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Investigations of Smooth Functions and Analytic Sets Using Fractal Dimensions.

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Sunto. – *Si parte dal seguente problema: data una funzione* $f : [0, 1] \to [0, 1]$, *cosa si può dire riguardo l'insieme dei punti nel codominio in cui gli insiemi di livello sono grandi secondo una opportuna definizione. Ciò porta alla necessità di analizzare la struttura degli insiemi di livello per funzioni di classe* C^n. *Analogo problema viene affrontato per le funzioni di classe* $C^{n,\alpha}$ *che sono in un certo senso intermedie fra quelle di classe* C^n *e quelle di classe* C^{n+1}. *I risultati coinvolgono strumenti di analisi reale, teoria geometrica della misura e teoria descrittiva classica degli insiemi.*

Summary. – *We start from the following problem: given a function* $f : [0, 1] \to [0, 1]$ *what can be said about the set of points in the range where level sets are «big» according to an opportune definition. This yields the necessity of an analysis of the structure of level sets of* C^n *functions. We investigate the analogous problem for* $C^{n,\alpha}$ *functions. These are in a certain way intermediate between* C^n *and* C^{n+1} *functions. The results involve a mixture of Real Analysis, Geometric Measure Theory and Classical Descriptive Set Theory.*

1. – Introduction.

Some nineteenth century mathematicians were aware of the existence of continuous functions that had no point of differentiability. Constructions of such functions involved summations of infinite series whose successive terms contributed increasingly to the nondifferentiability of their sum. Perhaps the first such construction was given by K. Weierstrass around 1985 [3].

In early 1900's H. Lebesgue proved *The Lebesgue Differentiation Theorem*, i.e. every real-valued function of bounded variation defined on the line must be differentiable almost everywhere. From this we have that a nowhere differentiable function can not be of bounded variation.

We recall that if a property is valid for all points in a complete metric

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space except for a set of the first category, we say that the property holds *tipically.*

Use of *The Baire Category Theorem* to prove the existence of continuous functions without points of differentiability had to wait until 1931, at which time S. Banach [1] and S. Mazurkiewicz [14], in separate papers published in the journal *Studia Mathematica*, provided such proofs. They proved that a typical $f \in C([0, 1])$ (here $C([0, 1])$ is given with the sup norm) is nowhere differentiable. Banach also characterized functions of bounded variation in a theorem which has come to be known as *The Banach Indicatrix Theorem*:

THEOREM 1.1. – *A function $f : [0.1] \to \mathbb{R}$ is of bounded variation if and only if $\int_{\mathbb{R}} I(y)\, dy < +\infty$, where*

$$I(y) = \begin{cases} n & \text{if the cardinality of } f^{-1}(\{y\}) \text{ is } n \\ +\infty & \text{if } f^{-1}(\{y\}) \text{ is an infinite set.} \end{cases}$$

From these results it follows that a typical function $f \in C([0, 1])$ has to wiggle a lot, it has the property that for many $y's$ the set $f^{-1}(\{y\})$ must be «big».

In what follows we call $f^{-1}(\{y\})$ a level set of f.

Motivated by the above results, in 1977 Bruckner and Garg investigated level sets of a typical $f \in C([0, 1])$. They gave a full description of level sets: they showed that a typical continuous function has the property that all except possibly countably many of its level sets are perfect. Namely, they proved:

THEOREM 1.2. – ([4]) *For a typical $f \in C([0, 1])$ there exists a countable set $S_f \subset (\min f, \max f)$ such that the level set $f^{-1}(\{y\})$ is:*

1. *a nowhere dense perfect set if $y \notin S_f \cup \{\min f, \max f\}$,*
2. *a single point if $y = \min f$ or $\max f$, and*
3. *the union of a nowhere dense perfect set and an isolated point of $f^{-1}(\{y\})$ if $y \in S_f$.*

Function f in the above theorem has a local extremum at $(p, f(p))$ if p is an isolated point of $f^{-1}(\{p\})$. Hence, what this theorem implies is that a typical function $f \in C([0, 1])$ must oscillate a lot since all but two of its level sets contain a portion which is homeomorphic to the Cantor set.

