} X(n-1, k)\, x, B_{n-1} X(n-1, k)\, y \rangle) \\ &= E\langle A_{n-1}^* S A_{n-1} X(n-1, k)\, x, X(n-1, k)\, y \rangle \\ &\quad + b_{n-1} E\langle B_{n-1}^* S B_{n-1} X(n-1, k)\, x, X(n-1, k)\, y \rangle . \end{aligned}$$

It follows

$$E\langle SX(n, k)\, x, X(n, k)\, y \rangle = E\langle Q_{n-1}(S)\, X(n-1, k)\, x, X(n-1, k)\, y \rangle \tag{11}$$

for all $x, y \in H$. Let us consider the operator $V(n, k) : \mathcal{H} \to \mathcal{H}$,

$$\langle V(n, k)(S)\, x, y \rangle = E\langle SX(n, k)\, x, X(n, k)\, y \rangle \tag{12}$$

for all $S \in \mathcal{H}$ and $x, y \in H$.

It is easy to see that $V(n, k)$ is well defined because the right member of this equality is a symmetric bilinear form, which also defines a unique linear, bounded and self-adjoint operator on H.

From (11) and (12) we obtain $V(n, k)(S) = V(n-1, k)\, Q_{n-1}(S)$ if $n - 1 \geq k$ and $V(k, k) = I$. Now it is easy to see that $V(n, k) = T(n, k)$ and it follows (10). ■

Since $Q_p(\mathcal{K}) \subset \mathcal{K}$ for all $p \in \mathbf{N}$ we deduce that $T(n, k)(\mathcal{K}) \subset \mathcal{K}$.

4. – Theorems which characterize exponential and uniform exponential stability.

We need the following definitions.

DEFINITION 7. – *We say that the system* (3) *is uniformly exponential stable if there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \mathbf{N}$ *such that we have*

$$E\|X(n, k)\, x\|^2 \leq \beta a^{n-k} \|x\|^2 \tag{13}$$

for all $n \geq k \geq n_0$ *and* $x \in H$.

DEFINITION 8. – *The system* (3) *is exponentially stable if there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \mathbf{N}$ *such that we have*

$$E\|X(n, 0)\, x\|^2 \leq \beta a^{n-k} E\|X(k, 0)\, x\|^2 \tag{14}$$

for all $n \geq k \geq n_0$ *and* $x \in H$.

First, we establish a necessary and sufficient condition for the uniform exponential stability (resp. exponential stability) of system (3) by using the evolution operator $Y(n, k) \in L(\mathcal{N})$.

THEOREM 9. – *The system* (3) *is uniformly exponential stable if and only if the system* (6) *is uniformly exponential stable on* $\mathcal{N}$ *or equivalently if and only if there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \mathbf{N}$ *such that*

$$\|Y(n, k)\|_1 \leq \beta a^{n-k} \tag{15}$$

for all $n \geq k \geq n_0$, *where* $\|Y(n, k)\|_1 = \sup\limits_{T \in \mathcal{N},\, \|T\|_1 = 1} \|Y(n, k)(T)\|_1$.

PROOF. – From (8) and the Definition 7 it follows that the uniform exponential stability of system (3) is equivalent with the following assertion: there exist $\beta \geq 1$, $a \in (0, 1)$ and $n_0 \in \mathbf{N}$ such that we have

$$\|Y(n, k)(x \otimes x)\|_1 \leq \beta a^{n-k} \|x \otimes x\|_1 \tag{16}$$

for all $n \geq k \geq n_0$ and $x \in H$.

Because the implication «$\Leftarrow$» is obviously true, we only prove the converse.

