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Fibred Closed Braids with Disc-Band Fibre Surfaces.

MARTA RAMPICHINI

Sunto. – *Un risultato classico di Stallings fornisce una condizione necessaria e sufficiente per stabilire se una data superficie immersa senza autointersezioni in S^3 è una fibra per $S^3 - \partial S$. In questo articolo si descrive come trovare una possibile fibra per un link presentato come treccia chiusa. Si descrive anche un algoritmo, implementato al calcolatore, che permette di trovare i principali ingredienti per verificare la condizione necessaria e sufficiente di Stallings, cioè una presentazione del gruppo fondamentale della superficie e del suo complementare in S^3, e una espressione esplicita dell'omomorfismo indotto in omotopia dalla mappa di push-off. L'articolo termina con una discussione di particolari proprietà della presentazione del gruppo $\pi_1(S^3 \setminus S_W)$.*

Summary. – *A classical result by Stallings provides a necessary and sufficient condition to decide whether a given embedded surface S is a fibre in $S^3 \setminus \partial S$. In this paper it is described how to find a candidate fibre surface for a a link presented as a closed braid. Also it is described an implemented algorithm to find the main ingredients of the necessary and sufficient condition of Stallings, namely presentations of the fundamental groups of the surface and of its complement in S^3, and an explicit expression of the homomorphism induced in homotopy by the push-off map. The paper ends with a discussion of the particular properties of the presentation of $\pi_1(S^3 \setminus S_W)$.*

1. – Introduction.

Fibred three-manifolds are very nice topological objects: they can be described as a cartesian product of a surface cross an interval, $S \times [0, 1]$, modulo a gluing map h which identifies the two copies of the surface, $h : S_0 \to S_1$. When a surface S is embedded in S^3 so that the complement of its boundary is a three-manifold fibred over S^1, then S is said to be a *fibre surface*, and its boundary is called a *fibred link*. The unknot is fibred, since its complement in S^3 can be filled by a continuous family of discs, all with the given unknot as boundary. Famous examples of fibred links are the algebraic links [13] and the closure of homogeneous braids [24]. Harer gave instructions for constructing all possible fibred links starting from the unknot, by modifying their fibre surfaces [10].

Montesinos and Morton constructed all fibred links by using branched coverings on closed braids [14].

Gabai gave a necessary and sufficient topological condition for a link to be fibred by looking at a candidate fibre surface [7]. The question of detecting whether a given link is fibred is related to the problem of finding a candidate fibre surface. An algorithm for finding minimal genus fibred surfaces, based on normal surfaces with respect to a triangulation of the three manifold, is given in [25, 16].

In this paper, I am particularly interested in finding fibre surfaces for links presented as closed braids, using the explicit expression of the braid for finding a fibre surface, looking for direct connections between algebraic properties of the braid and topological properties of the surface.

Some necessary conditions for a Seifert surface of a link to be a fibre are summarized in the following:

THEOREM 1 (cf. [23, 6] Prop. 4.1). – *If S is a fibre surface then it is connected and it is a Seifert surface of minimal genus for its boundary. If L is a fibred link, then all its minimal genus Seifert surfaces are fibre surfaces and they are all isotopic.*

There are partial results about the problem of finding minimal genus Seifert surfaces for particular links, see [8, 9, 26], but for general links the problem is still open.

A classical result by Stallings [24] gives a necessary and sufficient algebraic condition for a given surface to be a fibre. It involves the fundamental groups of the surface and of its complement in S^3. It consists in deciding whether the particular homomorphism induced in homotopy by the push-off map is an isomorphism. However, if we want to use this result, we first need to find a candidate fibre surface.

Another way to attack the problem is that of representing links as closed braids, which allows to deal with them as algebraic objects: this is not completely equivalent to the previous setting, since any link can be represented as a closed braid in infinitely many different ways. But if we fix a braid axis for the link, then we will be dealing with a single conjugacy class in a specified braid group [15].

In order to make the paper self-contained, I have recalled known results where useful. In order to make my algorithm more clear, I have considered it along with an example, throughout the paper.

The paper is organized as follows.

In Section 2 some interesting properties of closed braids are collected, and some Seifert surfaces embedded in a particular way are shown. I will define a fibred closed braid and in Proposition 5 I will prove how to find a candidate fibre surface for a closed braid.

In Section 3 I explain an algorithm to find the ingredients of Stallings theorem, namely: presentations for the fundamental groups of the surface and of its complement in S^3, and an explicit expression for the homomorphism. Also I prove that the homomorphism is always injective.

In Section 4 I explain some properties of the presentation of $\pi_1(S^3 \backslash S)$, in order to attack the problem (which in general is not solvable) of determining whether the homomorphism is an isomorphism.

In the Appendix I explain an algorithm for computing the Alexander polynomial for a braid given as a word in disc-band generators.

Throughout the paper, all topological objects under consideration (links, surfaces, braids) are **oriented**.

The algorithms presented in this paper have been implemented using GAP, a programming language for Group Theory (1), with the help of P. Boldi (2) and S. Vigna (2).

2. – Fibred closed braids.

The band presentation of the braid group given by Birman, Ko and Lee in [2] provides a translation of many combinatorial and algebraic properties of braid words and braids into topological properties of *disc-band surfaces*, which are Seifert surfaces for the closed braids, embedded in a particular way.

In this section I will recall the band presentation of B_n, the definition of disc-band surface, and some useful properties.

The *band generators* are a generalisation of the classical Artin generators: for B_n there are $\binom{n}{2}$ generators $a_{j,i}$ with $1 \leqslant j < i \leqslant n$, where $a_{j,i}$ is the braid in which the i^{th} strand crosses over the j^{th} strand, both going over the other intervening strands. The set of band generators includes the set of classical Artin generators, in fact $\sigma_i = a_{i,i+1}$. To simplify the notation, I will write (j, i) for $a_{j,i}$ and $\overline{(j, i)}$ for $a_{j,i}^{-1}$.

