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Metrizability of the Unit Ball of the Dual of a Quasi-Normed Cone.

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Sunto. – *Dimostriamo teoremi di metrizzabilità e di quasi metrizzabilità per alcune topologie di tipo debole* sulla palla unitaria del duale di un cono quasi normato separabile. Ciò è ottenuto grazie a un'opportuna versione del teorema di Alaoglu, anch'essa dimostrata nel presente lavoro.*

Summary. – *We obtain theorems of metrization and quasi-metrization for several topologies of weak* type on the unit ball of the dual of any separable quasi-normed cone. This is done with the help of an appropriate version of the Alaoglu theorem which is also obtained here.*

1. – Introduction and preliminaries.

Throughout this paper the letters $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of nonnegative real numbers and the set of positive integers numbers, respectively. Our main references for quasi-pseudo-metric spaces are [3] and [8].

Recall that a *monoid* is a semigroup $(X, +)$ with neutral element 0.

According to [6], a *cone* (on $\mathbb{R}^+$) is a triple $(X, +, \cdot)$ such that $(X, +)$ is an Abelian monoid, and $\cdot$ is a function from $\mathbb{R}^+ \times X$ to X such that for all $x, y \in X$ and $r, s \in \mathbb{R}^+$: (i) $r\cdot(s\cdot x) = (rs)\cdot x$; (ii) $r\cdot(x+y) = (r\cdot x) + (r\cdot y)$; (iii) $(r+s)\cdot x = (r\cdot x) + (s\cdot x)$; (iv) $1\cdot x = x$.

A cone $(X, +, \cdot)$ is called *cancellative* if for all $x, y, z \in X$, $z + x = z + y$ implies $x = y$.

Obviously, every linear space $(X, +, \cdot)$ can be considered as a cancellative cone when we restrict the operation $\cdot$ to $\mathbb{R}^+ \times X$.

A *quasi-norm* on a cone $(X, +, \cdot)$ is a function $q : X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$:

(i) $x = 0$ if and only if there is $-x \in X$ and $q(x) = q(-x) = 0$; (ii) $q(r\cdot x) = rq(x)$; (iii) $q(x+y) \leqslant q(x) + q(y)$.

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If the quasi-norm q satisfies: (i′) $q(x) = 0$ if and only if $x = 0$, then q is called a *norm* on the cone $(X, +, \cdot)$.

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In our context, by a *quasi-metric on* X we mean a quasi-pseudo-metric d on X that satisfies the following condition: $d(x, y) = d(y, x) = 0$ if and only if $x = y$.

We will also consider *extended quasi-(pseudo-)metrics.* They satisfy the three above axioms, except that we allow $d(x, y) = +\infty$.

Each extended quasi-pseudo-metric d on a set X induces a topology $\tau(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X: d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

Observe that if d is an extended quasi-metric, then $\tau(d)$ is a T_0 topology.

The following well-known example will be useful later on. For each $x, y \in \mathbb{R}$ let $u(x, y) = (y - x) \vee 0$. Then u is clearly a quasi-metric on $\mathbb{R}$. Its restriction to $\mathbb{R}^+$ will be also denoted by u.

A (n *extended*) *quasi-(pseudo-)metric space* is a pair (X, d) such that X is a set and d is a (n extended) quasi-(pseudo-)metric on X.

In this paper we prove an analogue to the celebrated Alaoglu theorem for topologies of weak* type defined on the (positive) dual of a quasi-normed cone, and derive theorems of metrization and quasi-metrization for these topologies when the quasi-normed cone is separable. These results extend important theorems on metrizablity of weak* topologies for separable normed linear spaces (compare Chapter IIA of [13]).

It seems interesting to point out that quasi-normed cones and other related «nonsymmetric» structures from topological algebra and functional analysis, have been successfully applied, in the last years, to several problems in theoretical computer science and approximation theory, respectively (see Sections 11 and 12 of [8], and also [1], [2], [4], [6], [10], [11], [12], etc.).

2. – Generating extended quasi-metrics from quasi-norms on cones.

