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Properties of (Λ, δ)-closed Sets in Topological Spaces.

D. N. Georgiou - S. Jafari - T. Noiri

Sunto. – *In questo articolo vengono presentate e studiate le nozioni di insieme Λ_δ e di insieme (Λ, δ)-chiuso. Inoltre, vengono introdotte le nozioni di (Λ, δ)-continuità, (Λ, δ)-compatezza e (Λ, δ)-connessione e vengono fornite alcune caratterizzazioni degli spazi $\delta - T_0$ e $\delta - T_1$. Infine, viene mostrato che gli spazi (Λ, δ)-connessi e (Λ, δ)-compatti vengono preservati mediante suriezioni δ-continue.*

Summary. – *We present and study the notions of Λ_δ-sets and (Λ, δ)-closed sets. Moreover, we introduce the notions of (Λ, δ)-continuity, (Λ, δ)-compactness and (Λ, δ)-connectedness. Characterizations of $\delta - T_0$ and $\delta - T_1$ spaces are given. It is shown that (Λ, δ)-connected and (Λ, δ)-compact spaces are preserved under δ-continuous surjections.*

1. – Preliminaries.

The notions of δ-closed sets was introduced by Veličko [5] and is widely investigated in the literature. In this paper, we define and study some sets, spaces and functions by using the notion of δ-closed sets.

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$, respectively. A point $x \in X$ is called the *δ-cluster* point of A if $A \cap \mathrm{Int}(Cl(U)) \neq \emptyset$ for every open set U of X containing x. The set of all δ-cluster points of A is called the *δ-closure* of A, denoted by $Cl_\delta(A)$. A subset A is called *δ-closed* if $A = Cl_\delta(A)$. The complement of a δ-closed set is called *δ-open*. We denote the collection of all δ-open (resp. δ-closed) sets by $\delta(X, \tau)$ (resp. $\mathrm{Cl}_\delta(X, \tau)$). The set $\{x \in X \mid x \in U \subset \mathrm{Int}(Cl(U)) \subset A\}$ for some open set U of X is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$. Recall that a topological space is called *Alexandroff* if every point has a minimal neighborhood, or equivalently, has a unique minimal base.

In section 2, we consider the notion of Λ_δ-sets. By definition a subset A of a space (X, τ) is called a Λ_δ-set if A is the intersection of all δ-open sets containing A. It turns out that the family τ^{Λ_δ} of Λ_δ-sets of a space (X, τ) is a topology for X. Moreover, we introduce and investigate the notion of (Λ, δ)-closed sets. The definition is as follows: A is (Λ, δ)-closed if $A = T \cap C$, where T is a Λ_δ-set and C is a δ-closed set. In section 3, it is shown that a space (X, τ) is $\delta - T_0$ (resp. $\delta - T_1$) if and only if for each $x \in X$ the singleton $\{x\}$ is (Λ, δ)-closed (resp. a Λ_δ-set). In section 4, we define a function $f:(X, \tau) \to (Y, \sigma)$ to be (Λ, δ)-continuous if $f^{-1}(\sigma^{\Lambda_\delta}) \subset \tau^{\Lambda_\delta}$ and obtain their characterizations. It is shown that if $f:(X, \tau) \to (Y, \sigma)$ is a δ-continuous function, then it is (Λ, δ)-continuous. In the last section, we present the notions of (Λ, δ)-compactness and (Λ, δ)-connectedness and show that (Λ, δ)-compactness (resp. (Λ, δ)-connectedness) is preserved by (Λ, δ)-continuous (hence δ-continuous) surjections.

2. – (Λ, δ)-closed sets.

DEFINITION 1. – *Let A be a subset of a topological space (X, τ). A subset $\Lambda_\delta(A)$ is defined as follows: $\Lambda_\delta(A) = \cap\{O \in \delta(X, \tau) \mid A \subset O\}$.*

LEMMA 2.1. – *For subsets A, B and A_i $(i \in I)$ of a topological space (X, τ), the following hold:*

(1) $A \subset \Lambda_\delta(A)$.

(2) $\Lambda_\delta(\Lambda_\delta(A)) = \Lambda_\delta(A)$.

(3) *If* $A \subset B$, *then* $\Lambda_\delta(A) \subset \Lambda_\delta(B)$.

(4) $\Lambda_\delta(\cap\{A_i \mid i \in I\}) \subset \cap\{\Lambda_\delta(A_i) \mid i \in I\}$.

(5) $\Lambda_\delta(\cup\{A_i \mid i \in I\}) = \cup\{\Lambda_\delta(A_i) \mid i \in I\}$.