There are two types of relations (see Fig. 1):

1. $(j, i)(k, h) = (k, h)(j, i)$ whenever $(i-h)(i-k)(j-h)(j-k) > 0$, the condition meaning that the two pairs of indices are not interleaved, and

2. $(j, i)(k, j) = (k, j)(k, i) = (k, i)(j, i)$ whenever $n \geqslant i > j > k \geqslant 1$.

(1) [GAP 99] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.1; Aachen, St Andrews, 1999. The package is available at http://www-gap.dcs.st-and.ac.uk/ gap/gap.html.

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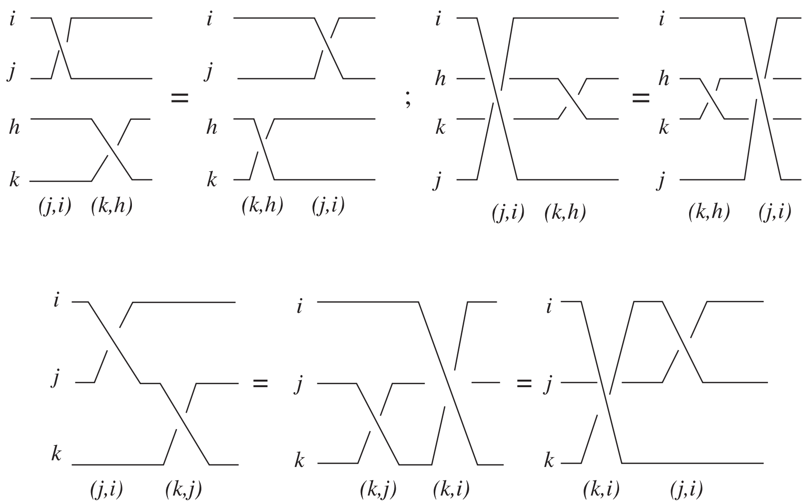

Fig. 1. – Relations of band generators.

To each word in band generators we can associate a Seifert surface for the closed braid embedded in a particularly nice way.

DEFINITION 2. – *Given a word W expressed in the band generators of B_n, we associate to it the disc-band surface S_W, constructed in the following way:*

- *take n discs D_k, each pierced once by the braid axis, all in the same direction*;
- *for any letter $a_{i,j}^{\varepsilon}$ of W, connect the discs D_i and D_j by an half-twisted band, the twist depending on the sign of the letter.*

If β is the braid represented by W, the boundary of S_W is the closure $\widehat{\beta}$, and the sign of each letter corresponds to the sign of the associated crossing.

See Fig. 2 for an example. I will go on with this example throughout the paper.

Disc-band surfaces are also called *braided surfaces* in [19, 20, 21, 22]. Following Rudolph [19] let me represent each half-twisted band as a half curl, so that we always see the positive side of the surface, as in Fig. 3.

There are infinitely many disc-band surfaces associated to a given closed braid: in fact there are closed braids with more than three strands that can be represented by infinitely many different conjugacy classes of braids [4]; moreover any braid can always be represented by infinitely many different words in the band generators. Nevertheless, many algebraic

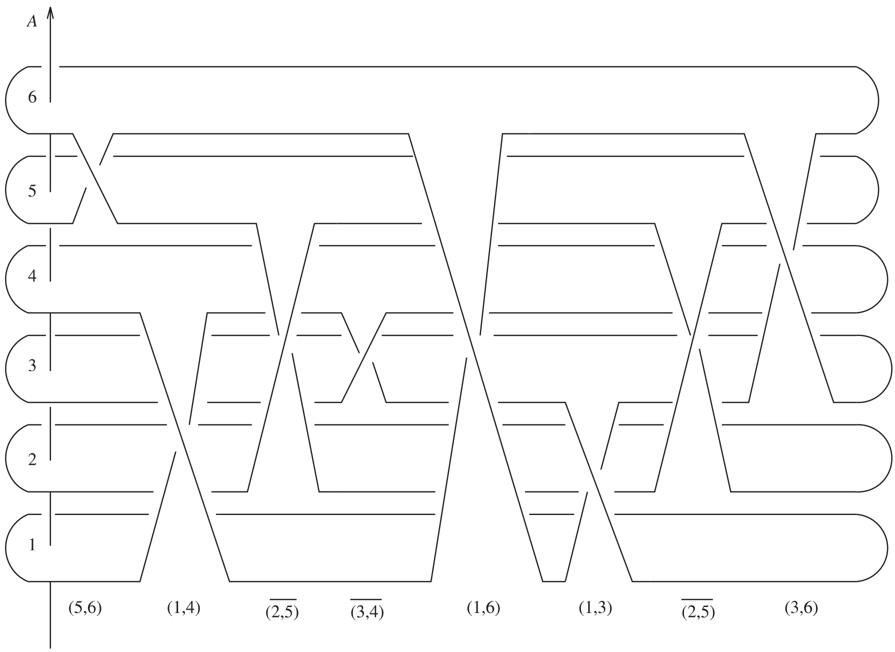

Fig. 2. – The disc-band surface S_W for the braid word $W = (5,6)(1,4)\overline{(2,5)}\overline{(3,4)}(1,6)(1,3)\overline{(2,5)}(3,6)$.

and combinatorial equivalences of braids and braid words can be easily translated into isotopies of these surfaces.

