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On the Measurability of the Conic Sections' Family in the Projective Space $\mathcal{P}_n$.

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Sunto. – *In questo lavoro proviamo che la famiglia delle sezioni coniche non degeneri nello spazio proiettivo $\mathcal{P}_n$ è misurabile.*

Summary. – *In this paper we prove that the non degenerate conic sections' family in the projective space $\mathcal{P}_n$ is measurable.*

In the previous papers [2], p. 919, and [1], p. 537, it is proved the measurability [1] of the families of non degenerate conic sections in the projective plane $\mathcal{P}_2$ and in the projective space $\mathcal{P}_3$.

In this work we study the measurability of the non degenerate conic sections' family in the projective space $\mathcal{P}_n$, $(n > 3)$.

THEOREM. – *The non degenerate conic sections' family of the projective space $\mathcal{P}_n$ is measurable.*

PROOF. – Let $\mathcal{P}_n$ be the n-dimensional projective space, whose non homogeneous projective coordinates we denote by $X_1, X_2, \ldots, X_n$. A non-degenerate conic section in $\mathcal{P}_n$ can be viewed as the intersection between a non-degenerate quadratic cone an a hyperplane which does not contain its vertex. Let $x_1, x_2, \ldots, x_n$ be the non-homogeneous projective coordinates of the cone vertex and

$$\sum_{i=1}^{n-1} a_{ii} X_i^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij} X_i X_j + 2 \sum_{i=1}^{n-1} a_{in} X_i + 1 = 0$$

be the equation of the cone directrix in the $X_n = 0$ hyperplane. Then the cone

(1) For the notion of measurability of a family of varieties and for all the other notions of integral geometry used in this work, see ([3]).

equation is:

$$\sum_{i=1}^{n-1} a_{ii}\left(x_i - x_n \frac{X_i - x_i}{X_n - x_n}\right)^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij}\left(x_i - x_n \frac{X_i - x_i}{X_n - x_n}\right)\left(x_j - x_n \frac{X_j - x_j}{X_n - x_n}\right) + 2 \sum_{i=1}^{n-1} a_{in}\left(x_i - x_n \frac{X_i - x_i}{X_n - x_n}\right) + 1 = 0\,,$$

that is

$$\text{(1)} \quad \sum_{i=1}^{n-1} a_{ii} X_i^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij} X_i X_j - \frac{2}{x_n} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} a_{ij} x_j + a_{in}\right) X_i X_n + \frac{1}{x_n^2}\left(\sum_{i=1}^{n-1} a_{ii} x_i^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij} x_i x_j + 2 \sum_{i=1}^{n-1} a_{in} x_i + 1\right) X_n^2 + 2 \sum_{i=1}^{n-1} a_{in} X_i - \frac{2}{x_n}\left(\sum_{i=1}^{n-1} a_{in} x_i + 1\right) X_n + 1 = 0, \qquad (x_n \neq 0),$$

with the condition

$$\text{(1')} \qquad \Delta = \det \|a_{hk}\| \neq 0, \qquad (h,\, k = 1, \dots, n;\; a_{hk} = a_{kh},\; a_{nn} = 1).$$

The equation of a hyperplane not passing through the cone vertex is:

$$\text{(2)} \qquad \sum_{h=1}^{n} A_h X_h + 1 = 0\,,$$

with the condition:

$$\text{(2')} \qquad \sum_{h=1}^{n} A_h x_h + 1 \neq 0\,.$$

Hence the non degenerate conic sections' family of the projective space $\mathcal{P}_n$ is defined by equations (1) and (2) together with conditions (1') and (2'). We denote this family by $\mathcal{F}_{\frac{n^2+5n-2}{2}}$ since it is characterized by the $\frac{n^2+5n-2}{2}$ parameters: $x_1, \dots, x_n, A_1, \dots, A_n, a_{11}, a_{12}, \dots, a_{1n}, a_{22}, a_{23}, \dots, a_{2n}, \dots, a_{n-1,\,n}$.