PROOF. – We prove only statements (4) and (5).

(4) Suppose that $x \notin \cap\{\Lambda_\delta(A_i) \mid i \in I\}$. There exists $i_0 \in I$ such that $x \notin \Lambda_\delta(A_{i_0})$ and there exists a δ-open set O such that $x \notin O$ and $A_{i_0} \subset O$. We have $\bigcap_{i \in I} A_i \subset A_{i_0} \subset O$ and $x \notin O$. Therefore, $x \notin \Lambda_\delta(\cap\{A_i \mid i \in I\})$.

(5) First $A_i \subset \Lambda_\delta(A_i) \subset \Lambda_\delta\left(\bigcup_{i \in I} A_i\right)$ and hence $\Lambda_\delta(A_i) \subset \Lambda_\delta\left(\bigcup_{i \in I} A_i\right)$. Therefore, we obtain $\bigcup_{i \in I} \Lambda_\delta(A_i) \subset \Lambda_\delta\left(\bigcup_{i \in I} A_i\right)$. Conversely, suppose that $x \notin \bigcup_{i \in I} \Lambda_\delta(A_i)$. Then $x \notin \Lambda_\delta(A_i)$ for each $i \in I$ and hence there exists $V_i \in \delta(X, \tau)$ such that $A_i \subset V_i$ and $x \notin V_i$ for each $i \in I$. We have $\bigcup_{i \in I} A_i \subset \bigcup_{i \in I} V_i$ and $\bigcup_{i \in I} V_i$ is a δ-open set which does not contain x. Therefore, $x \notin \Lambda_\delta\left(\bigcup_{i \in I} A_i\right)$. This shows that $\Lambda_\delta\left(\bigcup_{i \in I} A_i\right) \subset \bigcup_{i \in I} \Lambda_\delta(A_i)$.

REMARK 2.2. – *In Lemma* 2.1(4), *the converse is not always true as the following example shows.*

EXAMPLE 2.3. – *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{a\}\}$. *Now put* $B = \{b\}$ *and* $C = \{c\}$. *Then* $\Lambda_\delta(B \cap C) = \Lambda_\delta(\emptyset) = \emptyset$, $\Lambda_\delta(B) \cap \Lambda_\delta(C) = X$ *and* $\Lambda_\delta(B) \neq B$.

DEFINITION 2. – *A subset A of a topological space* (X, τ) *is called a* Λ_δ*-set* if $A = \Lambda_\delta(A)$.

LEMMA 2.4. – *For subsets A and* A_i $(i \in I)$ *of a topological space* (X, τ), *the following hold*:

(1) $\Lambda_\delta(A)$ *is a* Λ_δ*-set.*

(2) *If A is* δ*-open, then A is a* Λ_δ*-set.*

(3) *If* A_i *is a* Λ_δ*-set for each* $i \in I$, *then* $\bigcap_{i \in I} A_i$ *is a* Λ_δ*-set.*

(4) *If* A_i *is a* Λ_δ*-set for each* $i \in I$, *then* $\bigcup_{i \in I} A_i$ *is a* Λ_δ*-set.*

PROOF. – This follows readily from Lemma 2.1. ■

THEOREM 2.5. – *For a topological space* (X, τ), *we put* $\tau^{\Lambda_\delta} = \{A \mid A$ *is a* Λ_δ*-set of* $X\}$. *Then the pair* $(X, \tau^{\Lambda_\delta})$ *is an Alexandroff space.*

PROOF. – This is an immediate consequence of Lemma 2.4. ■

DEFINITION 3. – *Let A be a subset of a topological space* (X, τ). *A set* $\Lambda^*_\delta(A)$ *is defined as follows:* $\Lambda^*_\delta(A) = \cup\{B \in Cl_\delta(X, \tau) \mid B \subset A\}$.

DEFINITION 4. – *A subset A of a topological space* (X, τ) *is called a* Λ^*_δ*-set if* $A = \Lambda^*_\delta(A)$.

We obtain the following two lemmas which are similar to Lemma 2.1 and Lemma 2.4.

LEMMA 2.6. – *For subsets* A, B *and* A_i $(i \in I)$ *of a topological space* (X, τ) *the following properties hold:*

(1) $\Lambda^*_\delta(A) \subseteq A$.

(2) *If* $A \subseteq B$, *then* $\Lambda^*_\delta(A) \subseteq \Lambda^*_\delta(B)$.

(3) *If A is* δ*-closed, then* $\Lambda^*_\delta(A) = A$.

(4) $\Lambda^*_\delta(\cap\{A_i : i \in I\}) = \cap\{\Lambda^*_\delta(A_i) : i \in I\}$.

