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On some numerical properties of Fano varieties


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On Some Numerical Properties of Fano Varieties.

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Sunto. – Questa nota è il testo di una conferenza tenuta al XVII Convegno dell’Unione Matematica Italiana, tenutosi a Milano, 8-13 settembre 2003. Parlo di alcune congettture e teoremi sulle relazioni tra l’indice, lo pseudo-index e il numero di Picard di una varietà di Fano. I risultati in questione fanno parte di un lavoro in collaborazione con Bonavero, Debarre e Druel.

Summary. – This is the text of a talk given at the XVII Convegno dell’Unione Matematica Italiana held at Milano, September 8-13, 2003. I would like to thank Angelo Lopez and Ciro Ciliberto for the kind invitation to the conference. I survey some numerical conjectures and theorems concerning relations between the index, the pseudo-index and the Picard number of a Fano variety. The results I refer to are contained in the paper [3], wrote in collaboration with Bonavero, Debarre and Druel.

1. – Introduction.

Let $X$ be a smooth, complex projective variety of dimension $n$. Recall that the Picard group Pic $X$ is the group of isomorphism classes of line bundles on $X$, and the anticanonical bundle $-K_X \in \text{Pic } X$ is the determinant of the tangent bundle of $X$. $X$ is called a Fano variety if $-K_X$ is ample, or equivalently if $c_1(X)$ is represented by a positive form. When $X$ is Fano, Pic $X = H^2(X, \mathbb{Z})$ is a free abelian group of rank $q$, the Picard number of $X$.

Examples of Fano varieties are:

1) the projective space $\mathbb{P}^n$;

2) the complete intersections $X = Y_1 \cap \ldots \cap Y_r$, $Y_i$ a generic hypersurface of degree $d_i$ in $\mathbb{P}^N$, with $d_1 + \ldots + d_r \leq N$;

3) homogeneous varieties, namely varieties acted on transitively by a connected linear algebraic group (for instance, grassmannians and flag varieties);

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4) any degree $d$ Galois cyclic cover $X \to \mathbb{P}^n$, ramified over a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $dh$, with $h(d - 1) \leq n$;

5) the moduli spaces $M(r, L, C)$ of stable vector bundles of rank $r$ on a fixed curve $C$ (smooth, of genus at least 2), with determinant a fixed line bundle $L \in \text{Pic } C$ such that $(\deg L, r) = 1$;

6) all (finite) products of Fano varieties.

Fano varieties have a very rich geometry and have been classically intensively studied, see the book [IP99] for a complete survey on the subject.

Up to dimension 3, Fano varieties are classified: in dimension 1 there is only $\mathbb{P}^1$. In dimension 2, there are 10 deformation types: $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-ups of $\mathbb{P}^2$ in $d$ generic points, $d \in \{0, \ldots, 8\}$. For $n = 3$ there are 105 deformation types (the classification is due to Iskovskikh, 1977-78, in the case $q = 1$; to Mori and Mukai, 1981 (1), in the case $q \geq 2$; see [IP99], Ch. 4 and §7.1).

It is well-known that for $n \geq 3$, not all Fano varieties are rational. For instance, the generic cubic hypersurface in $\mathbb{P}^4$ is not rational (Clemens-Griffiths, 1972; see [IP99], Ch. 8 and [Kol96], V.5). Anyway, Fano varieties are close to the projective space in the sense that they contain «lots» of rational curves (by a rational curve we mean the image of a non-constant morphism $\mathbb{P}^1 \to X$). This is formalized saying that every Fano variety is rationally connected (Campana and Kollár-Miyaoka-Mori, 1992; see [IP99], Corollary 6.2.11 and [Kol96], V.2), namely any two points in $X$ can be joined by a rational curve.

This result implies that in any dimension $n$ there is a finite number of deformation types of Fano varieties, with an explicit bound in $n$ (Nadel, Campana, Kollár-Miyaoka-Mori, 1990-1992; see [IP99], §6.2 and [Kol96], V.2.2.4 for a history of the result).

2. – Toric Fano varieties.

A toric variety is a normal, complex algebraic variety, acted on by the group $(\mathbb{C}^*)^n$, and having a dense orbit. (Toric varieties do not need to be Fano, they don't even need to be projective.)