«$\Rightarrow$» Let $T \in \mathcal{N}$, $\|T\|_1 = 1$. If $T = \sum\limits_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ is the Hilbert-Schmidt (Theorem 1) decomposition of T, where $\{e_i\}_{i \in \mathbf{N}^*} \subset H$, is an orthonormal basis, then we use Corollary 3 and the boundedness of $Y(n, k)$ and we have

$$\left\| Y(n, k)\left(\sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i\right) \right\|_1 =$$

$$\left\| \sum_{i=1}^{\infty} \lambda_i Y(n, k)(e_i \otimes e_i) \right\|_1 \leq \sum_{i=1}^{\infty} |\lambda_i| \|Y(n, k)(e_i \otimes e_i)\|_1.$$

Since the system $\{e_i\}_{i \in \boldsymbol{N}^*}$ is orthonormal we deduce from the hypothesis and (16) that there exist $\beta \geq 1$, $a \in (0, 1)$ and $n_0 \in \boldsymbol{N}$ such that

$$\|Y(n, k)(e_i \otimes e_i)\|_1 \leq \beta a^{n-k}$$

for all $n \geq k \geq n_0$. Thus $\|Y(n, k)(T)\|_1 \leq \beta a^{n-k} \sum_{i=1}^{\infty} |\lambda_i|$.

By Proposition 2 we get $\|Y(n, k)(T)\|_1 \leq \beta a^{n-k} \|T\|_1 = \beta a^{n-k}$. Now we obtain the conclusion. ■

THEOREM 10. – *The system* (3) *is exponentially stable if and only if there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \boldsymbol{N}$ *such that we have*

$$\|Y(n, 0)(T)\|_1 \leq \beta a^{n-k} \|Y(k, 0)(T)\|_1 \tag{17}$$

for all $n \geq k \geq n_0$ *and* $T \in \mathcal{K}_1$.

PROOF. – «$\Leftarrow$» We consider (17) for $T = x \otimes x$ and we have $\|Y(n, 0)(x \otimes x)\|_1 \leq \beta a^{n-k} \|Y(k, 0)(x \otimes x)\|_1$. By (8) and Definition 8 we obtain the conclusion.

«$\Rightarrow$» Let $T \in \mathcal{K}_1$ and $T = \sum_{i=1}^{\infty} \lambda_i (e_i \otimes e_i)$ be its Hilbert-Schmidt decomposition. Then $\lambda_i \geq 0$ for all $i = 1, 2 \dots$. It follows from the definition of $\|\,.\,\|_1$ that if $T_1, T_2 \in \mathcal{K}_1$ and c, d are real, nonnegative numbers, then $\|cT_1 + dT_2\|_1 = c\|T_1\|_1 + d\|T_2\|_1$.

Thus, if the system (3) is exponentially stable, we use Corollary 3, the boundedness of $Y(n, k)$ and the above property of $\|\,.\,\|_1$ and we have:

$$\|Y(n, 0)(T)\|_1 = \left\| \sum_{i=1}^{\infty} \lambda_i Y(n, 0)(e_i \otimes e_i) \right\|_1 =$$

$$\sum_{i=1}^{\infty} \lambda_i \|Y(n, 0)(e_i \otimes e_i)\|_1 \leq \sum_{i=1}^{\infty} \lambda_i \beta a^{n-k} \|Y(k, 0)(e_i \otimes e_i)\|_1 =$$

$$\beta a^{n-k} \sum_{i=1}^{\infty} \lambda_i \|Y(k, 0)(e_i \otimes e_i)\|_1 = \beta a^{n-k} \|Y(k, 0)(T)\|_1$$

for all $n \geq k \geq n_0$. The proof is finished. ■

The following lemma is known (see [10]).

LEMMA 11. – *Let* $T \in L(\mathcal{H})$. *If* $T(\mathcal{K}) \subset \mathcal{K}$ *then* $\|T\| = \|T(I)\|$, *where* I *is the identity operator on* H.

PROOF. – It is obviously true that $\|T(I)\| \leqslant \|T\|$ and we only will prove the converse.

Let $S \in \mathcal{H}$ such as $\|S\| \leqslant 1$. Then $\|S\| = \sup_{\|x\|=1} |\langle Sx, x\rangle|$ and $-I \leqslant S \leqslant I$. Since $-T(I) \leqslant T(S) \leqslant T(I)$, we have $|\langle T(S)\, x, x\rangle| \leqslant \langle T(I)\, x, x\rangle$ for all $x \in H$. Thus, $\|T(S)\| \leqslant \|T(I)\|$ for all $S \in \mathcal{H}$ such as $\|S\| \leqslant 1$ and we deduce that $\|T\| \leqslant \|T(I)\|$. ■

The following theorem establishes a relation between the operator $T(n, k)$ and the evolution operator $Y(n, k)$.