The above result of Bruckner and Garg has inspired further research concerning level sets of functions ([10], [6], [7], [9], [8]). Darji and Morayne [10] investigated the analogue of Bruckner-Garg theorem for smooth functions. Consider $C^n([0, 1])$, the space of n-times continuously differentiable functions endowed with the usual norm. They showed that a typical $f \in C^1([0, 1])$

is either strictly monotone or f has uncountably many level sets having exactly one accumulation point and all other level sets of f are finite. Namely, they proved:

THEOREM 1.3. – *A typical* $f \in C^1([0, 1])$ *is either strictly monotone or there exist a perfect nowhere dense set* $P_f \subset (\min f, \max f)$ *and a countable dense set* $D_f \subset P_f$ *such that the level set* $f^{-1}(\{y\})$ *is*:

1. *a set with exactly one accumulation point if* $y \in P_f \setminus D_f$,
2. *a finite set if* $y \in D_f \cup ((\min f, \max f) \setminus P_f)$, *and*
3. *a single point if* $y \in \{\min f, \max f\}$.

Moreover, they showed that the Lebesgue measure of $P_f \setminus D_f$ is zero. It is also shown in their paper that for $n \geq 2$ a typical function in $C^n([0, 1])$ has the property that all of its level sets are finite.

The results mentioned above motivate the following natural question:

QUESTION. – What can we say in the opposite direction, i.e. what is the «worst» case behavior of C^n functions as far as the level structure is concerned?

Our results in this direction involve a mixture of Real Analysis, Geometric Measure Theory and Classical Descriptive Set Theory.

2. – The C^n case $(0 \leq n \leq +\infty)$.

The proofs of the results mentioned in this section are contained in a joint paper with U. B. Darji [7].

The general problem motivating our results is the following:

Given a function $f \in C^n([0, 1])$, $f : [0, 1] \to [0, 1]$, what can be said about the sets of points in the range where level sets of f are «big»? Of course, one has to decide what «big» means and what it means «to describe» this set.

It is a classical result of Mazurkiewicz and Sierpinski [15] that $M \subseteq [0, 1]$ is analytic if and only if M is equal to the set $\{y : f^{-1}(\{y\}) \textit{ is uncountable}\}$ for some continuous function f. We characterize the set of points where level sets of continuous functions are perfect. Namely, we prove:

THEOREM 2.1. – *Let* $M \subseteq [0, 1]$. *The following are equivalent*:

1. M *is the union of a* G_δ *set and a countable set*,
2. *there exists a continuous function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is perfect for all* $y \in M$ *and* $f^{-1}(\{y\})$ *is finite otherwise.*

In order to characterize the set of points where level sets of a C^n function $(1 \leqslant n \leqslant +\infty)$ are perfect and the set of points where level sets of a C^n function $(1 \leqslant n \leqslant +\infty)$ are uncountable we first need few definitions and some terminology.

DEFINITION 2.2. – *Let f be a C^n function $(1 \leqslant n < \infty)$. For a positive integer i, we let $f^{(i)}$ be the i-th derivative of f and $f^{(0)} = f$. We let $Z_{(f,n)}$ denote the set*

$$\{x : f^{(i)}(x) = 0 \ \textit{for all} \ 1 \leqslant i \leqslant n\}.$$

We call $Z_{(f,n)}$ a zero-derivative set. *We use $\|f\|_n$ to denote n-*norm, *i.e.* $\sum_{i=0}^{n} \|f^{(i)}\|$, *where $\|\cdot\|$ denotes the sup norm. If $f \in C^\infty$, then $Z_{(f,\infty)}$ is simply*

$$\{x : f^{(i)}(x) = 0 \ \textit{for all} \ i \geqslant 1\}.$$

One of the main tools is the characterization of the following class $\mathcal{A}_n$, whose proof is geometric and requires somewhat delicate estimates.

DEFINITION 2.3. – *We define $\mathcal{A}_n$ $(1 \leqslant n \leqslant \infty)$ to be the collection of all sets $P \subseteq [0, 1]$ with the property that there is a C^n function f from $[0, 1]$ into $[0, 1]$ such that $P = f(Z_{(f,n)})$.*

REMARK 2.4. – *We point out that $\mathcal{A}_1$ is the collection of all closed subsets of $[0, 1]$ with Lebesgue measure zero.*

DEFINITION 2.5. – *If $M \subseteq \mathbb{R}$ and $s > 0$, then $\mathcal{H}^s(M)$ is the s-dimensional Hausdorff measure of M. We use $\lambda(A)$ to denote the Lebesgue measure of A.*

Our characterization of $\mathcal{A}_n$ involves Hausdorff measures and the condition β defined below.