If q is a quasi-norm on a cone $(X, +, \cdot)$, then we can construct in a natural way a topology on X, for which the collection of sets of the form $x + \{y \in X : q(y) < \varepsilon\}$, $\varepsilon > 0$, is a basis of neighborhoods of x, for all $x \in X$.

Next we show that this topology can be induced by a subinvariant extended quasi-pseudo-metric, where, similarly to [7], an extended quasi-metric d on a cone $(X, +, \cdot)$ is said to be *subinvariant* if for each $x, y, z \in X$ and $r > 0$, $d(x + z, y + z) \leq d(x, y)$ and $d(rx, ry) = rd(x, y)$.

Given a cone $(X, +, \cdot)$, for each $x \in X$ we define $x + X = \{x + y : y \in X\}$.

PROPOSITION 1. – *Let q be a quasi-norm on a cone $(X, +, \cdot)$. Then the function e_q defined on $X \times X$ by*

$e_q(x, y) = \inf\{q(a): y = x + a\}$ *if* $y \in x + X$,

$e_q(x, y) = +\infty$ *if* $y \notin x + X$,

is a subinvariant extended quasi-pseudo-metric on X.

Furthermore for each $x \in X$, $r > 0$ and $\varepsilon > 0$, $rB_{e_q}(x, \varepsilon) = rx + \{y \in X: q(y) < r\varepsilon\}$, and the translations with respect to $+$ and $\cdot$ are $\tau(e_q)$-open.

PROOF. – For each $x \in X$ we obviously have $e_q(x, x) = q(0) = 0$.

Next we show that for all $x, y, z \in X$, $e_q(x, z) \leqslant e_q(x, y) + e_q(y, z)$.

Note that it suffices to consider only the case that $y \in x + X$ and $z \in y + X$. Choose an arbitrary $\varepsilon > 0$. Then, there exist $a, b \in X$ such that $y = x + a$, $z = y + b$, $q(a) < e_q(x, y) + \varepsilon$ and $q(b) < e_q(y, z) + \varepsilon$. Consequently $z = x + a + b$, and thus

$$e_q(x, z) \leqslant q(a + b) \leqslant q(a) + q(b) < e_q(x, y) + e_q(y, z) + 2\varepsilon .$$

We conclude that e_q is an extended quasi-pseudo-metric on X.

Now we show that e_q is subinvariant.

Let $x, y, z \in X$. If $y \notin x + X$, $e_q(x, y) = +\infty$. Otherwise, for each $a \in X$ such that $y = x + a$ we have $y + z = x + z + a$, so $e_q(x + z, y + z) \leqslant q(a)$. Therefore

$$e_q(x + z, y + z) \leqslant \inf\{q(a): y = x + a\} = e_q(x, y).$$

On the other hand, it is easy to check that, for each $x, y \in X$ and $r > 0$, $e_q(rx, ry) = re_q(x, y)$. We conclude that e_q is subinvariant.

Finally note that for each $x \in X$ and $r > 0$, $e_q(0, x) = q(x)$, so for each $\varepsilon > 0$, we have $B_{e_q}(0, \varepsilon) = \{x \in X: q(x) < \varepsilon\}$ and $rB_{e_q}(0, \varepsilon) = B_{e_q}(0, r\varepsilon)$. It immediately follows that for each $x \in X$ and each $\varepsilon > 0$

$$rB_{e_q}(x, \varepsilon) = rx + B_{e_q}(0, r\varepsilon),$$

and thus the translations with respect to $+$ and $\cdot$ are $\tau(e_q)$-open. ■

EXAMPLE 1. – Let $q : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $q(x) = x$ for all $x \in \mathbb{R}^+$. Clearly q is a (quasi-)norm on the cone $\mathbb{R}^+$ with $e_q(x, y) = y - x$ if $x \leqslant y$, and $e_q(x, y) = +\infty$, otherwise. So e_q is an extended quasi-metric on $\mathbb{R}^+$ that induces the Sorgenfrey topology on $\mathbb{R}^+$.