THEOREM 12. – *If* H *is a real Hilbert space then*

$$\|Y(n, k)(x \otimes x)\|_1 = \langle T(n, k)(I)\, x, x\rangle \tag{18}$$

and

$$\|T(n, k)\| = \|Y(n, k)\|_1, \tag{19}$$

where $\|Y(n, k)\|_1 = \sup_{T \in \mathcal{N},\, \|T\|_1 = 1} \|Y(n, k)(T)\|_1$ *and* I *is the identity operator on* H.

PROOF. – From Theorem 6 we have

$$\langle T(n, k)(I)\, x, x\rangle = E\|X(n, k)\, x\|^2.$$

Now we use (8) and we obtain (18). From (18) we deduce

$$\begin{aligned}
\|T(n, k)(I)\| &= \sup_{x \in H,\, \|x\|=1} \langle T(n, k)(I)\, x, x\rangle = \\
&= \sup_{x \in H,\, \|x\|=1} \|Y(n, k)(x \otimes x)\|_1 \\
&= \sup_{x \otimes x \in \mathcal{N},\, \|x \otimes x\|_1 = 1} \|Y(n, k)(x \otimes x)\|_1 \leqslant \\
&\leqslant \sup_{T \in \mathcal{N},\, \|T\|_1 = 1} \|Y(n, k)(T)\|_1 = \|Y(n, k)\|_1 .
\end{aligned}$$

Now, we prove the opposite inequality. Let $T \in \mathcal{N}$ and $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ be its Hilbert-Schmidt decomposition (Theorem 1). Arguing as in the proof of Theorem 9 we get

$$\|Y(n, k)\|_1 = \sup_{T \in \mathcal{N},\, \|T\|_1 = 1} \|Y(n, k)(T)\|_1 = \sup_{T \in \mathcal{N},\, \|T\|_1 = 1} \left\| \sum_{i=1}^{\infty} \lambda_i Y(n, k)(e_i \otimes e_i) \right\|_1 .$$

From (18), Lemma 11 and Proposition 2 we obtain

$$\begin{aligned}
\|Y(n, k)\|_1 &\leq \sup_{T \in \mathcal{N}, \|T\|_1 = 1} \sum_{i=1}^{\infty} |\lambda_i| \|Y(n, k)(e_i \otimes e_i)\|_1 \\
&= \sup_{T \in \mathcal{N}, \|T\|_1 = 1} \sum_{i=1}^{\infty} |\lambda_i| \langle T(n, k)(I)\, e_i, e_i \rangle \\
&\leq \sup_{T \in \mathcal{N}, \|T\|_1 = 1} \sum_{i=1}^{\infty} |\lambda_i| \|T(n, k)(I)\| \|e_i\|^2 \\
&= \|T(n, k)(I)\| \sup_{T \in \mathcal{N}, \|T\|_1 = 1} \sum_{i=1}^{\infty} |\lambda_i| \\
&= \|T(n, k)(I)\| \|T\|_1 = \|T(n, k)(I)\| = \|T(n, k)\| .
\end{aligned}$$

The proof is complete. ■

The results from above allow us to give characterizations of the exponential and uniform exponential stability of system (3) by using both operators $Y(n, k)$ and $T(n, k)$.

THEOREM 13. – *The following statements are equivalent*:
a) *the system* (3) *is uniformly exponential stable*;
b) *there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \boldsymbol{N}$ *such that*

$$\|Y(n, k)\|_1 \leq \beta a^{n-k} \tag{20}$$

for all $n \geq k \geq n_0$;
c) *there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \boldsymbol{N}$ *such that*

$$\|T(n, k)\| \leq \beta a^{n-k} \tag{21}$$

for all $n \geq k \geq n_0$.