(5) $\cup\{\Lambda^*_\delta(A_i) : i \in I\} \subseteq \Lambda^*_\delta(\cup\{A_i : i \in I\})$.

(6) $\Lambda_\delta(X - A) = X - \Lambda^*_\delta(A)$ *and* $\Lambda^*_\delta(X - A) = X - \Lambda_\delta(A)$.

LEMMA 2.7. – *For subsets* A, B *and* A_i $(i \in I)$ *of a topological space* (X, τ) *the following properties hold:*

(1) $\Lambda_\delta^*(A)$ *is a* Λ_δ^**-set.*

(2) *If* A *is a* δ*-closed, then* A *is a* Λ_δ^**-set.*

(3) *If* A_i *is a* Λ_δ^**-set for each* $i \in I$*, then* $\cup\{A_i \mid i \in I\}$ *and* $\cap\{A_i \mid i \in I\}$ *are* Λ_δ^**-sets.*

REMARK 2.8. – *For a topological space* (X, τ)*, we set* $\tau^{\Lambda_\delta^*} = \{A \mid A$ *is a* Λ_δ^**-set of* $X\}$*, then the pair* $(X, \tau^{\Lambda_\delta^*})$ *is an Alexandroff space.*

DEFINITION 5. – *A subset* A *of a topological space* (X, τ) *is called* (Λ, δ)*-closed* if $A = T \cap C$, where T is a Λ_δ-set and C is a δ-closed set.

THEOREM 2.9. – *The following statements are equivalent for a subset* A *of a topological space* (X, τ):

(1) A *is* (Λ, δ)*-closed*;

(2) $A = T \cap Cl_\delta(A)$*, where* T *is a* Λ_δ*-set*;

(3) $A = \Lambda_\delta(A) \cap Cl_\delta(A)$.

PROOF. – (1) $\Rightarrow$ (2): Let $A = T \cap C$, where T is a Λ_δ-set and C is a δ-closed set. Since $A \subset C$, we have $Cl_\delta(A) \subset C$ and $A = T \cap C \supset T \cap Cl_\delta(A) \supset A$. Therefore, we obtain $A = T \cap Cl_\delta(A)$.

(2) $\Rightarrow$ (3): Let $A = T \cap Cl_\delta(A)$, where T is a Λ_δ-set. Since $A \subset T$, we have $\Lambda_\delta(A) \subset \Lambda_\delta(T) = T$ and hence $A \subset \Lambda_\delta(A) \cap Cl_\delta(A) \subset T \cap Cl_\delta(A) = A$. Therefore, we obtain $A = \Lambda_\delta(A) \cap Cl_\delta(A)$.

(3) $\Rightarrow$ (1): Since $\Lambda_\delta(A)$ is a Λ_δ-set, $Cl_\delta(A)$ is δ-closed and $A = \Lambda_\delta(A) \cap Cl_\delta(A)$. ■

LEMMA 2.10. – *Every* Λ_δ*-set (resp.* δ*-closed set) is* (Λ, δ)*-closed.*

DEFINITION 6. – *A subset* A *of a topological space* (X, τ) *is said to be* (Λ, δ)*-open if the complement of* A *is* (Λ, δ)*-closed.*

THEOREM 2.11. – *Let* A_i $(i \in I)$ *be a subset of a topological space* (X, τ).

(1) *If* A_i *is* (Λ, δ)*-closed for each* $i \in I$*, then* $\cap\{A_i \mid i \in I\}$ *is* (Λ, δ)*-closed.*

(2) *If* A_i *is* (Λ, δ)*-open for each* $i \in I$*, then* $\cup\{A_i \mid i \in I\}$ *is* (Λ, δ)*-open.*

PROOF. – (1) Suppose that A_i is (Λ, δ)-closed for each $i \in I$. Then, for each i, there exist a Λ_δ-set T_i and a δ-closed set C_i such that $A_i = T_i \cap C_i$. We have $\bigcap_{i \in I} A_i = \bigcap_{i \in I}(T_i \cap C_i) = \left(\bigcap_{i \in I} T_i\right) \cap \left(\bigcap_{i \in I} C_i\right)$. By Lemma 2.4, $\bigcap_{i \in I} T_i$ is a Λ_δ-set and $\bigcap_{i \in I} C_i$ is a δ-closed. This shows that $\bigcap_{i \in I} A_i$ is (Λ, δ)-closed.