REMARK 1. – Note that if q is a norm on a linear space $(X, +, \cdot)$, then the (extended) metric e_q of the above theorem is the classical metric on X generated by q, i.e. $e_q(x, y) = q(y - x)$ for all $x, y \in X$.

In the light of the preceding facts we suggest the following notion.

DEFINITION 1. – A *quasi-normed* cone is a pair (X, q) such that X is a cone and q is a quasi-norm on X such that e_q is a (n extended) quasi-metric on X. In the case that q is a norm we will say that (X, q) is a *normed cone.*

Note that for the norm q of Example 1, $(\mathbb{R}^+, q)$ is a normed cone. Actually, it is not hard to see that if X is a cancellative cone, then for any quasi-norm q on X, e_q is a (n extended) quasi-metric, so (X, q) is a quasi-normed cone.

In connection with Proposition 1 and Example 1 above, some results for quasi-normed monoids may be found in [9].

3. – The dual space of a quasi-normed cone.

A mapping from a quasi-normed cone (X, q) to a topological space (Y, τ) will be called *continuous* if it is continuous from $(X, \tau(e_q))$ to (Y, τ).

Given a quasi-normed cone (X, q) let

$$X^* = \{f : (X, q) \rightarrow (\mathbb{R}^+, u) : f \text{ is linear and continuous}\}.$$

Obviously, X^* is a cone for the usual pointwise operations. Note that $f : X \rightarrow \mathbb{R}^+$ is in X^* if and only if it is a linear and upper semicontinuous non-negative real-valued function on (X, q).

The next result is essentially known. For the sake of completeness we give its easy proof.

PROPOSITION 2. – *Let (X, q) be a quasi-normed cone and let $f : X \rightarrow \mathbb{R}^+$ be linear. Then $f \in X^*$ if and only if there is $M > 0$ such that $f(x) \leq Mq(x)$ for all $x \in X$.*

PROOF. – Suppose that $f \in X^*$. Then there is $\delta > 0$ such that $f(B_{e_q}(0, \delta)) \subseteq [0, 1[$. Put $M = 2/\delta$. Fix $x \in X$. If $q(x) = 0$, then $f(x) = 0$ (indeed, if $f(x) > 0$, we have $q(x/f(x)) = 0$ but $f(x/f(x)) = 1$, a contradiction). If $q(x) > 0$, then $x/Mq(x) \in B_{e_q}(0, \delta)$ and thus $f(x/Mq(x)) < 1$. We conclude that $f(x) \leq Mq(x)$ for all $x \in X$.

Conversely, if $e_q(x, x_n) \rightarrow 0$, there is a sequence $(a_n)_{n \in \mathbb{N}}$ in X such that $x_n = x + a_n$ for all $n \in \mathbb{N}$ and $q(a_n) \rightarrow 0$. Since $f(x_n) = f(x) + f(a_n) \leq f(x) + Mq(a_n)$ for all $n \in \mathbb{N}$, it follows that $u(f(x), f(x_n)) \rightarrow 0$, so f is continuous from $(X, \tau(e_q))$ to $(\mathbb{R}^+, u)$, i.e. $f \in X^*$. ■

PROPOSITION 3. – *Let (X, q) be a quasi-normed cone. For each $f \in X^*$ set*

$$q^*(f) = \sup\{f(x) : q(x) \leq 1\}.$$

Then (X^*, q^*) is a quasi-normed cone.

PROOF. – First observe that, by Proposition 2, there is $M>0$ such that $q^*(f) \leqslant M$ for all $f \in X^*$. So q^* is well-defined. Furthermore $q^*(0)=0$, and if $f \in X^*$ satisfies that $-f \in X^*$, then $f \equiv 0$. Finally, it is clear that for each $f, g \in X^*$ and $r \in \mathbb{R}^+$ we have $q^*(af) = aq^*(f)$ and $q^*(f+g) \leqslant q^*(f) + q^*(g)$. Consequently q^* is a quasi-norm on X^*. (Note also that q^* is a norm on X^* whenever q is a norm on X.)