PROOF. – The equivalence between a) and b) is given by Theorem 9 and the equivalence «b) ⇔ c)» follows from the above theorem. ■

THEOREM 14. – *The following statements are equivalent*:
a) *the system* (3) *is exponentially stable*;
b) *there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \boldsymbol{N}$ *such that we have*

$$\|Y(n, 0)(T)\|_1 \leq \beta a_1^{n-k} \|Y(k, 0)(T)\| \tag{22}$$

for all $n \geq k \geq n_0$ *and* $T \in \mathcal{K}_1$;

c) *there exist* $\beta \geq 1$, $a \in (0, 1)$ *and* $n_0 \in \boldsymbol{N}$ *such that we have*

$$\langle T(n, 0)(I)\, x, x \rangle \leq \beta a^{n-k} \langle T(k, 0)(I)\, x, x \rangle \tag{23}$$

for all $n \geq k \geq n_0$ *and* $x \in H$, *where* $I \in L(H)$ *is the identity operator.*

PROOF. – The equivalence between a) and b) is a consequence of the Theorem 10 and the equivalence «a) ⇔ c)» follows from the Definition 8 and from (10). The proof is complete. ■

The following remark is a consequence of theorems T.14 and T.13.

REMARK 15. – *If the system* (3) *is uniformly exponential stable, then it is exponentially stable.*

PROOF. – Since (3) is uniformly exponential stable, we deduce from Theorem 13 b) that there exist $\beta \geq 1$, $a \in (0, 1)$ and $n_0 \in \boldsymbol{N}$ such that $\|Y(n, k)(T)\|_1 \leq \beta a^{n-k} \|T\|_1$ for all $n \geq k \geq n_0$ and $T \in \mathcal{N}$. Taking $T = Y(k, 0)(S)$, where $S \in \mathcal{K}_1$ is arbitrary we obtain (22) and it follows the conclusion. ■

5. – The uniform exponential stability and the Lyapunov equations.

On the space $\mathcal{H}$ we consider the Lyapunov equation

$$P_n = A_n^* P_{n+1} A_n + b_n B_n^* P_{n+1} B_n + W_n, \tag{24}$$

where $\{W_n\}$ is a sequence in $\mathcal{H}$ with the property that there are $u, v > 0$ such that we have

$$u\|x\|^2 \leq \langle W_n x, x \rangle \leq v\|x\|^2 \tag{25}$$

for all $n \in \boldsymbol{N}$ and $x \in H$. It is easy to see that if (25) holds, then $\|W_n\| \leq v$ for all $n \in \boldsymbol{N}$. Now we can prove the following theorem:

THEOREM 16. – *The system* (3) *is uniformly exponential stable if and only if the equation* (24) *has a unique solution* $P = (P_n)_{n \in \boldsymbol{N}}$ *with the property that there exist* $m, M > 0$ *such that*

$$m\|x\|^2 \leq \langle P_n x, x \rangle \leq M\|x\|^2 \tag{26}$$

for all $n \in \boldsymbol{N}$ *and* $x \in H$.

PROOF. – Let us prove the implication «⇒». If Q_n is the linear bounded operator given by (9) then we introduce the linear operator

$$P_n = \sum_{k=n+1}^{\infty} Q_n \dots Q_{k-1}(W_k) + W_n = \sum_{k=n}^{\infty} T(k, n)(W_k).$$

Since the series $\sum_{k=n}^{\infty} \|T(k, n)(W_k)\|$ converges in $\boldsymbol{R}$, it follows that P_n is well-defined.

Indeed, if $n \geqslant n_0$ we deduce from Theorem 13 and the hypothesis that there exist $\beta \geqslant 1$, $a \in (0, 1)$ and $n_0 \in \boldsymbol{N}$ such that

$$\sum_{k=n}^{\infty} \|T(k, n)(W_k)\| \leqslant \sum_{k=n}^{\infty} \beta a^{k-n} \|W_k\| \leqslant v \sum_{k=n}^{\infty} \beta a^{k-n} = \frac{v\beta}{1-a} < \infty .$$

If $n < n_0$, we use again the Theorem 13 and we have

$$\sum_{k=n}^{\infty} \|T(k, n)(W_k)\| \leqslant v \left(\sum_{k=n}^{n_0} \|T(k, n)\| + \|T(n_0, n)\| \sum_{k=n_0}^{\infty} \beta a^{k-n_0} \right) < \infty .$$