(2) Let A_i is (Λ, δ)-open for each $i \in I$. Then $X - A_i$ is (Λ, δ)-closed and $X - \bigcup_{i \in I} A_i = \bigcap_{i \in I}(X - A_i)$. Therefore, by (1) $\bigcup_{i \in I} A_i$ is (Λ, δ)-open. ■

DEFINITION 7. – *A subset A of a topological space (X, τ) is called a (δ, δ)-* generalized-closed *set (briefly (δ, δ)-g-closed) if $Cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ). A subset A is said to be (δ, δ)-g-open if $X - A$ is (δ, δ)-g-closed.*

The following two lemmas are obtained easily from the definitions.

LEMMA 2.12. – *For a subset A of a topological space (X, τ), the following properties hold*:

(1) *A is (δ, δ)-g-closed if and only if $Cl_\delta(A) \subset \Lambda_\delta(A)$.*

(2) *A is δ-closed if and only if A is (δ, δ)-g-closed and (Λ, δ)-closed.*

LEMMA 2.13. – *For a subset A of a topological space (X, τ), the following properties hold*:

(1) *A is (δ, δ)-g-open if and only if $\Lambda_\delta^*(A) \subset \mathrm{Int}_\delta(A)$.*

(2) *A is δ-open if and only if A is (δ, δ)-g-open and (Λ, δ)-open.*

THEOREM 2.14. – *For a subset A of a topological space (X, τ), the following are equivalent*:

(1) *A is (Λ, δ)-open*;

(2) *$A = T \cup C$, where T is a Λ_δ^*-set and C is δ-open*;

(3) *$A = T \cup \mathrm{Int}_\delta(A)$, where T is a Λ_δ^*-set*;

(4) *$A = \Lambda_\delta^*(A) \cup \mathrm{Int}_\delta(A)$.*

PROOF. – (1) $\Rightarrow$ (2): Suppose that A is (Λ, δ)-open. Then $X - A$ is (Λ, δ)-closed and $X - A = K \cap D$, where K is a Λ_δ-set and D is a δ-closed set. Hence, we have $A = (X - A) \cup (X - D)$, where $X - K$ is a Λ_δ^*-set and $X - D$ is δ-open set.

(2) $\Rightarrow$ (3): Let $A = T \cup C$, where T is a Λ_δ^*-set and C is δ-open. Since $C \subset A$ and C is δ-open, $C \subset \mathrm{Int}_\delta(A)$ and hence $A = T \cup C \subset T \cup \mathrm{Int}_\delta(A) \subset A$. Therefore, we obtain $A = T \cup \mathrm{Int}_\delta(A)$.

(3) $\Rightarrow$ (4): Let $A = T \cup \mathrm{Int}_\delta(A)$, where T is a Λ_δ^*-set. Since $T \subset A$, we have $\Lambda_\delta^*(A) \supset \Lambda_\delta^*(T)$ and hence $A \supset \Lambda_\delta^*(A) \cup \mathrm{Int}_\delta(A) \supset \Lambda_\delta^*(T) \cup \mathrm{Int}_\delta(A) = T \cup \mathrm{Int}_\delta(A) = A$. Therefore, we obtain $A = \Lambda_\delta^*(A) \cup \mathrm{Int}_\delta(A)$.

(4) $\Rightarrow$ (1): Let $A = \Lambda_\delta^*(A) \cup \mathrm{Int}_\delta(A)$. Then, we have $X - A = (X - \Lambda_\delta^*(A)) \cap (X - \mathrm{Int}_\delta(A)) = \Lambda_\delta(X - A) \cap Cl_\delta(X - A)$. By Lemma 2.4, $\Lambda_\delta(X -$

A) is a Λ_δ-set and $Cl_\delta(X-A)$ is δ-closed. Therefore, $X-A$ is a (Λ, δ)-closed set and A is (Λ, δ)-open. ■

3. – Properties of (Λ, δ)-closed Sets.

DEFINITION 8. – *A topological space (X, τ) is called a $\delta - R_0$ space if for each δ-open set U and each $x \in U$, $Cl_\delta(\{x\}) \subset U$.*

DEFINITION 9. – *A topological space (X, τ) is said to be*

(1) $\delta - T_0$ [1] *if for any distinct pair of points in X, there is a δ-open set containing one of the points but not the other.*

(2) $\delta - T_1$ [1] if for any distinct pair of points x and y in X, there is a δ-open U in X containing x but not y and a δ-open set V in X containing y but not x.

(3) $\delta - T_2$ [1] if for any distinct pair of points x and y in X, there are δ-open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.