By Proposition 1 e_{q^*} is an extended quasi-pseudo-metric on X^*. It remains to show that e_{q^*} is actually an extended quasi-metric. Let $f, g \in X^*$ be such that $e_{q^*}(f, g) = e_{q^*}(g, f) = 0$. Then there exist two sequences $(f_n)_{n \in \mathbb{N}}$, $(g_n)_{n \in \mathbb{N}}$ in X^* such that $g = f + f_n$, $f = g + g_n$, $q^*(f_n) < 1/n$ and $q^*(g_n) < 1/n$ for all $n \in \mathbb{N}$. So $f_n(x) < 1/n$ and $g_n(x) < 1/n$ whenever $q(x) \leqslant 1$. Choose an arbitrary point $x \in X$. If $q(x) \leqslant 1$ we have $g(x) = f(x) + f_n(x) < f(x) + 1/n$ for all $n \in \mathbb{N}$, and thus $g(x) \leqslant f(x)$. Similarly, we show that $f(x) \leqslant g(x)$. If $q(x) > 1$, the above argument shows that $f(x/q(x)) = g(x/q(x))$, so $f(x) = g(x)$. Therefore $f = g$. The proof is complete. ■

The quasi-normed cone (X^*, q^*) will be called the *positive dual quasi-normed cone* of (X, q), or simply the *dual cone* of (X, q).

4. – Weak topologies and the Alaoglu theorem.

We start this section with the definitions of the weak* topologies which will be used to obtain our version of Alaoglu's theorem.

DEFINITION 2. – We define the *weak* topology* for X^* as the one that has as a basis of neighborhoods of each $f \in X^*$ the subsets of the form $V_{\varepsilon, x_1, \dots, x_n}(f)$, where $n \in \mathbb{N}$, $x_1, \dots, x_n$, are points of X, ε is a positive real number and

$$V_{\varepsilon, x_1, \dots, x_n}(f) = \{ g \in X^* : |g(x_1) - f(x_1)| < \varepsilon, \dots, |g(x_n) - f(x_n)| < \varepsilon \}.$$

Note that, as in the classical case of the dual of a normed linear space, the weak* topology coincides with the topology of pointwise convergence on X^*.

DEFINITION 3. – We define the *weak* positive topology* on X^* as the one that has as a basis of neighborhoods of each $f \in X^*$ the subsets of the form $V^+_{\varepsilon, x_1, \dots, x_n}(f)$, where $n \in \mathbb{N}$, $x_1, \dots, x_n$, are points of X, ε is a positive real number and

$$V^+_{\varepsilon, x_1, \dots, x_n}(f) = \{ g \in X^* : g(x_1) - f(x_1) < \varepsilon, \dots, g(x_n) - f(x_n) < \varepsilon \}.$$

DEFINITION 4. – We define the *weak* negative topology* on X^* as the one that has as a basis of neighborhoods of each $f \in X^*$ the subsets of the form $V^-_{\varepsilon, x_1, \dots, x_n}(f)$, where $n \in \mathbb{N}$, $x_1, \dots, x_n$, are points of X, ε is a positive real number and

$$V^{-}_{\varepsilon, x_1, \dots, x_n}(f) = \{g \in X^*: f(x_1) - g(x_1) < \varepsilon, \dots, f(x_n) - g(x_n) < \varepsilon\}.$$

The weak* topology, the weak* positive topology and the weak* negative topology on X^*, will be denoted by τ_{weak^*}, $\tau_{\text{weak}^*_+}$ and $\tau_{\text{weak}^*_-}$, respectively. The following result is an immediate consequence of the above definitions.