The conclusion follows. More, if

$$M = v \max \left\{ \max_{n<n_0} \left\{ \sum_{k=n}^{n_0} \|T(k, n)\| + \|T(n_0, n)\| \frac{\beta}{1-a} \right\}, \frac{\beta}{1-a} \right\}$$

then we have $\|P_n\| \leqslant M$. Since $T(n, k) \in L(\mathcal{H})$ and $W_k \in \mathcal{H}$ for all $n \geqslant k \geqslant 0$, we deduce $P_n \in \mathcal{H}$; hence $\langle P_n x, x \rangle \leqslant M \|x\|^2$ for all $n \geqslant k \geqslant 0$ and $x \in H$.

By (25) and since $T(n, k)(\mathcal{K}) \subset \mathcal{K}$, we get $\langle P_n x, x \rangle \geqslant \langle W_n x, x \rangle \geqslant u \|x\|^2$. We take $m = u$ and we deduce that (26) holds. Computing we have

$$Q_n(P_{n+1}) + W_n = \sum_{k=n+2}^{\infty} Q_n Q_{n+1} \dots Q_{k-1}(W_k) + Q_n(W_{n+1}) + W_n =$$

$$\sum_{k=n+1}^{\infty} Q_n \dots Q_{k-1}(W_k) + W_n = P_n.$$

Therefore P_n is a solution of (24).

Now we prove the *uniqueness* of the solution. Let us assume that R_n is another solution of (24), which satisfies (26). Then we have $P_n - R_n = Q_n(P_{n+1} - R_{n+1})$ and, by induction, $P_n - R_n = T(n+k, n)(P_{n+k} - R_{n+k})$.

By (26), we have

$$\|P_n - R_n\| \leqslant \|T(n+k, n)\| \|P_{n+k} - R_{n+k}\| \leqslant 2M \|T(n+k, n)\| \tag{27}$$

From the hypotheses and from Theorem 13 it follows $\sup_{k \to \infty} \|T(n+k, n)\| = 0$ for all $n \in \boldsymbol{N}$. As $k \to \infty$ in (27) we get $P_n = R_n$ for all $n \in \boldsymbol{N}$.

«⇐» If P_n is the solution of the equation (24) which satisfies (26), then

$P_n = T(n+1, n)(P_{n+1}) + W_n$. Thus

$$E\langle P_n X(n, k)\, x, X(n, k)\, x\rangle =$$

$$E\langle T(n+1, n)(P_{n+1})\, X(n, k)\, x, X(n, k)\, x\rangle + E\langle W_n X(n, k)\, x, X(n, k)\, x\rangle$$

for all $n \geqslant k$. From Theorem 6 we obtain

$$\begin{aligned} E\langle T(n+1,n)(P_{n+1})\, X(n,k)\, x, X(n,k)\, x\rangle &= \langle T(n,k)\, T(n+1,n)(P_{n+1})\, x, x\rangle \\ &= \langle T(n+1, k)(P_{n+1})\, x, x\rangle \\ &= E\langle P_{n+1} X(n+1, k)\, x, X(n+1, k)\, x\rangle. \end{aligned}$$

By (25) and (26) we obtain

$$E\langle P_n X(n, k)\, x, X(n, k)\, x\rangle \geqslant$$

$$E\langle P_{n+1} X(n+1, k)\, x, X(n+1, k)\, x\rangle + \frac{u}{M} E\langle P_n X(n, k)\, x, X(n, k)\, x\rangle.$$

From (24), (25), (26) and since P_n is nonnegative we deduce $\frac{u}{M} < 1$. (If $\frac{u}{M} = 1$ we obtain the trivial case). We have

$$\left(1 - \frac{u}{M}\right) E\langle P_n X(n, k)\, x, X(n, k)\, x\rangle \geqslant E\langle P_{n+1} X(n+1, k)\, x, X(n+1, k)\, x\rangle$$

and, by induction

$$\left(1 - \frac{u}{M}\right)^{n+1-k} \langle P_k x, x\rangle \geqslant E\langle P_{n+1} X(n+1, k)\, x, X(n+1, k)\, x\rangle.$$

From (26) it follows $mE\|X(n+1, k)x\|^2 \leqslant M\left(1 - \frac{u}{M}\right)^{n+1-k} \|x\|^2$. If we take $\beta = \frac{M}{m} \geqslant 1$, $\alpha = 1 - \frac{u}{M}$ and $n_0 = 0$ we obtain the conclusion. The proof is complete. ■

6. – The time-invariant case.