Toric Fano varieties are very special among Fano varieties; here are some of their properties:

1) there is a finite number of them in each dimension (Batyrev, 1982, see [Bat99] and references therein);

2) they are classified up to dimension 4 (for $n = 3$ the classification is due to Batyrev, 1981, and Watanabe-Watanabe, 1982, see [Oda88], §2.3 p. 90;

(1) Mori and Mukai noticed in 2002 that there is a family missing from their original list.
for $n = 4$ the classification is due to Batyrev [Bat99], see also [Sat00], example 4.7 for a missing case in Batyrev’s list);

3) they are rational;

4) they are rigid, namely they do not have non-trivial infinitesimal deformations. This is because for any smooth toric projective variety $X$, the Bott vanishing holds (see [Oda88], §3.3), namely $H^p(\Omega^q_X \otimes L) = 0$ for any $p > 0$, $q \geq 0$ and $L \in \text{Pic} X$ ample. If $X$ is Fano, this gives $H^1(X, T_X) = 0$ ($T_X$ the tangent bundle of $X$).

Some examples of toric Fano varieties are: $\mathbb{P}^n$; the blow-up of $\mathbb{P}^2$ in 1, 2 or 3 points; the blow-up of $\mathbb{P}^n$ along a linear subspace; any (finite) product of toric Fano varieties.

To any toric Fano variety one can associate an $n$-dimensional convex polytope (a so-called Fano polytope), in such a way that the variety is determined by its polytope. Hence, when studying toric Fano varieties, one can use – together with the standard geometric techniques – also their combinatorial features. This makes toric Fano varieties easier and more explicit to study; their are a good testing ground for conjectures about general Fano varieties. For more on toric Fano varieties, see the surveys [Deb03, Wiś02] and references therein.

3. – Index and pseudo-index of a Fano variety.

An important invariant of Fano varieties is the index, defined as

$$r := \max \{ m \in \mathbb{Z} \mid \text{there exists } H \in \text{Pic} X \text{ such that } -K_X = mH \}.$$ 

It is well known that (Kobayashi-Ochiai, 1970, see [IP99], Corollary 3.1.15):

1) $r \in \{ 1, \ldots, n + 1 \}$;

2) $r = n + 1$ if and only if $X = \mathbb{P}^n$;

3) $r = n$ if and only if $X \subseteq \mathbb{P}^{n+1}$ is a smooth quadric.

There are other classified cases:

4) $r = n - 1$: this case has been classified by Iskovskikh in dimension 3 and by Fujita for general $n$ (see [IP99], §3.2); for $n \geq 7$ there are only 4 deformation types in any dimension, all with $q = 1$.

5) $r = n - 2$: the classification is due Wiśniewski in the case $q \geq 2$ (see [IP99], Theorems 7.2.1 and 7.2.2) and mainly to Mukai in the case $q = 1$ (see [IP99], §5.2). Again, for $n \geq 11$ there are only 5 deformation types in any dimension, all with $q = 1$. 

Observe that in dimension 4, the only non classified case is $r = 1$.

The criterion that emerges from these results is that: *Fano varieties with bigger index are simpler.* In 1988 Mukai formulated the following:

**Conjecture M ([Muk88]).** – Let $X$ be a Fano variety of dimension $n$, Picard number $q$ and index $r$. Then

$$q(r - 1) \leq n,$$

and equality holds if and only if $X = (\mathbb{P}^{r-1})^q$.

In 1990 Wiśniewski [Wis'90], proving a case of Conjecture M (property (c) below), introduced a new invariant of $X$, closely related to the index. This is the *pseudo-index*, defined as:

$$i := \min \{-K_X \cdot C | C \text{ rational curve in } X\}.$$

Observe that $i \geq 1$ by Kleiman’s criterion of ampleness. Moreover $r$ divides $i$, because $-K_X = rH$, so for any curve $C$ in $X$ you have

$$-K_X \cdot C = r(H \cdot C).$$

It can be $r < i$: for instance, $\mathbb{P}^1 \times \mathbb{P}^2$ has index 1 and pseudo-index 2.

Basic properties of $i$ are:

(a) $i \leq n + 1$ (Mori, 1979, see [Kol96], Theorem V.1.1.6);
(b) $i = n + 1$ if and only if $X = \mathbb{P}^n$ [CMSB02];
(c) if $i > \frac{1}{2}n + 1$, then $q = 1$ [Wis'90].

This last property, as Wiśniewski implicitly noticed in [Wis'90], leads to formulate the following stronger conjecture:

**Conjecture GM ([BCDD03]).** – Let $X$ be a Fano variety of dimension $n$, Picard number $q$ and pseudo-index $i$. Then

$$q(i - 1) \leq n,$$

and equality holds if and only if $X = (\mathbb{P}^{i-1})^q$.

Observe that the inequality is meaningful only if $i > 1$.

Observe also that, by properties (a) and (b), Conjecture GM holds if $q = 1$.

If $q = 2$, property (c) gives the inequality $i \leq \frac{1}{2}n + 1$. If moreover $i = \frac{1}{2}n + 1$, then $X = (\mathbb{P}^{n/2})^2$ (this is due to Wiśniewski [Wis'90] if $r = i$ and to Occhetta [Oce03] in general). Hence Conjecture GM holds for $q = 2$ too.