The following algebraic changes in a braid word W correspond to isotopies of the associated disc-band surface S_W in the complement of the braid axis:

1. the disc-band relations; an example of the corresponding isotopy is shown in Fig. 4;

2. conjugations by $\delta = (n-1, n)(n-2, n-1)\dots(1, 2)$, which have the effect of shifting indices by 1 (mod n) on each band generator: $\delta^{-1}(j, i)\delta = (j+1, i+1)$; the corresponding isotopy is a cycling of the discs D_k of S_W along the axis;

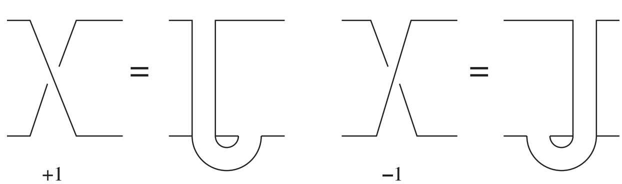

Fig. 3. – How to see the half-twisted bands as half curls.

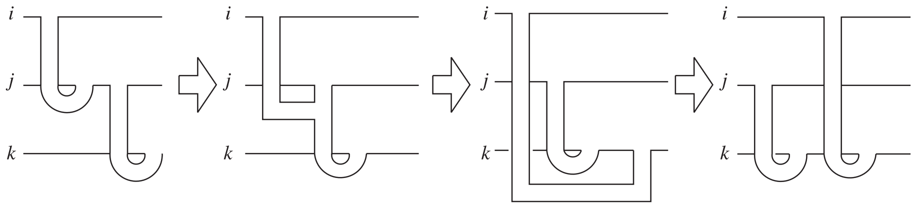

Fig. 4. – How to see the relation $(j, i)(k, j) = (k, j)(k, i)$ as isotopy of disc-band surfaces.

3. conjugations by initial or final letters of W, which correspond to cycling the letters of W; the corresponding isotopy is a cicling of the bands around the discs of S_W.

I call 2 and 3 *easy conjugations.*

It is still open the question whether all isotopies among two disc-band surfaces in the complement of the axis can be expressed by a finite chain of moves 1, 2 and 3 above, cf Rudolph [22], p. 263.

We can also read from a word W an important topological property of the surface S_W. The Euler characteristic of S_W is related to the number n of strands of the braid and the length $l(W)$ of the word in the band generators:

$$\chi(S_W) = n - l(W);$$

this can be seen for instance by retracting S_W on a graph Γ_W as shown in figures 7 and 8.

Notice that all relations and easy conjugations are length preserving. Instead, the free reduction of words is not length preserving, and in fact does not correspond to an isotopy of disc-band surfaces.

In what follows, any link L will be seen as a closed braid $\widehat{\beta}$, with both the axis A and the number $n = lk(L, A)$ of strands fixed. This will allow me to study L as an algebraic object, the conjugacy class $[\beta]$ in B_n. This will not compromise my study, thanks to the following:

THEOREM 3 (Rudolph, Proposition 1 of [20]). – *For any fixed braid axis A (i.e. for any fixed embedded unknot, with the fibration by discs of $S^3 \setminus A$ fixed), any given embedded oriented connected fibre surface S can be isotoped in S^3 so that it lies as a disc-band surface relative to the chosen braid axis A.*

In what follows the term «closed braid» will mean both the isotopy class of a closed braid in $S^3 \setminus A$, or the corresponding conjugacy class in B_n.

DEFINITION 4. – *A closed braid* $\widehat{\beta}$ *is called a* fibred closed braid *if it has a disc-band surface which is a fibre.*

PROPOSITION 5. – *A closed braid* $\widehat{\beta}$ *is fibred if and only if for each word W of minimal length in the conjugacy class of* β *the surface* S_W *is a fibre surface.*

PROOF. – If $\widehat{\beta}$ is fibred, then all its fibres are isotopic and have minimal genus, and by definition there exists at least one which is a disc-band surface. Since the genus of a disc-band surface is minimal (among disc-band surfaces) when the length of the word is minimal in the conjugacy class, there is at least one of these minimal-length words such that S_W is a fibre. But for all such words the corresponding surfaces are isotopic, since they have the same genus and isotopic boundary; therefore they all are fibre surfaces. ■

Notice that any fibred link L can be seen as a fibred closed braid $\widehat{\beta}$, with $\beta \in B_n$, but usually it is not known for which n this is possible. The theorem of Bennequin studied in [3] guarantees that all braids with at most three strands have minimal genus disc-band Seifert surfaces, but this is no longer true in B_n with $n \geqslant 4$: Ko and Lee in [11] give an example by Morton of a 4-braid whose closure is the unknot (therefore it is a fibred link), which has not a disc-band surface of minimal genus, because the minimal length of a word in its conjugacy class is 7, while it should be 3. Anyway in B_4 there are conjugacy classes representing the unknot and with minimal length 3.

So, if we are given a braid and we want to find a candidate fibre surface we have to look for a minimal-length word in its conjugacy class, whose disc-band surface is connected.

The general problem of finding a minimal-length word in a given conjugacy class of B_n is not solved, but for fibred closed braids we can use the Alexander polynomial of the closed braid to know what the minimal length should be.

PROPOSITION 6. – *The following are necessary conditions for a closed braid* $\widehat{\beta}$ *to be fibred:*

1. *the minimal length* l *for a word in its conjugacy class is such that*

$$l \geqslant \deg \Delta_{\widehat{\beta}}(t) + n - 1, \quad \text{if } \widehat{\beta} \text{ has one component}, \quad \text{and}$$

$$l \geqslant \deg \Delta_{\widehat{\beta}}(t) + n, \quad \text{if } \widehat{\beta} \text{ has more than one component};$$

2. *in the Alexander polynomial* $\Delta_{\widehat{\beta}}(t)$ *the coefficients of the extreme powers of* t *are* ± 1.

PROOF. – This is a corollary of the following:

THEOREM 7 (cf. [17, 18]). – *A necessary condition for a link L to be fibred is that the Alexander polynomial $\Delta_L(t)$ satisfies the following conditions:*

1. *if S is a fibre, then*

$$\deg \Delta_L(t) \leq 1 - \chi(S), \quad \text{if } L \text{ has one component};$$

$$\deg \Delta_L(t) \leq -\chi(S), \quad \text{if } L \text{ has more than one component};$$

2. *in the Alexander polynomial $\Delta_L(t)$ the coefficients of the extreme powers of t are* ± 1. ■

In the Appendix I explain an algorithm for computing the Alexander polynomial of a braid expressed by a word in band generators.