PROPOSITION 4. – *Let (X, q) be a quasi-normed cone. Then $\tau_{\text{weak}^*_+} \vee \tau_{\text{weak}^*_-} = \tau_{\text{weak}^*}$ on X^*.*

PROPOSITION 5. – *Let (X, q) be a quasi-normed cone. Then $\tau_{\text{weak}^*_+}$ (resp. $\tau_{\text{weak}^*_-}$) is the coarsest topology on X^* that makes upper (resp. lower) semicontinuous the functionals $x: X^* \to \mathbb{R}^+$, defined by $x(f) := f(x)$ for all $x \in X$.*

PROOF. – First let us show that every functional defined on X^* by an element $x \in X$, is upper semicontinuous for $\tau_{\text{weak}^*_+}$. Indeed, given $r > 0$, consider the open set $[0, r)$ in $(\mathbb{R}^+, u)$. Let $f \in x^{-1}([0, r))$. Put $\varepsilon = r - f(x)$. It is clear that $x(V^+_{\varepsilon, x}(f)) \subseteq [0, r)$, and hence the functional is upper semicontinuous. On the other hand, let τ be a topology on X^* that makes upper semicontinuous the functional $x(f) := f(x)$. Thus $x^{-1}([0, r)) \in \tau$ for all $r > 0$. Since for each $f \in X^*$, $x \in X$ and $\varepsilon > 0$, we have $f \in x^{-1}([0, f(x) + \varepsilon)) \subseteq V^+_{\varepsilon, x}(f)$, it follows that $\tau_{\text{weak}^*_+}$ is the coarsest topology that makes upper semicontinuous the functionals $x(f) := f(x)$ for all $x \in X$. The parenthetical result is proved similarly. ■

Combining Propositions 4 and 5 we immediately deduce the following.

COROLLARY. – *Let (X, q) be a quasi-normed cone. Then τ_{weak^*} is the coarsest topology on X^* that makes continuous the functionals $x: X^* \to \mathbb{R}^+$, defined by $x(f) := f(x)$ for all $x \in X$.*

Let (X, q) be a quasi-normed cone. Denote by B_{X^*} the unit ball in the quasi-normed cone (X^*, q^*), i.e. $B_{X^*} = \{f \in X^*: q^*(f) \leq 1\}$.

The next result provides an extension of the celebrated Alaoglu theorem to our context. (A generalization of Alaoglu's theorem to asymmetric normed linear spaces was obtained in [5]. Since each asymmetric normed linear space can be considered as a quasi-normed cone, our result generalizes the corresponding result of [5] to the case of linear and upper semicontinuous nonnegative real-valued functions.)

THEOREM 1. – *Let (X, q) be a quasi-normed cone. Then B_{X^*} is compact in $(X^*, \tau_{\text{weak}^*})$.*

PROOF. – Let $x \in X$ and $f \in B_{X^*}$. If $q(x) = 0$, then $f(x) = 0$ by Proposition 2. If $q(x) > 0$, then $f(x/q(x)) \leqslant q^*(f)$ by Proposition 3. Since $q^*(f) \leqslant 1$, we deduce that $f(x) \leqslant q(x)$. Therefore $f(x) \in [0, q(x)]$ for all $x \in X$ and $f \in B_{X^*}$.

Now, consider the product space $H = \Pi_{x \in X}[0, q(x)]$ endowed with the product topology. Identify each function $f \in B_{X^*}$ with its range $(f(x))_{x \in X} \in H$.

Clearly, the restriction of the product topology to the subset of H, $\{(f(x))_{x \in X} : f \in B_{X^*}\}$, coincides with the restriction of τ_{weak^*} to it.

Since, by the Tychonoff theorem, the product space H endowed with the product topology is compact, it will be suffices to prove that $\{(f(x))_{x \in X} : f \in B_{X^*}\}$ is a closed subset of H. To this end, fix $x, y \in X$. Let us define the function $\Psi_{x,y} : H \to \mathbb{R}^+$ by $\Psi_{x,y}(f) = f(x) + f(y) - f(x+y)$ for all $f \in H$.

This function is obviously continuous for the product topology, since its definition only depends on two elements of X.

On the other hand, fix $a \in \mathbb{R}^+$ and $x \in X$. Define the function $\Phi_{a,x} : H \to \mathbb{R}^+$ by $\Phi_{a,x}(f) = af(x) - f(ax)$ for all $f \in H$.

Clearly $\Phi_{a,x}$ is also continuous for the product topology.