The maximum invariance group of the family is the projective group $G_{n(n+2)}$, of equations:

$$\text{(3)} \qquad X_i = \frac{\sum_{j=1}^{n} \alpha_{ij} X_j' + \alpha_i}{\sum_{j=1}^{n} \alpha_{n+1,\,j} X_j' + 1}\,, \qquad (i = 1, \dots, n)\,,$$

with:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \alpha_1 \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} & \alpha_n \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,n} & 1 \end{vmatrix} \neq 0 .$$

Under the action of the group $G_{n(n+2)}$, the family defined by (1) and (2) transforms into:

$$\sum_{i=1}^{n-1} a'_{ii} X_i'^2 + 2 \sum_{i<j}^{1,\, n-1} a'_{ij} X'_i X'_j - \frac{2}{x'_n} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} a'_{ij} x'_j + a'_{in} \right) X'_i X'_n +$$
$$\frac{1}{x_n'^2} \left(\sum_{i=1}^{n-1} a'_{ii} x_i'^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij} x_i x_j + 2 \sum_{i=1}^{n-1} a'_{in} x'_i + 1 \right) X_n'^2 +$$
$$2 \sum_{i=1}^{n-1} a'_{in} X'_i - \frac{2}{x'_n} \left(\sum_{i=1}^{n-1} a'_{in} x'_i + 1 \right) X'_n + 1 = 0 ,$$

$$\sum_{h=1}^{n} A'_h X'_h + 1 = 0 ,$$

where

$$x_h = \frac{\sum_{k=1}^{n} \alpha_{hk} x'_k + \alpha_h}{\sum_{k=1}^{n} \alpha_{n+1,\, k} x'_k + 1} , \qquad (h = 1, \ldots, n), \tag{4}$$

$$A'_h = \frac{\sum_{k=1}^{n} \alpha_{hk} A_k + \alpha_{n+1,\, h}}{\sum_{k=1}^{n} \alpha_k A_k + 1} , \qquad (h = 1, \ldots, n) \tag{5}$$

$$\begin{aligned} a'_{uu} = \frac{1}{d} \Bigg[& \sum_{i=1}^{n-1} \alpha_{iu}^2 a_{ii} + 2 \sum_{i<j}^{1,\, n-1} \alpha_{iu} \alpha_{ju} a_{ij} - \frac{2}{x_n} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} a_{ij} x_j + a_{in} \right) \alpha_{iu} \alpha_{nu} \\ & + \frac{1}{x_n^2} \left(\sum_{i=1}^{n-1} a_{ii} x_i^2 + 2 \sum_{i<j}^{1,\, n-1} a_{ij} x_i x_j + 2 \sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \alpha_{nu}^2 \\ & + 2 \left(\sum_{i=1}^{n-1} \alpha_{iu} a_{in} \right) \alpha_{n+1,\, u} - \frac{2}{x_n} \left(\sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \alpha_{nu} \alpha_{n+1,\, u} + \alpha_{n+1,\, u}^2 \Bigg] \end{aligned} \tag{6'}$$

$$(u = 1, \ldots, n-1)$$

$$(7)\qquad a'_{vw} = \frac{1}{d}\Bigg[\sum_{i=1}^{n-1} \alpha_{iv}\alpha_{iw} a_{ii} + \sum_{i<j}^{1,\,n-1} (\alpha_{iv}\alpha_{jw} + \alpha_{iw}\alpha_{jv})\, a_{ij}$$

$$- \frac{1}{x_n}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} a_{ij}x_j + a_{in}\right)(\alpha_{iv}\alpha_{nw} + \alpha_{iw}\alpha_{nv})$$

$$+ \frac{1}{x_n^2}\left(\sum_{i=1}^{n-1} a_{ii}x_i^2 + 2\sum_{i<j}^{1,\,n-1} a_{ij}x_ix_j + 2\sum_{i=1}^{n-1} a_{in}x_i + 1\right)\alpha_{nv}\alpha_{nw}$$