Once we have computed the Alexander polynomial, we can easily check the required property on the coefficients of extreme powers of t. If this is satisfied, we can look for a word with prescribed length and with the same exponent sum as β; then we can check conjugacy *e.g.* by the algorithm of [2]. The set of words of B_n with prescribed length and exponent sum is finite, so either we end with a negative answer, or we find a word W giving a candidate fibre surface S_W, to which we will apply the Stallings theorem.

3. – The algorithm.

The aim of this section is to explain how to find in our settings the main ingredients of Stallings' necessary and sufficient condition for a given surface to be a fibre. We first need the following:

DEFINITION 8. – *Given a connected oriented surface S embedded in S^3, the* push-off map $i^+ : S \to S^3 \setminus S$ *is the continuous map (defined up to isotopy in $S^3 \setminus \partial S$) which sends S in a homeomorphic copy S^+, which lies in $S^3 \setminus S$, obtained from S by pushing each point off from S along the positive normal direction.*

THEOREM 9 (Stallings, [24]). – *A given connected surface S embedded in S^3 is a fibre in $S^3 \setminus \partial S$ if and only if the homomorphism*

$$i_*^+ : \pi_1(S) \to \pi_1(S^3 \setminus S)$$

induced in homotopy by the push-off map is an isomorphism.

- The fundamental group $\pi_1(S_W)$ is a free group on $1 - \chi(S_W)$ generators, but we need an explicit expression of such generators in topological terms to

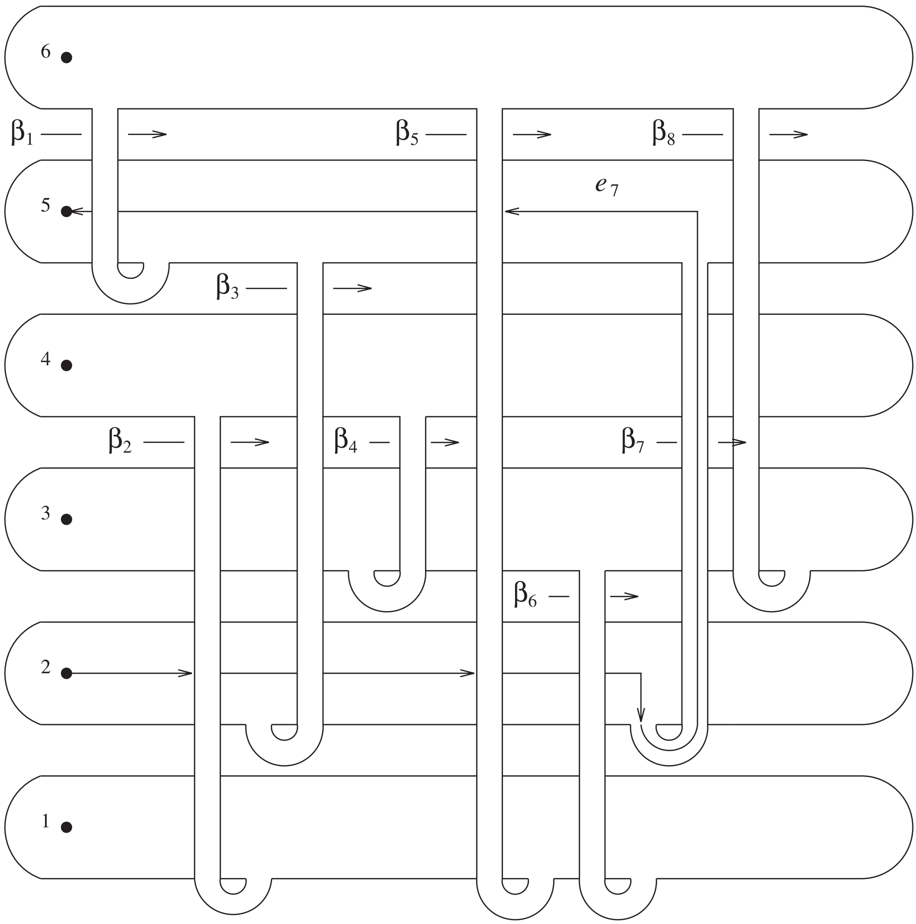

Fig. 5. – The surface S_W of Fig. 2, flattened, with the vertices of the graph Γ_W embedded as intersections of S_W with the braid axis. The edge e_7 is shown as an example. Each generator β_j of $\pi_1(S^3\backslash S_W)$ is a loop from your eye to the arrow and back to your eye.

know how the homomorphism i_*^+ will act on them. We can read them explicitly from W: look at Fig. 5, where the surface of Fig. 2 has been flattened as explained in Fig. 3.

To find an explicit expression of the generators, we can consider a graph Γ_W which is a strong deformation retract of S_W, look at a spanning tree T_W of Γ_W and then read a loop α_k for each edge in $\Gamma_W\backslash T_W$. All this information can be read directly from W. Choose a base point on D_1 in a small neighbourhood of the point where the braid axis intersects the disc. Suppose $W=\prod_{j=1}^{l(W)} w_j^{\varepsilon_j}$,

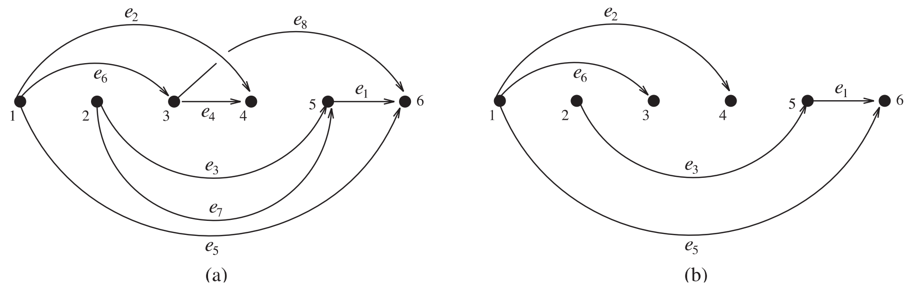

Fig. 6. – The graph Γ_W (a) and a spanning tree T_W for it (b).