DEFINITION 2.6. – *Suppose that I is a closed interval, P is a closed set and $1 \leqslant n < \infty$. We use $\beta_n(P, I)$ to denote the number $\sum_{i=1}^{\infty} \lambda(S_i)^{1/n}$, where $S_1, S_2, \dots$ are components of $I \setminus P$.*

THEOREM 2.7. – *Let $P \subseteq [0, 1]$. Suppose $1 \leqslant n < \infty$. Then, the following are equivalent*:

1. $P \in \mathcal{A}_n$,
2. *P is a closed set with $\mathcal{H}^{1/n}(P) = 0$ and $\beta_n(P, [0, 1]) < \infty$.*

Moreover, if $P \subseteq [0, 1]$ satisfies Condition 2 then there is a C^n homeomorphism from $[0, 1]$ onto $[0, 1]$ such that $P = f(Z_{(f,n)})$ and $\lambda(Z_{(f,n)}) = 0$.

We point out that if $P \in \mathcal{A}_n$ then P is closed as it is the continuous image of a compact set and, by Sard's Theorem, $\mathcal{H}^{1/n}(P) = 0$.

Now we consider the C^∞ case.

THEOREM 2.8. – *Let $P \subseteq [0, 1]$. Then, the following are equivalent*:

1. $P \in \mathcal{A}_\infty$.
2. *P is a closed set with $\mathcal{H}^{1/n}(P) = 0$ and $\beta_n(P, [0, 1]) < \infty$ for all n.*

Moreover, if $P \subseteq [0, 1]$ satisfies Condition 2 then there is a C^∞ homeomorphism from $[0, 1]$ onto $[0, 1]$ such that $P = f(Z_{(f, \infty)})$ and $\lambda(Z_{(f, \infty)}) = 0$.

Next two results give characterizations of the set of points where levels sets of C^n functions are perfect and of the set of points where levels sets of C^n functions are uncountable.

THEOREM 2.9. – *Let $M \subseteq [0, 1]$ and $1 \leqslant n \leqslant \infty$. Then, the following are equivalent*:

1. *M is the union of a G_δ set and a countable set and there is $P \in \mathcal{A}_n$ such that $M \subseteq P$.*
2. *There is a C^n function $f : [0, 1] \to [0, 1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

THEOREM 2.10. – *Let $M \subseteq [0, 1]$ and $1 \leqslant n \leqslant \infty$. Then, the following are equivalent*:

1. *M is analytic and there is $P \in \mathcal{A}_n$ such that $M \subseteq P$,*
2. *there is a C^n function $f : [0, 1] \to [0, 1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

The proofs of the above two theorems use Theorem 2.7, Theorem 2.8 and useful results of Bruckner-Goffman, Laczkovich-Preiss, and Lebedev. Let us describe the Bruckner-Goffman-Laczkovich-Preiss-Lebedev results.

DEFINITION 2.11. – *Let $f : I \to \mathbb{R}$ be a continuous function. Let G be the union of all open (relative to I) subintervals S such that f is monotone on S. We call $p \in I \setminus G$ a turning point of f. We use T_f to denote the union of the set of all turning points of f and the end-points of I.*

DEFINITION 2.12. – *Let $f : I \to \mathbb{R}$ be a continuous function. Then, n-variation of f is defined to be*

$$V_n(f) = \sup\left\{ \sum_{i=1}^{k} |f(x_{i+1}) - f(x_i)|^{1/n} \right\},$$

where $x_1 < x_2 < x_3 < \ldots < x_{k+1}$, with $x_i \in T_f$ for all i.

REMARK 2.2. – *We remark here that $V_1(f)$ is equal to $V(f)$, the usual variation of f* ([12], *Theorem* 2.3).

Henceforth, we shall denote by *CBV* continuous functions of bounded variation and by *BV* functions of bounded variation. The following result is one of the essential tools for our proofs. The C^1 case was proved by Bruckner and Goffman in [5] and the general case was proved independently by Laczkovich and Preiss in [12] and by Lebedev in [13].

THEOREM 2.14. – *Let $f:[0,1]\to\mathbb{R}$ be a CBV function with $V_n(f)<\infty$. Then*

1. [5] *if $n=1$ and $\lambda(f(T_f))=0$, there is a homeomorphism $h:[0,1]\to[0,1]$ such that $f\circ h$ is C^1,*
2. ([12], [13]) *if $n>1$, there is a homeomorphism $h:[0,1]\to[0,1]$ such that $f\circ h$ is C^n.*

The following is the C^∞ version of the above theorem.

THEOREM 2.15. – ([12], [13]) *Suppose that $f:[0,1]\to\mathbb{R}$ is a CBV function with $V_n(f)<\infty$, for every n. Then, there is a homeomorphism $h:[0,1]\to[0,1]$ such that $f\circ h$ is C^∞.*

3. – The $C^{n,\alpha}$ case $(1\leq n<+\infty,\ 0<\alpha\leq 1)$.