Conjecture GM remains open in full generality, but it has been proved in a number of cases:
Theorem 1 ([BCDD03]). – Let $X$ be a Fano variety of dimension $n$, Picard number $\rho$ and pseudo-index $i$. Conjecture GM holds in the following cases:

1) $n \leq 4$;

2) $X$ is toric and $n \leq 7$;

3) $X$ is toric and $i \geq \frac{1}{3}n + 1$;

4) $X$ is a homogeneous variety.

Recently, Andreatta, Occhetta and Chierici have proved some more cases:

Theorem 2 ([ACO03]). – Let $X$ be a Fano variety of dimension $n$, Picard number $\rho$ and pseudo-index $i$. Conjecture GM holds in the following cases:

1) $n = 5$;

2) $i \geq \frac{1}{3}n + 1$ and $X$ has a fiber type extremal contraction;

3) $i \geq \frac{1}{3}n + 1$ and $X$ has no small extremal contractions.

4. – Families of rational curves.

The basic tool in the proof of Theorem 1 is Mori theory, and more generally the study of families of rational curves on $X$. We describe here a part of our approach to the problem. The reference for this subject is the book [Kol96].

Let $X$ be a smooth, complex projective variety of dimension $n$. There is a variety $\text{RatCurves}^n(X)$ parametrizing birational morphisms $\mathbb{P}^1 \to X$, modulo automorphisms of $\mathbb{P}^1$. This is constructed as follows: consider the Hilbert scheme $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ of birational morphisms from $\mathbb{P}^1$ to $X$ and consider its normalization. Then $\text{RatCurves}^n(X)$ is the quotient of this normalization under the action of $\text{Aut}(\mathbb{P}^1)$.

An irreducible component $V$ of $\text{RatCurves}^n(X)$ is called a family of rational curves; curves parametrized by $V$ are all deformation of a same rational curve in $X$, so they are algebraically and numerically equivalent. Hence, they all have the same anticanonical degree, which we denote by $\deg_{-K_X} V$.

The family $V$ is called unsplit if and only if $V$ is proper (compact) as a variety; this is equivalent to asking that curves parametrized by $V$ do not deform to reducible curves in $X$.

Being unsplit is a very strong property. If $X$ is Fano, a family $V$ such that
deg_{-K_X} V < 2\ell is necessarily unsplit: indeed, if a rational curve $C$ deform to a reducible curve $C_1 \cup C_2$, then $-K_X \cdot C \geq -K_X \cdot C_1 - K_X \cdot C_2 \geq 2\ell$.

Conversely, an unsplit family can not have «too high» anticanonical degree:

**Theorem 3 ([BCDD03]).** – Let $X$ be a smooth, complex projective variety of dimension $n$. Let $V_1, \ldots, V_k$ be unsplit families of rational curves in $X$ such that the classes of $V_1, \ldots, V_k$ are algebraically independent. For any $x \in X$ define

$L(V_1, \ldots, V_k)_x := \{ y \in X | \text{there exist curves } C_1, \ldots, C_k \text{ in } X \text{ such that } x \in C_i \text{ and } y \in C_k, C_j \text{ is in } V_j \text{ and } C_j \cap C_{j+1} \neq \emptyset \text{ for all } j \}$.

If $L(V_1, \ldots, V_k)_x \neq \emptyset$, then $\deg_{-K_X} V_1 + \ldots + \deg_{-K_X} V_k \leq \dim L(V_1, \ldots, V_k)_x + k$.

Theorem 3 gives the following general approach to Conjecture GM:

**Corollary 4.** – Let $X$ be a Fano variety of Picard number $\rho$. Assume that there exist unsplit families $V_1, \ldots, V_\rho$ of rational curves in $X$ such that

(i) the classes of $V_1, \ldots, V_\rho$ are algebraically independent;

(ii) there exists curves $C_1, \ldots, C_\rho$ in $X$ such that $C_i$ is in $V_i$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $j$.

Then Conjecture GM holds for $X$.

**Proof.** – By (ii), there exists $x_1 \in X$ such that $L(V_1, \ldots, V_{\rho})_{x_1} \neq \emptyset$. If $\ell$ is the pseudo-index of $X$, we have $\deg_{-K_X} V_j \geq \ell$ for all $j$, so Theorem 3 yields

$\rho \ell \leq \deg_{-K_X} V_1 + \ldots + \deg_{-K_X} V_\rho \leq \dim L(V_1, \ldots, V_{\rho})_{x_1} + \rho \leq n + \rho$,

namely $\rho (\ell - 1) \leq n$. Assume now that $\rho (\ell - 1) = n$. Then $n + \rho = \rho \ell$, hence all inequalities above are equalities. In particular we have $\deg_{-K_X} V_j = \ell$ for all $j$ and $\dim L(V_1, \ldots, V_{\rho})_{x_1} = n$, so $L(V_1, \ldots, V_{\rho})_{x_1} = X$ ($L(V_1, \ldots, V_{\rho})_{x_1}$ is a closed subset, see [BCDD03], §5). This means that for every point $y \in X$ there is a curve belonging to $V_{\rho}$ and passing through $y$, namely that $V_{\rho}$ is a covering family.