where each w_j is a band generator and $\varepsilon_j = \pm 1$. The graph Γ_W has one vertex v_j on each disc D_j of S_W, that is one vertex for each index $j \in \{1, 2, \dots n\}$ (they are marked by «big dots» in Fig. 5). The edges of Γ_W are one for each band of S_W, that is one edge e_i for each letter w_i. If $w_i = (h, k)$, then e_i is a simple arc which joins v_h to v_k going along D_h, then along the i^{th} band then along D_k (in Fig. 5, the edge e_7 is shown as an example). We can see a schematic picture of Γ_W in Fig. 6 (a) and a spanning tree in Fig. 6 (b). Our GAP procedures BuildGraph(W, n) and SpTree(G) give automatically the graph and the spanning tree from the braid word W.

It is convenient for the implementation of the algorithm to write the $1 - \chi(S_W)$ generators α_h's as words in the letters e_i's, with the convention that each edge is directed from the smallest to the greatest of its two vertices. This is done by our GAP procedure ReadGenerators(G, T), which for our example gives

$$\alpha_1 = e_6 e_4 e_2^{-1},$$
$$\alpha_2 = e_5 e_1^{-1} e_3^{-1} e_7 e_1 e_5^{-1},$$
$$\alpha_3 = e_6 e_8 e_5^{-1}.$$

- We can apply the well known Wirtinger algorithm to find a presentation of the *fundamental group* $\pi_1(S^3 \setminus S_W)$. Look at Fig. 5: choose one generator β_i for each band of S_W, i.e. for each letter w_i of W; each generator is a loop surrounding exactly one band, as indicated schematically by the arrow in Fig. 5. To read the relations, the surface can be retracted as in Fig. 7 and 8.

In order to write the relations, for each generator β_k I define two conjugates, γ_k and δ_k, which can be seen in the corner of Fig. 8: γ_k (respectively δ_k) is the loop on the same strand as β_k, but as near as possible to the initial (respectively final) vertex of the simple arc e_k. Their expression as conjugates of

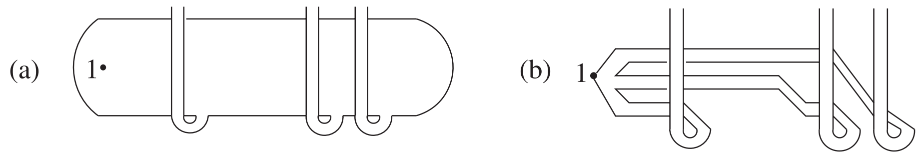

Fig. 7. – (a) The disc D_1 of Fig. 5 and (b) its retract.

the β_k's can be obtained from W as follows: suppose $w_k = (i, j)$ and let $w_r = (p, q)$ be another generic letter. Then

$$\gamma_k = P_k \beta_k P_k^{-1} \text{ and } \delta_k = Q_k \beta_k Q_k^{-1},$$

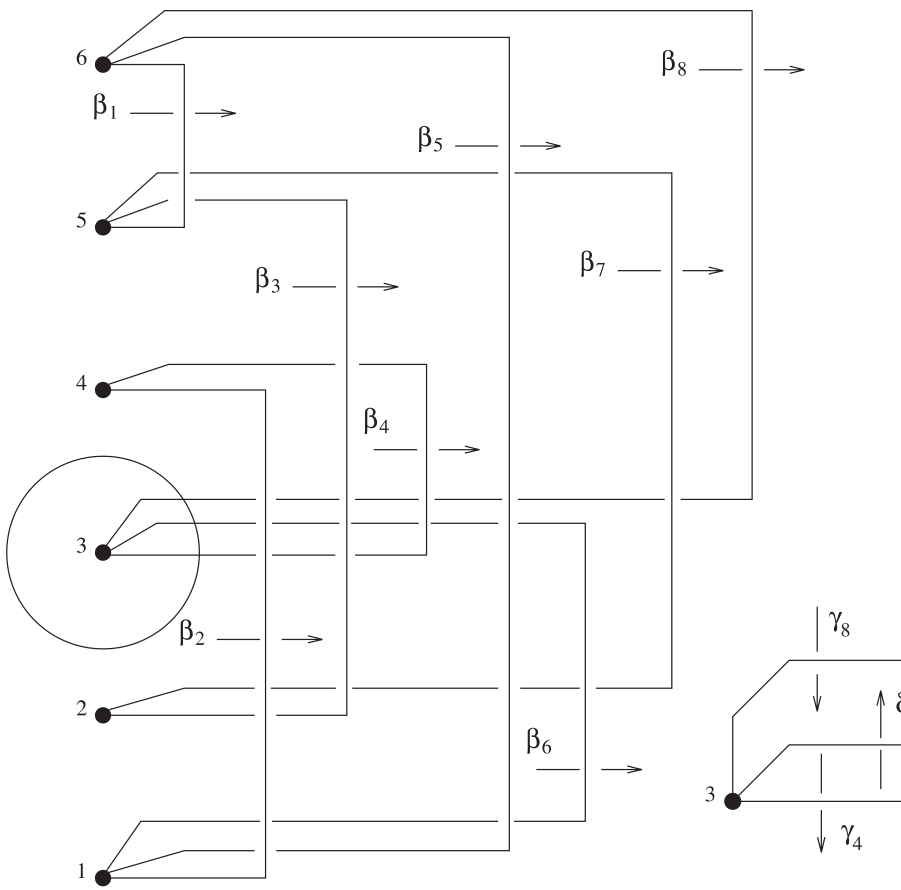

Fig. 8. – The graph Γ_W as deformation retract of the surface S_W, with arrows indicating the generators of $\pi_1(S^3 \setminus S_W)$. In a corner, a zoom of a neighbourhood of a vertex.