The results in the previous section lead to another natural question.

QUESTION. – And when n is not an integer what can we say? What can be said about the sets of points in the range where level sets of f are «big»? Is it possible «to parametrize» Hausdorff dimension of analytic sets using $C^{n,\alpha}$ functions?

This section is devoted to answering this question. The proofs of the results are contained in [9].

DEFINITION 3.1. – *If $0<\alpha\leq 1$, we denote by $C^{0,\alpha}(I)$ the space of Hölder functions on a closed interval I, i.e. such that*

$$[f]_{0,\alpha}=\sup_{\substack{x,y\in I\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty .$$

More generally, we denote by $C^{n,\alpha}(I)$ the space of C^n functions with Hölder n-th derivatives. Clearly, $C^{0,1}(I)$ is the space of Lipschitz functions on I.

DEFINITION 3.2. – *We define* $\mathcal{A}_{n,\alpha}$ $(1 \leq n < \infty,\ 0 < \alpha \leq 1)$ *to be the collection of all sets* $P \subseteq [0,1]$ *such that there is a* $C^{n,\alpha}$ *function* f *from* $[0,1]$ *into* $[0,1]$ *such that* $P = f(Z_{(f,n)})$.

We provide a characterization of $\mathcal{A}_{n,\alpha}$ in terms of Hausdorff measures and the condition β defined below.

DEFINITION 3.3. – *Suppose that* I *is a closed interval and* P *is a closed set. We use* $\beta_{n,\alpha}(P, I)$ *to denote the number* $\sum_{i=1}^{\infty} \lambda(S_i)^{1/(n+\alpha)}$, *where* $S_1, S_2, \dots$ *are components of* $I \setminus P$.

THEOREM 3.4. – *Let* $P \subseteq [0,1]$. *Suppose* $1 \leq n < \infty$ *and* $0 < \alpha \leq 1$. *Then, the following are equivalent*:

1. $P \in \mathcal{A}_{n,\alpha}$,
2. P *is a closed set with* $\beta_{n,\alpha}(P,[0,1]) < \infty$ *and* $\mathcal{H}^{1/(n+\alpha)}(P) = 0$.

Moreover, if $P \subseteq [0,1]$ *satisfies Condition* 2 *then there is a* $C^{n,\alpha}$ *homeomorphism from* $[0,1]$ *onto* $[0,1]$ *such that* $P = f(Z_{(f,n)})$ *and* $\lambda(Z_{(f,n)}) = 0$.

We point out that if $P \in \mathcal{A}_{n,\alpha}$ then it clearly is closed and, by Sard's Theorem, $\lambda(P) = 0$. By a result of Besicovitch and Taylor ([2]: Lemma 2), $\mathcal{H}^{1/(n+\alpha)}(P) = 0$.

EXAMPLE 3.5. – *Denote by* $\dim_{\mathcal{H}}$ *the Hausdorff dimension. Let* C_γ *be the «Cantor sets» obtained by removing the middle* γ*-th percentage every time. It is easy to compute that* $\dim_{\mathcal{H}}(C_\gamma) = -\frac{\log 2}{\log[(1-\gamma)/2]}$. *Clearly, if* $\gamma > 1 - \frac{1}{2^{n+\alpha-1}}$ *then* $\mathcal{H}^{1/(n+\alpha)}(C_\gamma) = 0$ and $\beta_{n,\alpha}(C_\gamma,[0,1]) < \infty$, hence $C_\gamma \in \mathcal{A}_{n,\alpha}$.

REMARK 3.6. – *From Theorem* 2.7 *and Theorem* 2.8, *it follows that* $\bigcap_n \mathcal{A}_n$ *coincides with* $\mathcal{A}_\infty$. *From Theorem* 2.7 *and Theorem* 3.4, *it follows that* $\mathcal{A}_{n+1}$ *coincides with* $\mathcal{A}_{n,1}$. *On the other hand, it is clear that* $\bigcap_n \mathcal{A}_{n,\alpha} = \mathcal{A}_\infty$.

Next follow the characterizations of the set of points where level sets of a $C^{n,\alpha}$ function are perfect and the set of points where level sets of a $C^{n,\alpha}$ function are uncountable.