Then, the set A defined as

$$A = \Big(\bigcap_{x, y \in X} \Psi_{x,y}^{-1}(\{0\})\Big) \cap \Big(\bigcap_{a \in \mathbb{R}^+,\, x \in X} \Phi_{a,x}^{-1}(\{0\})\Big)$$

is closed in H, since it is the intersection of a family of closed subsets. Moreover, A is clearly the representation of the unit ball B_{X^*} *via* the range $(f(x))_{x \in X}$ of each function f. Therefore B_{X^*} is compact in $(X^*, \tau_{\text{weak}^*})$. ■

5. – Metrizability and quasi-metrizability of the unit ball.

We say that a quasi-normed cone (X, q) is *separable* if $(X, \tau(e_q))$ is a separable topological space.

Let us recall that a topological space (X, τ) is *sub(quasi-)metrizable* provided that there is a (quasi-)metric d on X such that $\tau(d) \subseteq \tau$.

If d is a quasi-metric on a set X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X called the *conjugate* of d (see [3]). Moreover, the function d^s defined on $X \times X$ by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$, is, obviously, a metric on X.

Our next result shows that if (X, q) is a separable quasi-normed cone, then $(X^*, \tau_{\text{weak}^*_+})$ is subquasi-metrizable via a quasi-metric d such that the topology $\tau(d^{-1})$ is weaker than $\tau_{\text{weak}^*_-}$. We also prove that this quasi-metric induces $\tau_{\text{weak}^*_+}$ on B_{X^*} and its conjugate induces $\tau_{\text{weak}^*_-}$ on B_{X^*}, and thus $(B_{X^*}, \tau_{\text{weak}^*})$ is metrizable via the metric d^s.

THEOREM 2. – *Let (X, q) be a separable quasi-normed cone. Then, there is a quasi-metric d on X^* such that*:

(1) $\tau(d) \subseteq \tau_{\text{weak}^*_+}$ *and* $\tau(d^{-1}) \subseteq \tau_{\text{weak}^*_-}$ *on* X^*.

(2) $\tau(d) = \tau_{\text{weak}^*_+}$ *and* $\tau(d^{-1}) = \tau_{\text{weak}^*_-}$ *on* B_{X^*}.

PROOF. – Let $A = \{x_n : n \in \mathbb{N}\}$ be a (countable) subset of X that is dense in (X, q). Define a nonnegative real-valued function d on $X^* \times X^*$ by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n}[\{(g(x_n) - f(x_n)) \vee 0\} \wedge 1].$$

We shall show that d is a quasi-metric on X^* such that $\tau(d)$ is weaker than $\tau_{\text{weak}^*_+}$ and $\tau(d^{-1})$ is weaker than $\tau_{\text{weak}^*_-}$ on X^*.

Clearly $d(f, f) = 0$ for all $f \in X^*$ and d satisfies the triangle inequality. Now let $f, g \in X^*$ such that $d(f, g) = d(g, f) = 0$. Then $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Choose an arbitrary point $x \in X$ and an $\varepsilon > 0$. Since f and g are continuous functions from (X, q) to $(\mathbb{R}^+, u)$ and A is dense in (X, q), there exist $\delta > 0$ and $x_n \in A$ such that $e_q(x, x_n) < \delta$ and $f(x_n) - f(x) < \varepsilon/2$ and $g(x_n) - g(x) < \varepsilon/2$. Let $a_n \in X$ be such that $x_n = x + a_n$ and $q(a_n) < \delta$. Then $f(x_n) = f(x) + f(a_n)$ and $g(x_n) = g(x) + g(a_n)$, so $f(a_n) < \varepsilon/2$ and $g(a_n) < \varepsilon/2$. Since $f(x) + f(a_n) = g(x) + g(a_n)$, we deduce that $f(x) < g(x) + \varepsilon/2$ and $g(x) < f(x) + \varepsilon/2$. Therefore $f(x) = g(x)$. We conclude that $f = g$, and thus d is a quasi-metric on X^*.