where

$$P_k = \prod_{r=1}^{k-1} \beta_r^{s_r} \qquad \text{and} \qquad Q_k = \prod_{r=1}^{k-1} \beta_r^{t_r},$$

and

$$s_r = \begin{cases} 1 & \text{if } p \leqslant i < q \\ 0 & \text{otherwise}, \end{cases} \qquad t_r = \begin{cases} 1 & \text{if } p \leqslant j < q \\ 0 & \text{otherwise}. \end{cases}$$

For instance, from Fig. 8 it is easy to read $\gamma_4 = \beta_2\beta_3\beta_4\beta_3^{-1}\beta_2^{-1}$.

The conjugates γ_k and δ_k are useful in order to read relations of the group $\pi_1(S^3 \setminus S_W)$; look at a neighbourhood of each vertex (as at the corner of Fig. 8): we get

$$r_m = \prod_{k=1}^{l} \gamma_k^{x_k} \delta_k^{y_k},$$

where

$$x_k = \begin{cases} -1 & \text{if } i = m \\ 0 & \text{otherwise}, \end{cases} \qquad y_k = \begin{cases} 1 & \text{if } j = m \\ 0 & \text{otherwise}. \end{cases}$$

For instance, from Fig. 8 it is easy to read $r_3 = \gamma_4^{-1}\delta_6\gamma_8^{-1}$.

Our GAP procedure CompleteHomotopyRelations(F, W, n) (here F is the free group on $l(W)$ generators) gives

$$\begin{aligned}
r_1 &= \beta_6^{-1}\beta_5^{-1}\beta_2^{-1}, \\
r_2 &= \beta_2\beta_5\beta_6\beta_7^{-1}\beta_6^{-1}\beta_5^{-1}\beta_3^{-1}\beta_2^{-1}, \\
r_3 &= \beta_2\beta_3\beta_5\beta_6\beta_7\beta_8^{-1}\beta_7^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_2^{-1}, \\
r_4 &= \beta_2\beta_3\beta_4\beta_3^{-1}, \\
r_5 &= \beta_3\beta_5\beta_7\beta_5^{-1}\beta_1^{-1}, \\
r_6 &= \beta_1\beta_5\beta_8.
\end{aligned}$$

There exists a GAP procedure which simplifies presentations (using Tietze's moves); in our example, it immediately gives a free group on three generators. We get $\pi_1(S^3 \setminus S_W) = \langle \beta_2, \beta_3, \beta_5 \,|\, - \rangle$, with

$$\begin{aligned}
\beta_1 &= \beta_3\beta_5\beta_2\beta_3^{-1}\beta_2^{-1}\beta_5^{-1} \\
\beta_4 &= \beta_3^{-1}\beta_2^{-1}\beta_3 \\
\beta_6 &= \beta_5^{-1}\beta_2^{-1} \\
\beta_7 &= \beta_2\beta_3^{-1}\beta_2^{-1} \\
\beta_8 &= \beta_2\beta_3\beta_2^{-1}\beta_5^{-1}\beta_3^{-1}.
\end{aligned}$$

• *The homomorphism* i_*^+. For each arc e_k one can see in Fig. 5 how to read its image $i^+(e_k)$ as a product of loops β_j's: it is sufficient to read one β_j each time e_k passes under the corresponding band in the same direction as β_j, and the inverse when it passes in the opposite direction. All this information can be read directly from W as follows:

$$i^+(e_k) = P_k \beta_k^{\varepsilon} Q_k^{-1},$$

where $\varepsilon = \dfrac{\varepsilon_k + 1}{2}$ and P_k, Q_k have been defined above. Our GAP procedure IPlusEks(W) computes the list of images of the e_k's. In our example we get

$$\begin{aligned}
i^+(e_1) &= \beta_1\\
i^+(e_2) &= \beta_2\\
i^+(e_3) &= \beta_2\beta_1^{-1}\\
i^+(e_4) &= \beta_2\beta_3\beta_3^{-1} = \beta_2\\
i^+(e_5) &= \beta_2\beta_5\\
i^+(e_6) &= \beta_2\beta_5\beta_6\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_2^{-1}\\
i^+(e_7) &= \beta_2\beta_3\beta_5\beta_6\beta_5^{-1}\beta_1^{-1}\\
i^+(e_8) &= \beta_2\beta_3\beta_4\beta_5\beta_7\beta_8.
\end{aligned}$$

Now we can use the expression of the generators α_j as products of e_k's and substitute their images $i^+(e_k)$ to get the images $i_*^+(\alpha_j)$. Our GAP procedure IPlus(W, n, H) gives the list of images $i_*^+(\alpha_j)$, using the word W, the braid index n and the fundamental group $H = \pi_1(S^3 \setminus S_W)$ as found by our GAP procedure HomotopyGroup(W, n). In our example, we get:

$$\begin{aligned}
i_*^+(\alpha_1) &= \beta_2\beta_5\beta_6\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_2^{-1}\\
i_*^+(\alpha_2) &= \beta_2\beta_5\beta_3\beta_5\beta_6\beta_5^{-2}\beta_2^{-1}\beta_5^{-2}\beta_2^{-1}\\
i_*^+(\alpha_3) &= \beta_2\beta_5\beta_6\beta_7\beta_8\beta_5^{-1}\beta_2^{-1}.
\end{aligned}$$