THEOREM 3.7. – *Let* $M \subseteq [0,1]$. *The following are equivalent*:

1. M *is the union of a* G_δ *set and a countable set and there is* $P \in \mathcal{A}_{n,\alpha}$ *such that* $M \subseteq P$,
2. *there is a* $C^{n,\alpha}$ *function* $f : [0,1] \to [0,1]$ *such that* $f^{-1}(\{y\})$ *is perfect for every* $y \in M$ *and finite otherwise.*

THEOREM 3.8. – *Let* $M \subseteq [0, 1]$. *The following are equivalent*:

1. *M is analytic and there is* $P \in \mathcal{A}_{n,\alpha}$ *such that* $M \subseteq P$,
2. *there is a* $C^{n,\alpha}$ *function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is uncountable for every* $y \in M$ *and countable otherwise.*

We now point out below several important consequences of Remark 3.6 and of Theorems 2.9, 2.10. 3.7 and 3.8.

1. There exist a C^∞ function and a perfect set P such that $f^{-1}(\{y\})$ is perfect for all $y \in P$. This is another way of seeing that C^∞ functions are far from real analytic functions.
2. There are C^∞ functions from $[0, 1]$ into $[0, 1]$ such that $\{y : f^{-1}(\{y\}$ *is uncountable*$\}$ is analytic and not Borel.
3. These theorems can be viewed as «parametrization» of the Hausdorff dimension of analytic sets by smooth functions and Hölder functions.
4. C^{n+1} functions and $C^{n,1}$ functions have the same «worst» possible behaviour as far as the level sets structure is concerned.

Consequence (4) listed above is expressed in the next two results:

THEOREM 3.9. – *Let* $M \subseteq [0, 1]$ *and* $1 \leqslant n < \infty$. *Then, the following are equivalent*:

1. *there is a* $C^{n,1}$ *function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is perfect for every* $y \in M$ *and finite otherwise,*
2. *M is the union of a* G_δ *set and a countable set and there is* $P \in \mathcal{A}_{n,1}$ *such that* $M \subseteq P$,
3. *there is a* C^{n+1} *function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is perfect for every* $y \in M$ *and finite otherwise.*

THEOREM 3.10. – *Let* $M \subseteq [0, 1]$. *The following are equivalent*:

1. *there is a* $C^{n,1}$ *function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is uncountable for every* $y \in M$ *and countable otherwise,*
2. *M is an analytic set and there is* $P \in \mathcal{A}_{n,1}$ *such that* $M \subseteq P$,
3. *there is a* C^{n+1} *function* $f : [0, 1] \to [0, 1]$ *such that* $f^{-1}(\{y\})$ *is uncountable for every* $y \in M$ *and countable otherwise.*

4. – The Lipschitz case.

The following question naturally arises:

QUESTION. – And when $n = 0$ what happens, i.e. have $C^{0,1}$ functions and C^1 functions the same «worst» possible behaviour as far as the level sets structure is concerned?

This section is devoted to answering this question. The answer turns out to be negative. The proofs of the results are contained in [9] and [8].

In this section we characterize the set of points where level sets of a given Lipschitz function are perfect [9] and the set of points where level sets of a given Lipschitz function are uncountable [8].

A fundamental tool in order to obtain the main results is the following theorem proved by A. M. Bruckner and C. Goffman.

THEOREM 4.1. – ([5]: *Lemma* 1) *If f is a continuous function of bounded variation on* $[0, 1]$, *there exists a homeomorphism h of* $[0, 1]$ *onto itself such that $f \circ h$ is a Lipschitz function.*

PROPOSITION 4.2. – *Let $M \subseteq [0, 1]$ be the union of a G_δ set and a countable set with $\lambda(M) = 0$. Then there exists a Lipschitz function $f : [0, 1] \to [0, 1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

PROPOSITION 4.3. – *Let $M \subseteq [0, 1]$ be an analytic set with $\lambda(M) = 0$. Then there exists a Lipschitz function $f : [0, 1] \to [0, 1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

We point out that the difficult part in the proofs of Proposition 4.2 and Proposition 4.3 is to construct a *CBV* function with the required property for what concerns level sets as, once we have it, applying the above recalled result of Bruckner and Goffman, we can «transform» it into a Lipschitz function.

The following theorems are the goal of this section.

THEOREM 4.4. – *Let $M \subseteq [0, 1]$. Then the following are equivalent:*

1. *there is a Lipschitz function f from $[0, 1]$ into $[0, 1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise,*
2. *M is the union of a G_δ and a countable set and $\lambda(M) = 0$.*

THEOREM 4.5. – *Let $M \subseteq [0, 1]$. Then the following are equivalent*:

1. *there is a Lipschitz function f from $[0, 1]$ into $[0, 1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise,*
2. *M is analytic and $\lambda(M) = 0$.*

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