$$+ \sum_{i=1}^{n-1} a_{in}(\alpha_{iv}\alpha_{n+1,\,w} + \alpha_{iw}\alpha_{n+1,\,v})$$

$$- \frac{1}{x_n}\left(\sum_{i=1}^{n-1} a_{in}x_i + 1\right)(\alpha_{n+1,\,v}\alpha_{nw} + \alpha_{n+1,\,w}\alpha_{nv}) + \alpha_{n+1,\,v}\alpha_{n+1,\,w}\Bigg],$$

$(v < w = 1, \dots, n-1)$,

$$(8)\qquad a'_{un} = \frac{1}{d}\Bigg[\sum_{i=1}^{n-1} \alpha_{iu}\alpha_{i} a_{ii} + \sum_{i<j}^{1,\,n-1} (\alpha_{iu}\alpha_{j} + \alpha_{ju}\alpha_{i})\, a_{ij}$$

$$- \frac{1}{x_n}\sum\nolimits_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} a_{ij}x_j + a_{in}\right)(\alpha_{iu}\alpha_{n} + \alpha_{nu}\alpha_{i})$$

$$+ \frac{1}{x_i^2}\left(\sum_{i=1}^{n-1} a_{ii}x_i^2 + 2\sum_{i<j}^{1,\,n-1} a_{ij}x_ix_j + 2\sum_{i=1}^{n-1} a_{in}x_i + 1\right)\alpha_{nu}\alpha_{n}$$

$$+ \sum_{i=1}^{n-1} a_{in}(\alpha_{iu} + \alpha_{i}\alpha_{n+1,\,u})$$

$$- \frac{1}{x_n}\left(\sum_{i=1}^{n-1} a_{in}x_i + 1\right)(\alpha_{n+1,\,u}\alpha_{n} + \alpha_{nu}) + \alpha_{n+1,\,u}\Bigg]$$

where

$$d = \sum_{i=1}^{n-1} \alpha_i^2 a_{ii} + 2\sum_{i<j}^{1,\,n-1} \alpha_i\alpha_j a_{ij} - \frac{2}{x_n}\sum_{i=1}^{n-1}\alpha_i\alpha_n a_{in} + \frac{1}{x_n^2}\alpha_n^2$$

$$+2\sum_{i=1}^{n-1}\alpha_i a_{in} - \frac{2}{x_n}\left(\sum_{i=1}^{n-1} a_{in}x_i + 1\right) + 1\,.$$

The group $H_{n(n+2)}$, which is associated to the group $G_{n(n+2)}$ with respect to the conic sections' family in the space $\mathscr{P}_n$, has equations (4), (5), (6) and (7). The identity of the group $G_{n(n+2)}$, and consequently the identity I of the group

$H_{n(n+2)}$ can be obtained when:

$$I: \quad \alpha_{11} = \ldots = \alpha_{nn} = 1, \quad \alpha_{12} = \ldots = \alpha_{n-1,n} = \alpha_1 = \ldots = \alpha_n = 0 .$$

The coefficients of the infinitesimal transformations of the group $H_{n(n+2)}$ are:

$$\xi^i_{hk} = \left(\frac{\partial x'_i}{\partial \alpha_{hk}}\right)_I, \quad \xi^i_{n+1,k} = \left(\frac{\partial x'_i}{\partial \alpha_{n+1,k}}\right)_I, \quad \xi^i_h = \left(\frac{\partial x'_i}{\partial \alpha_h}\right)_I,$$

$$\eta^i_{hk} = \left(\frac{\partial A'_i}{\partial \alpha_{h,k}}\right)_I, \quad \eta^i_{n+1,k} = \left(\frac{\partial A'_i}{\partial \alpha_{n+1,k}}\right)_I, \quad \eta^i_h = \left(\frac{\partial A'_i}{\partial \alpha_h}\right)_I;$$