In this case, by substituting the expressions found above we get

$$\begin{aligned}
i_*^+(\alpha_1) &= \beta_5^{-1}\beta_3^{-1}\\
i_*^+(\alpha_2) &= \beta_2\beta_5\beta_3\beta_2^{-1}\beta_5^{-2}\beta_2^{-1}\beta_5^{-2}\beta_2^{-1}\\
i_*^+(\alpha_3) &= \beta_5^{-1}\beta_3^{-1}\beta_5^{-1}\beta_2^{-1},
\end{aligned}$$

from which we can easily get

$$\beta_1 = i_*^+ (\alpha_3^{-1} \alpha_1 \alpha_3^{-1} \alpha_1 \alpha_3 \alpha_2 \alpha_3^{-1} \alpha_1)$$
$$\beta_2 = i_*^+ (\alpha_1^{-1} \alpha_3^{-1} \alpha_1 \alpha_3 \alpha_2 \alpha_3^{-1} \alpha_1)$$
$$\beta_3 = i_*^+ (\alpha_1^{-1} \alpha_3 \alpha_2^{-1} \alpha_3^{-1} \alpha_1^{-1} \alpha_3).$$

To summarise, the algorithm presented in this section allows to find explicitly all the ingredients to apply Stallings' theorem, which are:

1) an expression of the generators of the free group $\pi_1(S_W)$ in terms of loops on S_W;

2) a presentation of the group $\pi_1(S^3 \setminus S_W)$;

3) an explicit expression of the homomorphism i_*^+.

One more step can be done:

PROPOSITION 10. – *If S_W is a connected surface of minimal genus for ∂S_W, then $i_*^+ : \pi_1(S_W) \to \pi_1(S^3 \setminus S_W)$ is injective.*

PROOF. – Since S_W is of minimal genus, it is incompressible; this means what follows: for any simple closed curve $\gamma \subset S_W$, boundary of a disc $D \subset S^3$, with $D \cap S_W = \gamma$, there is a disc $D' \subset S_W$ with $\partial D' = \gamma$. Since $i_*^+([\gamma]) = 1$ in $\pi_1(S^3 \setminus S_W)$ if and only if $i^+(\gamma)$ bounds a disc D embedded in $S^3 \setminus S_W$, one can attach to this disc the annulus $\boldsymbol{n} \times \gamma$ (where $\boldsymbol{n}$ is the normal positive vector used to define i^+): in this way one gets a disc in S^3 bounded by γ, therefore γ is nullhomotopic in S_W. ■

4. – Combinatorial properties of the presentation of $\pi_1(S^3 \setminus S_W)$.

To decide whether a braid is fibred or not, once we have found all ingredients of Stallings' theorem, we need to decide whether the homomorphism i_*^+ is surjective or not. This is an open problem of combinatorial group theory, which is known to be solvable in some large families of groups: the free groups for instance, and the groups satisfying the so called *small cancellation conditions.* A good reference is [12].

To give an idea about the small cancellation conditions we need some terminology of combinatorial group theory (cf. [12]). Suppose we are given a presentation $\langle X | R \rangle$ for a group G, where X is a set of generators and R a set of relations. Suppose $w = x_1 x_2 \dots x_k$ is a word in the generators (and their inverses): say that w is *cyclically reduced* if $x_k \neq x_1^{-1}$. Say that R is *symmetrized* if all elements of R are cyclically reduced and, for each $r \in R$, all cyclically reduced conjugates of r and of r^{-1} belong to R. Now suppose that there are two elements r_1, r_2 of R such that there are words b, c_1, c_2 such that $r_1 = bc_1$, $r_2 =$

bc_2. Then b is called a *piece relative to R.* The small cancellation conditions assert that pieces are relatively small. The most usual conditions are the following:

– *Condition* $C'(\lambda)$: for all $r \in R$, if $r = bc$ and b is a piece, then $|b| < \lambda |r|$, with $\lambda \in \boldsymbol{R}$, $\lambda > 0$.

– *Condition* $C(p)$: for all $r \in R$, r is the product of at least p pieces.

REMARK. – In both groups with free presentation and groups with presentations satisfying these small cancellation conditions for appropriate λ, p, the word problem is solvable.

In what follows I explain some combinatorial properties of the presentation of $\pi_1(S^3 \backslash S)$ found in the preceding section.

PROPOSITION 11. – *For a given word $W \in B_n$, of minimal length $l = l(W)$ in its conjugacy class, the presentation for $\pi_1(S^3 \backslash S_W)$ has l generators and n relations; each relation is the product of some conjugates γ_j, δ_j of the generators, with the following properties:*

1) *in each $\gamma_j = P_j \beta_j P_j^{-1}$, the generator β_j occurs only once; and a generator β_k can occur, at most once in P_j, only if $k < j$; similarly for $\delta_j = Q_j \beta_j Q_j^{-1}$;*

2) *if $w_j = (p, q)$, then the length of γ_j equals the number of letters w_h with $h < j$ such that $w_h = (r, s)$, $r \leqslant p < s$; similarly, the length of δ_j equals the number of letters w_h with $h < j$ such that $w_h = (r, s)$, $r \leqslant q < s$;*

3) *considering the set of n relations, each γ_j and each δ_j occur in one relation, and they do not occur in the same relation;*

4) *the number of γ_j's and δ_j's occurring in r_k equals the number of letters w_j in which the index k occurs.*

Summarising we get for all $j = 1, 2, \dots l$ and all $k = 1, 2, \dots n$

$$|\gamma_j| \leqslant 2j + 1$$
$$|\delta_j| \leqslant 2j + 1$$
$$|r_k| < l(2l + 1).$$

All these properties are clear from the explicit construction given in the preceding section.

Moreover, suppose there is one index $1 \leqslant i \leqslant n$ which only occurs in one letter of W. Then we can conjugate W by its initial letter until the letter with the index i becomes the first. Thus $r_i = \beta_1$, which means $\beta_1 = 1$, so it can be cancelled. This corresponds to the result of a Markov move on the surface.