REMARK 3.1. – *From Definition* 9, *we have the following diagram*:

$$\begin{array}{ccccc} \delta - T_2 & \rightarrow & \delta - T_1 & \rightarrow & \delta - T_0 \\ \downarrow & & \downarrow & & \downarrow \\ T_2 & \rightarrow & T_1 & \rightarrow & T_0 \end{array}$$

THEOREM 3.2. – *Let (X, τ) be a $\delta - R_0$ space. A singleton $\{x\}$ is (Λ, δ)-closed if and only if $\{x\}$ is δ-closed.*

PROOF. – *Necessity*. Suppose that $\{x\}$ is (Λ, δ)-closed. Then by Theorem 2.9 $\{x\} = \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. For any δ-open set U containing x, $Cl_\delta(\{x\}) \subset U$ and hence $Cl_\delta(\{x\}) \subset \Lambda_\delta(\{x\})$. Therefore, we have $\{x\} = \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\}) \supset Cl_\delta(\{x\})$. This shows that $\{x\}$ is δ-closed.

Sufficiency. Suppose that $\{x\}$ is δ-closed. Since $\{x\} \subset \Lambda_\delta(\{x\})$, we have $\Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\}) = \Lambda_\delta(\{x\}) \cap \{x\} = \{x\}$. This shows that $\{x\}$ is (Λ, δ)-closed. ■

THEOREM 3.3. – *A topological space (X, τ) is $\delta - T_0$ if and only if for each $x \in X$, the singleton $\{x\}$ is (Λ, δ)-closed.*

PROOF. – *Necessity*. Suppose that (X, τ) is $\delta - T_0$. For each $x \in X$, it is obvious that $\{x\} \subset \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. If $y \neq x$, (i) there exists a δ-open set V_x such that $y \notin V_x$ and $x \in V_x$ or (ii) there exists a δ-open set V_y such that $x \notin V_y$ and $y \in V_y$. In case of (i), $y \notin \Lambda_\delta(\{x\})$ and $y \notin \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. This shows that $\{x\} \supset \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. In case (ii), $y \notin Cl_\delta(\{x\})$ and $y \notin$

$\Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. This shows that $\{x\} \supset \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. Consequently, we obtain $\{x\} = \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$.

Sufficiency. Suppose that (X, τ) is not $\delta - T_0$. There exist two distinct points x, y such that (i) $y \in V_x$ for every δ-open set V_x containing x and (ii) $x \in V_y$ for every δ-open set V_y containing y. From (i) and (ii), we obtain $y \in \Lambda_\delta(\{x\})$ and $y \in Cl_\delta(\{x\})$, respectively. Therefore, we have $y \in \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$. By Theorem 2.9, $\{x\} = \Lambda_\delta(\{x\}) \cap Cl_\delta(\{x\})$ since $\{x\}$ is (Λ, δ)-closed. This is contrary to $x \neq y$. ■

COROLLARY 3.4. – *Let (X, τ) a δ-R_0 topological space. Then (X, τ) is δ-T_0 if for each $x \in X$, the singleton $\{x\}$ is δ-closed.*

PROOF. – It is an immediate consequence of Theorem 3.2 and Theorem 3.3. ■

THEOREM 3.5. – *A topological space (X, τ) is $\delta - T_1$ if and only if for each $x \in X$, the singleton $\{x\}$ is a Λ_δ-set.*

PROOF. – *Necessity*. Suppose that $y \in \Lambda_\delta(\{x\})$ for some point y distinct from x. Then $y \in \cap\{V_x \mid x \in V_x$ and V_x is δ-open$\}$ and hence $y \in V_x$ for every δ-open set V_x containing x. This contradicts that (X, τ) is a $\delta - T_1$.

Sufficiency. Suppose that $\{x\}$ is a Λ_δ-set for each $x \in X$. Let x and y be any distinct points. Then $y \notin \Lambda_\delta(\{x\})$ and there exists a δ-open set V_x such that $x \in V_x$ and $y \notin V_x$. Similarly, $x \notin \Lambda_\delta(\{y\})$ and there exists a δ-open set V_y such that $y \in V_y$ and $x \notin V_y$. This shows that (X, τ) is $\delta - T_1$. ■

THEOREM 3.6. – *A topological space (X, τ) is $\delta - T_2$ if and only if it is T_2.*

PROOF. – Every $\delta - T_2$ space is obviously T_2. Conversely, suppose that (X, τ) is T_2. Let x and y be any distinct points of X. There exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. We obtain $x \in U \subset \mathrm{Int}(Cl(U))$, $y \in V \subset \mathrm{Int}(Cl(V))$ and $\mathrm{Int}(Cl(U)) \cap \mathrm{Int}(Cl(V)) = \emptyset$. Every regular open set is δ-open. Therefore, (X, τ) is $\delta - T_2$. ■