Now choose a curve $C'_{\rho}$ in $V_{\rho}$ passing through $x_1$, and $x_{\rho} \in C'_{\rho}$. By construction $L(V_{\rho}, V_1, \ldots, V_{\rho-1})_{x_{\rho}} \neq \emptyset$, so applying again Theorem 3, we see that $L(V_{\rho}, V_1, \ldots, V_{\rho-1})_{x_{\rho}} = X$ and that $V_{\rho-1}$ is a covering family. Proceeding in this way, for each $j = 1, \ldots, 2$ we find $x_j$ such that $L(V_j, \ldots, V_{\rho}, V_1, \ldots, V_{\rho-1})_{x_j} = X$, so $V_{j-1}$ is a covering family.

Thus $V_1, \ldots, V_{\rho}$ are covering families of degree $\ell$, and Theorem 1 of [Occ03] yields $X \cong (\mathbb{P}^{n-1})^\rho$. 
5. – Other properties of the pseudo-index.

The pseudo-index has some remarkable properties also in relation to morphisms.

**Proposition 5** ([BCDD03]). – Let $X$ be a Fano variety of pseudo-index $i_X$, $Y$ a smooth variety and $f : X \rightarrow Y$ a surjective morphism with connected fibers.

If $\dim Y < i_X$, then $Y = \mathbb{P}^r$ and $X = F \times \mathbb{P}^r$, $F$ a smooth variety.

Again, we observe the principle that the bigger $i_X$ is, the stronger conditions we find on $X$.

Recently Bonavero has studied the behaviour of the pseudo-index under a smooth blow-up $X \rightarrow Y$. Assume $X$ and $Y$ are Fano and denote by $r_X$ and $i_X$ (respectively, $r_Y$ and $i_Y$) the index and the pseudo-index of $X$ (respectively, of $Y$). We have $r_X \leq r_Y$, and one would expect a similar behaviour for the pseudo-index. Quite surprisingly, it depends on the dimension of the center of the blow-up:

**Theorem 6** ([Bon03]). – Let $X$ and $Y$ be Fano varieties of dimension $n$, such that $X \rightarrow Y$ is the blow-up along a smooth subvariety $Z \subset Y$.

If $\dim Z < \frac{1}{2}(n + i_Y - 1)$ or $\dim Z > n - 2 - i_Y$, then $i_X \leq i_Y$.

These bounds are optimal: in [Bon03] you can find examples with $i_X > i_Y$ and $\dim Z = \frac{1}{2}(n + i_Y - 1)$ or $\dim Z = n - 2 - i_Y$.

6. – Related open questions.

6.1. – There are no known bounds (even conjecturally, to my knowledge) for the Picard number $\rho$ of an $n$-dimensional Fano variety $X$.

1) Conjecture GM would give $\rho \leq n$ if $\iota > 1$.

2) What happens when $\iota = 1$?

In the toric case, it is known that $\rho \leq 2n \sqrt{2n} + o(n^{3/2})$ [VK85, Deb03], but conjecturally the bound should be linear:

$$\rho \leq \begin{cases} 2n & \text{if } n \text{ is even}, \\ 2n - 1 & \text{if } n \text{ is odd}. \end{cases}$$

This bound holds for toric Fano varieties of dimension $n \leq 5$ [Bat99, Cas03b].
6.2. – Rational curves \( C \) in \( X \) having minimal anticanonical degree, namely such that \(-K_X \cdot C = i\), should be the analogue of lines in projective space. It is reasonable to expect that these curves have special properties:

CONJECTURE. – Let \( X \) be a Fano variety of pseudo-index \( i \) and \( C \subset X \) a rational curve. If \(-K_X \cdot C = i\), then \( C \) is extremal.

This conjecture has been proved for toric Fano varieties [Cas03a].

6.3. – We conclude with a conjecture about characterization of Fano varieties.

CONJECTURE ([Kol96], Conjecture III.1.2.5.4). – Let \( X \) be a smooth projective variety. If \(-K_X \cdot C > 0\) for any curve in \( X \), then \( X \) is Fano.

The conjecture is trivially true if \( X \) is a toric variety (see [Oda88], Theorem 2.18), and has been proved for Fano varieties of dimension \( n \leq 3 \) (Matsuki, 1987, see [Kol96], Remark III.1.2.5.5).

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