Now, we consider the time-invariant case when $A_n = A$, $B_n = B$ and $b_n = b$. In this case the operators U_n and Q_n given by (7) and (9) become $U_n(Y) = U(Y) = AYA^* + bBYB^*$, for all $Y \in \mathcal{N}$ and $Q_n(Y) = Q(Y) = A^*YA + bB^*YB$ for all $Y \in \mathcal{H}$. Thus we have

(28) $$Y(n, k)(Y) = Y(n-k, 0)(Y) \text{ and } Y(n, k)(Y) = U^{n-k}(Y)$$

for all $Y \in \mathcal{N}$ and respectively

$$T(n, k)(Y) = T(n-k, 0)(Y) \text{ and } T(n, k)(Y) = Q^{n-k}(Y) \tag{29}$$

for all $Y \in \mathcal{H}$.

The following theorem gives necessary and sufficient conditions for the uniform exponential stability of the system (3) in the time-invariant case and also, establishes the equivalence between the exponential stability and the uniform exponential stability in this case.

THEOREM 17. – *The following assertions are equivalent*:

a) *the system* (3) *is uniformly exponential stable*;

b) *there exist* $\beta \geq 1$ *and* $a \in (0, 1)$ *such that we have*

$$\|Y(n, 0)\|_1 \leq \beta a^n \text{ or equivalently } \|U^n\|_1 \leq \beta a^n \tag{30}$$

for all $n \in \boldsymbol{N}$;

c) *there exist* $\beta \geq 1$ *and* $a \in (0, 1)$ *such that we have*

$$\|T(n, 0)\| \leq \beta a^n \text{ or equivalently } \|Q^n\| \leq \beta a^n \tag{31}$$

for all $n \in \boldsymbol{N}$;

d) $\varrho(U) < 1$;

e) $\varrho(Q) < 1$;

f) $\lim_{n\to\infty} E\|X(n, 0)x\|^2 = 0$ *uniformly for* $x \in H$, $\|x\| = 1$;

g) $\lim_{n\to\infty} \|Y(n, 0)(x \otimes x)\|_1^2 = 0$ *uniformly for* $x \in H$, $\|x\| = 1$;

h) *the system* (3) *is exponentially stable.*

We denote by $\varrho(U)$ (*respectively* $\varrho(Q)$) *the spectral radius of* U (*respectively* Q).

PROOF. – From Theorem 13, (28) and (29) it results the equivalences «a) $\Leftrightarrow$ b)» and «a) $\Leftrightarrow$ c)». We will prove b) $\Leftrightarrow$ d).

«b) $\Rightarrow$ d)». From (30) we have $\|U^n\|_1 \leq \beta a^n$ and by using T.2.38 from [3] we see that $\varrho(U) = \lim_{n\to\infty} \sqrt[n]{\|U^n\|_1} \leq a < 1$.

«d) $\Rightarrow$ b)». Let $\varrho(U) = \lim_{n\to\infty} \sqrt[n]{\|U^n\|_1} = s < 1$ and let $\varepsilon > 0$ be such that $s + \varepsilon = a < 1$. Then, there exists $k_0 \in \boldsymbol{N}$ such that for all $n \geq k_0$ we have $\|U^n\|_1 \leq a^n$. If we take $\beta = \max\left\{1, \max_{p \in \boldsymbol{N}, p \leq k_0} \frac{\|U^p\|_1}{a^p}\right\}$, we obtain the conclusion. Analogously, we can show that «c) $\Leftrightarrow$ e)». The equivalence «f) $\Leftrightarrow$ g)» is a consequence of (8).

The implication «b) $\Rightarrow$ g)» is obviously true and, since a) $\Leftrightarrow$ b) and f) $\Leftrightarrow$ g), we obtain a) $\Rightarrow$ f).