DEFINITION 10. – *A function $f:(X, \tau) \to (Y, \sigma)$ is said to be*

(1) almost-continuous [4] *if for each $x \in X$ and each open set V of Y containing $f(x)$, there is an open set U containing x such that $f(U) \subset \mathrm{Int}(Cl(V))$,*

(2) δ-continuous [2] *if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}(Cl(U)) \subset \mathrm{Int}(Cl(V))$.*

THEOREM 3.7. – *For a topological space* (X, τ), *the following properties hold*:

(1) (X, τ) *is* $\delta - T_1$ *if and only if* $(X, \tau^{\Lambda_\delta})$ *is the discrete space.*

(2) *The indentity function* $id_X: (X, \tau^{\Lambda_\delta}) \to (X, \tau)$ *is almost-continuous.*

(3) *If* $(X, \tau^{\Lambda_\delta})$ *is connected, then* (X, τ) *is connected.*

PROOF. – (1) *Necessity*. Suppose that (X, τ) is $\delta - T_1$. Let x be any point of X. By Theorem 3.4, $\{x\}$ is a Λ_δ-set and $\{x\} \in \tau^{\Lambda_\delta}$. For any subset A of X, by Lemma 2.4 $A \in \tau^{\Lambda_\delta}$. This shows that $(X, \tau^{\Lambda_\delta})$ is discrete.

Sufficiency. For each $x \in X$, $\{x\} \in \tau^{\Lambda_\delta}$ and hence $\{x\}$ is Λ_δ-set. By Theorem 3.4, (X, τ) is $\delta - T_1$.

(2) Let V be any regular open set of (X, τ). Since V is δ-open, by Lemma 2.4 $(id_X)^{-1}(V) = V \in \tau^{\Lambda_\delta}$ and hence id_X is almost-continuous [4, Theorem 2.2].

(3) Suppose that (X, τ) is not connected. There exist nonempty open sets V_1, V_2 of (X, τ) such that $V_1 \cap V_2 = \emptyset$. Therefore, we obtain $\mathrm{Int}(Cl(V_1)) \cap \mathrm{Int}(Cl(V_2)) = \emptyset$, $\mathrm{Int}(Cl(V_1)) \cup \mathrm{Int}(Cl(V_2)) = X$ and $V_i \subset \mathrm{Int}(Cl(V_i)) \in \tau^{\Lambda_\delta}$ for $i = 1, 2$. This shows that $(X, \tau^{\Lambda_\delta})$ is not connected. ∎

THEOREM 3.8. – *If a function* $f: (X, \tau) \to (Y, \sigma)$ *is* δ*-continuous, then* $f: (X, \tau^{\Lambda_\delta}) \to (Y, \sigma^{\Lambda_\delta})$ *is continuous.*

PROOF. – Let V be any Λ_δ-set of (Y, σ), i.e. $V \in \sigma^{\Lambda_\delta}$. Then $V = \Lambda_\delta(V) = \cap \{W | V \subset W$ *and* W *is* δ*-open in* $(Y, \sigma)\}$. Since f is δ-continuous, $f^{-1}(W)$ is δ-open in (X, τ) for each W and hence we have $f^{-1}(V) = \cap \{f^{-1}(W) | f^{-1}(V) \subset f^{-1}(W)$ *and* W *is* δ*-open in* $(Y, \sigma)\} \supset \cap \{U | f^{-1}(V) \subset U$ *and* U *is* δ*-open in* $(X, \tau)\} = \Lambda_\delta(f^{-1}(V))$. On the other hand, by the definition $f^{-1}(V) \subset \Lambda_\delta(f^{-1}(V))$. Hence, we obtain $f^{-1}(V) = \Lambda_\delta(f^{-1}(V))$. Therefore, $f^{-1}(V) \in \tau^{\Lambda_\delta}$ and $f: (X, \tau) \to (Y, \sigma)$ is continuous. ∎

4. – (Λ, δ)-continuous functions.

DEFINITION 11. – *Let* (X, τ) *be a topological space and* $A \subseteq X$. *A point* $x \in X$ *is called a* (Λ, δ)*-cluster point of* A *if for every* (Λ, δ)*-open set* U *of* X *containing* x *we have* $A \cap U \neq \emptyset$. *The set of all* (Λ, δ)*-cluster points is called the* (Λ, δ)*-closed set of* A *and is denoted by* $A^{(\Lambda, \delta)}$.