Next, suppose that $(f_\lambda)_{\lambda \in \Lambda}$ is a net in X^* that converges to $f \in X^*$ with respect to $\tau_{\text{weak}^*_+}$. Then, for each $x_n \in A$, $f_\lambda(x_n) \to f(x_n)$ in $(\mathbb{R}^+, u)$ which clearly implies that $d(f, f_\lambda) \to 0$. Consequently, the topology $\tau(d)$ is weaker than $\tau_{\text{weak}^*_+}$ on X^*, so $(X^*, \tau_{\text{weak}^*_+})$ is subquasi-metrizable. Similarly, we prove that $\tau(d^{-1})$ is weaker than $\tau_{\text{weak}^*_-}$ on X^*.

Thus, statement (1) is proved. (Note that, in particular, $\tau(d) \subseteq \tau_{\text{weak}^*_+}$ and $\tau(d^{-1}) \subseteq \tau_{\text{weak}^*_-}$ on B_{X^*}.)

In order to prove (2) let $(f_n)_{n \in \mathbb{N}}$ be a sequence in B_{X^*} such that $d(f, f_n) \to 0$, where $f \in B_{X^*}$. We shall show that $(f_n)_{n \in \mathbb{N}}$ clusters to f with respect to $\tau_{\text{weak}^*_+}$.

Indeed, from our version of the Alaoglu theorem, the sequence $(f_n)_{n \in \mathbb{N}}$ clusters to some $g \in B_{X^*}$, with respect to τ_{weak^*}. Since by (1), $\tau(d^s) \subseteq \tau_{\text{weak}^*}$ on X^*, it follows that $(f_n)_{n \in \mathbb{N}}$ also clusters to g with respect to the topology $\tau(d^s)$. Let $(f_{n_m})_{m \in \mathbb{N}}$ be a subsequence of $(f_n)_{n \in \mathbb{N}}$ such that $d^s(g, f_{n_m}) \to 0$. By the triangle inequality $d(f, g) = 0$, and thus $g(x_n) \leq f(x_n)$ for all $n \in \mathbb{N}$.

Now fix $x \in X$ and let $\varepsilon > 0$. Then, there exists $m_0 \in \mathbb{N}$ such that $|f_{n_m}(x) - g(x)| < \varepsilon/2$ for all $m \geq m_0$. On the other hand, since A is dense in (X, q), there exist $\delta > 0$ and $x_k \in A$ such that $e_q(x, x_k) < \delta$, $f(x_k) - f(x) < \varepsilon/2$ and $g(x_k) - g(x) < \varepsilon/2$. Let $a_k \in X$ such that $x_k = x + a_k$ and $q(a_k) < \delta$. Since f and g are linear functions, we deduce that $f(a_k) < \varepsilon/2$ and $g(a_k) < \varepsilon/2$. Then

$$g(x) \leq g(x) + g(a_k) = g(x_k) \leq f(x_k) = f(x) + f(a_k) < f(x) + \varepsilon/2 .$$

Hence

$$f_{n_m}(x) - f(x) < f_{n_m}(x) - g(x) + \varepsilon/2 < \varepsilon ,$$

for all $m \geq m_0$. We conclude that $(f_{n_m})_{m \in \mathbb{N}}$ converges to f with respect to $\tau_{\mathrm{weak}^*_+}$. Consequently $(B_{X^*}, \tau_{\mathrm{weak}^*_+})$ is quasi-metrizable via the quasi-metric d. Similarly we show that $\tau(d^{-1}) = \tau_{\mathrm{weak}^*_-}$ on B_{X^*}. This concludes the proof. ■

From the above result we deduce the following metrization theorem.

THEOREM 3. – *Let (X, q) be a separable quasi-normed cone. Then there is a quasi-metric d on X^* such that $\tau(d^s) \subseteq \tau_{\mathrm{weak}^*}$ on X^*, and $\tau(d^s) = \tau_{\mathrm{weak}^*}$ on B_{X^*}. So $(B_{X^*}, \tau_{\mathrm{weak}^*})$ is metrizable.*

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