Conversely, from f) and (10) we deduce $\sup_{n \to \infty} \langle T(n, 0)(I)\, x, x \rangle = 0$, uniformly for $x \in H$, $\|x\| = 1$. Thus it exists $k_0 \in \boldsymbol{N}$ such that $\langle T(k_0, 0)(I)\, x, x \rangle < \frac{1}{2}$ for all $x \in H$, $\|x\| = 1$. Since $T(k_0, 0)(I) \geq 0$ and $\|T(k_0, 0)(I)\| = \|T(k_0, 0)\|$ we get $\|T(k_0, 0)\| < \frac{1}{2}$.

From (29) we deduce that there exits $k_0 \in \boldsymbol{N}$ such that $\|Q^{k_0}\| < \frac{1}{2}$. Let $n \in \boldsymbol{N}$. We have $n = k_0 c + r$, where $c, r \in \boldsymbol{N}$, $0 \leq r < k_0$ and $Q^n = (Q^{k_0})^c Q^r$.

Now we obtain $\|Q^n\| \leq \|Q^{k_0}\|^c \|Q^r\|$. Taking $a = \left(\frac{1}{2}\right)^{1/k_0}$ and $\beta = \max_{r \in N,\, r < k_0} \{2^{r/k_0} \|Q^r\|\}$, it follows «f) $\Rightarrow$ c)». Since a) $\Leftrightarrow$ c) we obtain «f) $\Rightarrow$ a)» and the equivalence a) $\Leftrightarrow$ f) is proved.

Finally, we show that a) $\Leftrightarrow$ h). The implication «a) $\Rightarrow$ h)» follows from Remark 15. Let us assume that h) holds. From Theorem 14 we see that there exist $\beta \geq 1$, $a \in (0, 1)$ and $n_0 \in \boldsymbol{N}$ such that we have $\langle T(n, 0)(I)x, x \rangle \leq \beta a^{n-k} \langle T(k, 0)(I)x, x \rangle$ for all $n \geq k \geq n_0$ and $x \in H$. By Lemma 11 we get $\|T(n, 0)\| \leq \beta a^{n-k} \|T(k, 0)\|$ for all $n \geq k \geq n_0$.

Now we use (29) and we obtain $\|Q^n\| \leq \beta a^{n-k} \|Q^k\|$. We take $k = n_0$ and we have $\sqrt[n]{\|Q^n\|} \leq a^{(n-n_0)/n} \sqrt[n]{\beta \|Q^{n_0}\|}$ for all $n \geq n_0$. As $n \to \infty$ in the last inequality we obtain $\varrho(Q) \leq a < 1$ and e) holds. Now we use the implication «e) $\Rightarrow$ a)» and the proof is finished. ∎

We consider the Lyapunov algebraic equation

$$P = Q(P) + J \tag{32}$$

on the space $\mathcal{H}$, where Q is the operator introduced above and $J \in \mathcal{H}$ is a positive operator. ($J \in \mathcal{H}$ is a positive operator if there exists $\gamma > 0$ such that $J > \gamma I$, where I is the identity operator on H.)

In the time-invariant case the Theorem 16 has the following corollary:

COROLLARY 18. – *If $A_n = A$, $B_n = B$ and $b_n = b$, then the solution of* (3) *is uniformly exponential stable if and only if the equation* (32) *has a unique positive solution.*

PROOF. – If (32) has a positive solution P then the equation (24) with $W_n = J$ has a solution $P_n = P$ which satisfies (26). By Theorem 16 it follows that (3) is uniformly exponential stable.

Conversely, if (3) is uniformly exponential stable then the Lyapunov equation (24) with $W_n = J$ has a unique solution $P_n = \sum_{k=0}^{\infty} Q^k(J)$ such as (26) holds. From Theorem 17 we deduce $\varrho(Q) < 1$ and consequently $P_n = (I - Q)^{-1}(J) \overset{not}{=} P$. Since P_n doesn't depend on n it is clear that P_n is the positive solution ($P_n \geqslant J$) of (32). If P_1 is another positive solution of (32) then it is also a solution of (24) which satisfy (26). By Theorem 16 it follows $P_1 = P$. The proof is complete. ■

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