LEMMA 4.1. – *Let* A *and* B *be subsets of a topological space* (X, τ). *For the* (Λ, δ)*-closure, the following properties hold:*

(1) $A \subset A^{(\Lambda, \delta)}$ *and* $(A^{(\Lambda, \delta)})^{(\Lambda, \delta)} = A^{(\Lambda, \delta)}$.

(2) $A^{(\Lambda, \delta)} = \cap \{F | A \subset F$ *and* F *is* (Λ, δ)*-closed*$\}$.

(3) *If* $A \subset B$, *then* $A^{(\Lambda, \delta)} \subset B^{(\Lambda, \delta)}$.

(4) *A is (Λ, δ)-closed if and only if $A = A^{(\Lambda, \delta)}$.*

(5) *$A^{(\Lambda, \delta)}$ is (Λ, δ)-closed.*

The proof of the above lemma is clear.

DEFINITION 12. – *Let (X, τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ (Λ, δ)-converges to x if for each (Λ, δ)-open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.*

LEMMA 4.2. – *Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A^{(\Lambda, \delta)}$ if and only if there exists a net $\{x_s, s \in S\}$ of A which (Λ, δ)-converges to x.*

The proof of the above lemma is clear.

DEFINITION 13. – *Let (X, τ) be a topological space, $\mathcal{F} = \{F_i : i \in I\}$ be a filterbase of X and $x \in X$. A filterbase $\mathcal{F}$ is said to converge to x (written $\mathcal{F} \to x$) if for each (Λ, δ)-open set U containing x there is a member $F_i \in \mathcal{F}$ such that $F_i \subseteq U$.*

DEFINITION 14. – *A function $f : (X, \tau) \to (Y, \sigma)$ is called (Λ, δ)-*continuous *if $f^{-1}(V)$ is a (Λ, δ)-open subset of X for every (Λ, δ)-open subset V of Y.*

THEOREM 4.3. – *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f is (Λ, δ)-continuous;*

(2) *$f^{-1}(B)$ is a (Λ, δ)-closed subset of X for every (Λ, δ)-closed subset B of Y;*

(3) *For each $x \in X$ and for each (Λ, δ)-open set V of Y containing $f(x)$ there exists a (Λ, δ)-open set U of X containing x and $f(U) \subseteq V$;*

(4) *$f(A^{(\Lambda, \delta)}) \subset [f(A)]^{(\Lambda, \delta)}$ for each subset A of X;*

(5) *$[f^{-1}(B)]^{(\Lambda, \delta)} \subset f^{-1}(B^{(\Lambda, \delta)})$ for each subset B of Y;*

(6) *For each $x \in X$ and each filterbase $\mathcal{F}$ which (Λ, δ)-converges to x, $f(\mathcal{F})$ (Λ, δ)-converges to $f(x)$.*

PROOF. – Obvious. ■

THEOREM 4.4. – *If $f : (X, \tau) \to (Y, \sigma)$ is a δ-continuous function, then it is (Λ, δ)-continuous.*

PROOF. – Let F be any (Λ, δ)-closed set of (Y, σ). Then there exist a Λ_δ-set T and a δ-closed set D such that $F = T \cap D$. Since f is δ-continuous, $f^{-1}(D)$ is δ-closed and $f^{-1}(T)$ is a Λ_δ-set of (X, τ) by Theorem 3.7. Therefore, $f^{-1}(F) =$

$f^{-1}(T) \cap f^{-1}(D)$ is a (Λ, δ)-closed set of (X, τ). By Theorem 4.3, f is (Λ, δ)-continuous. ■

5. – (Λ, δ)-compactness and (Λ, δ)-connectedness.

DEFINITION 15. – *A topological space (X, τ) is said to be*

(1) *(Λ, δ)-*compact *if every cover of X by (Λ, δ)-open sets of (X, τ) has a finite subcover,*

(2) nearly compact [3] *if every regular open cover of X has a finite subcover.*

THEOREM 5.1. – *A topological space (X, τ) is (Λ, δ)-compact if and only if for every family $\{A_i\colon i \in I\}$ of (Λ, δ)-closed sets in X satisfying $\cap\{A_i\colon i \in I\} = \emptyset$, there is a finite subfamily $A_{i_1}, \dots, A_{i_n}$ with $\cap\{A_{i_k}\colon k = 1, \dots, n\} = \emptyset$.*

PROOF. – Obvious. ■

THEOREM 5.2. – *For a topological space (X, τ), the following properties hold*:

(1) *If $(X, \tau^{\Lambda_\delta})$ is compact, then (X, τ) is nearly compact.*

(2) *If (X, τ) is (Λ, δ)-compact, then (X, τ) is nearly compact.*

(3) *If (X, τ) is (Λ, δ)-compact, then $(X, \tau^{\Lambda_\delta^*})$ is compact.*

PROOF. – (1) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any regular open cover of X. Since every regular open set is δ-open, by Lemma 2.4 V_α is a Λ_δ-set for each $\alpha \in \nabla$. Moreover, by the compactness of $(X, \tau^{\Lambda_\delta})$ there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that (X, τ) is nearly compact.