Therefore we can always suppose that each index $1 \leqslant i \leqslant n$ occurs at least twice in W; hence for all relations we get

$$|r_i| \geqslant 2.$$

Concerning the number of generators, we can say the following:

PROPOSITION 12. – *If W is a word of minimal length in the conjugacy class of β*

1) *if $|W| = n-1$ and $\widehat{\beta}$ has one component, then $\widehat{\beta}$ is a fibred braid*;

2) *if $|W| = n-1$ and $\widehat{\beta}$ has more than one component, or if $|W| < n-1$, then $\widehat{\beta}$ is neither a fibred braid nor a fibred link.*

PROOF. – In the first case $\chi(S_W) = 1$, therefore S_W is a disc, which is trivially a fibre surface. In the second case S_W is a minimal genus Seifert surface which is not connected (for a surface $S = S_1 \coprod S_2$, $\chi(S) = \chi(S_1) + \chi(S_2)$), therefore the boundary link cannot be fibred. ■

COROLLARY 13. – *In the presentation of $\pi_1(S^3 \backslash S_W)$ obtained as above, the number of generators is greater than or equal to the number of relations.*

PROOF. – We can always suppose $|W| \geqslant n$. ■

An interesting example. For the braid

$$W = (1,4)(2,6)(1,3)(2,4)(3,5)(1,4)(2,6)(1,3)(2,4)(3,5)$$

the Alexander polynomial

$$\Delta_{\widehat{\beta}}(t) = -1 + t^2 - t^4$$

satisfies the necessary conditions requested by Theorem 7. After some Tietze transformations, we get the presentation

$$\pi_1(S^3 \backslash S_W) = \langle \beta_1, \beta_2, \ldots, \beta_6 \,|\, r \rangle,$$

where the relation is

$$r = \beta_2 \beta_6^{-1} \beta_5^{-1} \beta_4^{-1} \beta_2^{-1} \beta_1^{-1} \beta_2 \beta_5 \beta_2^{-1} \beta_1 \beta_2 \beta_3 \beta_4 \beta_6 \beta_2^{-1} \beta_6^{-1} \beta_3^{-1} \beta_1^{-1},$$

with length $|r| = 18$, and pieces of maximal length $= 2$ ($\beta_1 \beta_2$, $\beta_1^{-1} \beta_2$ and $\beta_2 \beta_6^{-1}$, which occur in r and in r^{-1}), therefore it satisfies the small cancellation conditions $C'(1/8)$ and $C(12)$.
Open question Do these properties of $\pi_1(S^3 \backslash S_W)$ lead in general to some

small cancellation conditions, so that one can always solve the problem of deciding whether i_*^+ is surjective?

5. – Appendix: The Alexander polynomial.

The Alexander polynomial is a well-known invariant for knots and links. Definitions and results about it can be found e.g. in [1, 18]. We are interested in the reduced Alexander polynomial for a closed braid, that is the polynomial in which the same variable t has been associated to all components of the link.

Our algorithm, implemented in GAP, is adapted to links presented as closed braids written in band generators. Since the classical Artin generators form a subset of the band generators, it can be used also for braids written in the classical way.

The algorithm is based on the method of Seifert matrices:

DEFINITION 14. – *If S is a connected Seifert surface for a link L, define a Seifert matrix M as follows: let $a_1, a_2, \dots a_N$ be free abelian generators of $H_1(S)$ (here $N = 1 - \chi(S)$); then the elements of M are*

$$M_{ij} = lk(a_i, i_*^+(a_j)),$$

where lk denotes the linking number, i^+ is the push-off map, and the $i_*^+(a_j)$'s are elements of $H_1(S^3 \setminus S)$.

PROPOSITION 15. – *If M is any Seifert matrix for a given link L, then the Alexander polynomial is given by:*

$$\Delta_L(t) = \det(tM - M^T) \ \text{ if } L \text{ has one component};$$

$$\Delta_L(t) = \frac{\det(tM - M^T)}{1-t} \ \text{ if } L \text{ has more than one component}.$$

In order to find a Seifert matrix for a braid β given by a word W, we need to find a connected Seifert surface for it. The fact that the associated disc-band surface S_W is connected corresponds to the fact that the associated graph Γ_W is connected. If it is not connected, it is easy to find a connected one, by addying the necessary edges. This can be done on W by conjugating it by the letters corresponding to the added edges. Our GAP procedure ConnectBraid(W, n) gives a new word W' with connected $S_{W'}$ in case S_W was not connected.

We already know how to find generators of $\pi_1(S_W)$ in terms of edges e_k of Γ_W (see Section 3). The same loops α_j's (or their abelianisations a_j's) can be used as generators of $H_1(S_W)$. We also already know how to find their images

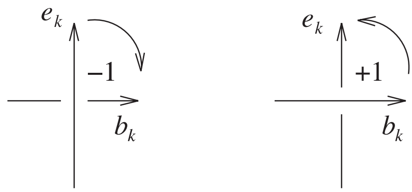

Fig. 9. – How to compute $lk(a_i, i_*^+(a_j))$.

$i_*^+(a_j)$ in $H_1(S^3 \backslash S)$, for instance by abelianising the images $i_*^+(\alpha_j)$ found in Section 3.

It is now very easy to read the elements M_{ij} (look at Figure 9):

$$lk(a_i, i^+(a_j)) = \sum_{k=1}^{l} -(\text{power of } e_k \text{ in } a_i)(\textit{power of } b_k \text{ in } i_*^+(a_j)).$$

This is computed by our GAP procedure SeifertMatrix (W, n).

The number of components of a closed braid can be easily read from W:

$$|\widehat{\beta}| = \text{ number of disjoint cycles in the induced permutation } \varrho(W) \in S_n$$

(where S_n is the symmetric group on n elements). This is an invariant of the closed braid. Our GAP procedure IsnCycle (p, n) finds «true» if p is a permutation of just one cycle on n points.

All the computations are performed in one step by our GAP procedure Alexander (W, n).

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