(2) Let $\{F_\alpha \mid \alpha \in \nabla\}$ be a family of regular closed sets of (X, τ) such that $\cap \{F_\alpha \mid \alpha \in \nabla\} = \emptyset$. Every regular closed set is δ-closed and by Theorem 2.9 F_α is a (Λ, δ)-closed set for each $\alpha \in \nabla$. By Theorem 5.1, there exists a finite subset ∇_0 of ∇ such that $\cap\{F_\alpha \mid \alpha \in \nabla_0\} = \emptyset$. It follows from [3, Theorem 2.1] that (X, τ) is nearly compact.

(3) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of X by Λ_δ^*-sets of (X, τ). Since $V_\alpha = V_\alpha \cup \emptyset$ and the empty set is δ-open, by Lemma 2.4 each V_α is (Λ, δ)-open in (X, τ). Since (X, τ) is (Λ, δ)-compact, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that $(X, \tau^{\Lambda_\delta^*})$ is compact. ■

THEOREM 5.3. – *If $f\colon (X, \tau) \to (Y, \sigma)$ is a (Λ, δ)-continuous surjection and (X, τ) is a (Λ, δ)-compact space, then (Y, σ) is (Λ, δ)-compact.*

PROOF. – Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any cover of Y by (Λ, δ)-open sets of (Y, σ). Since f is (Λ, δ)-continuous, by Theorem 4.3 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a cover of X

by (Λ, δ)-open sets of (X, τ). Thus, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$. Since f is surjective, we obtain $Y = f(X) = \cup \{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that (Y, σ) is (Λ, δ)-compact. ∎

COROLLARY 5.4. – *The (Λ, δ)-compactness is preserved by δ-continuous surjections.*

PROOF. – This is an immediate consequence of Theorem 4.4 and 5.3. ∎

DEFINITION 16. – *A topological space (X, τ) is called* (Λ, δ)-connected *if X cannot be written as a disjoint union of two non-empty (Λ, δ)-open sets.*

THEOREM 5.5. – *For a topological space (X, τ), the following statements are equivalent:*

(1) *(X, τ) is (Λ, δ)-connected;*

(2) *The only subsets of X, which are both (Λ, δ)-open and (Λ, δ)-closed are the empty set $\emptyset$ and X.*

PROOF. – Straightforward. ∎

THEOREM 5.6. – *For a topological space (X, τ), the following properties hold:*

(1) *If (X, τ) is (Λ, δ)-connected, then $(X, \tau^{\Lambda_\delta})$ is connected.*

(2) *If $(X, \tau^{\Lambda_\delta})$ is connected, then (X, τ) is connected.*

PROOF. – (1) Suppose that $(X, \tau^{\Lambda_\delta})$ is not connected. There exist nonempty Λ_δ-sets G, H of (X, τ) such that $G \cap H = \emptyset$ and $G \cup H = X$. By Lemma 2.10, G and H are (Λ, δ)-closed sets. This shows that (X, τ) is not (Λ, δ)-connected.

(2) Suppose that (X, τ) is not connected. There exist nonempty open sets G, H of (X, τ) such that $G \cap H = \emptyset$ and $G \cup H = X$. Every closed and open set is δ-open and G, H are Λ_δ-sets by Lemma 2.4. This shows that $(X, \tau^{\Lambda_\delta})$ is not connected. ∎

THEOREM 5.7. – *If $f : (X, \tau) \to (Y, \sigma)$ is a (Λ, δ)-continuous surjection and (X, τ) is (Λ, δ)-connected, then (Y, σ) is (Λ, δ)-connected.*

PROOF. – Suppose that (Y, σ) is not (Λ, δ)-connected. There exist nonempty (Λ, δ)-open sets G, H of Y such that $G \cap H = \emptyset$ and $G \cup H = Y$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Moreover, $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty (Λ, δ)-open sets of (X, τ). This shows that (X, τ) is not (Λ, δ)-connected. Therefore, (Y, σ) is (Λ, δ)-connected. ∎

COROLLARY 5.8. – *The (Λ, δ)-connectedness is preserved by δ-continuous surjections.*

PROOF. – This is an immediate consequence of Theorem 4.4 and Theorem 5.7. ∎

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