$$\zeta^{uu}_{hk} = \left(\frac{\partial a'_{uu}}{\partial \alpha_{hk}}\right)_I, \quad \zeta^{uu}_{n+1,h} = \left(\frac{\partial a'_{uu}}{\partial \alpha_{n+1,h}}\right)_I, \quad \zeta^{uu}_h = \left(\frac{\partial a'_{uu}}{\partial \alpha_h}\right)_I,$$

$$\zeta^{uv}_{hk} = \left(\frac{\partial a'_{uv}}{\partial \alpha_{hk}}\right)_I, \quad \zeta^{uv}_{n+1,h} = \left(\frac{\partial a'_{uv}}{\partial \alpha_{n+1,h}}\right)_I, \quad \zeta^{uv}_h = \left(\frac{\partial a'_{uv}}{\partial \alpha_h}\right)_I,$$

$$i, h, k = 1, \ldots, n; \quad u = 1, \ldots, n-1, \quad v = 1, \ldots, n; \quad u < v .$$

Writing equations (4) in the following way

$$\sum_{k=1}^{n} (\alpha_{hk} - \alpha_{n+1,k} x_h)\, x'_k = x_h - \alpha_h \qquad (h = 1, \ldots, n),$$

and taking the derivatives with respect to α_{lm} in I, we get:

$$\delta_{lh}\delta_{mk}(x'_k)_I + \sum_{k=1}^{n} (\alpha_{hk} - \alpha_{n+1,k} x_h)_I \left(\frac{\partial x'_k}{\partial \alpha_{lm}}\right)_I = 0 ,$$

then

(9) $$\xi^h_{lm} = -\delta_{lh} x_m, \qquad (h, l, m = 1, \ldots, n);$$

taking the derivatives with respect to $\alpha_{n+1,m}$ in I, we get:

$$-x_h (x'_m)_I + \sum_{k=1}^{n} (\alpha_{hk} - \alpha_{n+1,k} x_h)_I \left(\frac{\partial x'_k}{\partial \alpha_{n+1,m}}\right)_I = 0 ,$$

then

(10) $$\xi^h_{n+1,m} = x_h x_m;$$

taking the derivatives with respect to α_m in I, we get:

$$\sum_{k=1}^{n} \delta_{hk} \left(\frac{\partial x'_k}{\partial \alpha_m}\right)_I = -\delta_{hm},$$

then

$$\xi^h_m = -\delta_{hm}. \tag{11}$$

By deriving formulas (5) with respect to α_{lm} in I, we get:

$$\left(\frac{\partial A'_h}{\partial \alpha_{lm}}\right)_I = \sum_{k=1}^{n}\left(\frac{\partial \alpha_{kh}}{\partial \alpha_{lm}}\right)_I A_k,$$

then

$$\eta^h_{lm} = \delta_{hm} A_l; \tag{12}$$

by deriving with respect to $\alpha_{n+1,\,m}$ in I, we get:

$$\left(\frac{\partial A'_h}{\partial \alpha_{n+1,\,m}}\right)_I = \left(\frac{\partial \alpha_{n+1,\,h}}{\alpha_{n+1,\,m}}\right)_I,$$

then

$$\eta^h_{n+1,\,m} = \delta_{hm}; \tag{13}$$

by deriving with respect to α_m in I, we get:

$$\left(\frac{\partial A'_h}{\partial \alpha_m}\right)_I = -A_h \sum_{k=1}^{n}\left(\frac{\partial \alpha_k}{\partial \alpha_m}\right)_I A_k,$$

then

$$\eta^h_m = -A_h A_m. \tag{14}$$

By deriving the expressions (6) and (7) with respect to α_{hk}, $\alpha_{n+1,\,h}$, α_h in I, we get the following non identically zero ζ functions:

$$\zeta^{uu}_{vu} = 2a_{uv}, \qquad \zeta^{uu}_{nu} = -\frac{2}{x_n}\left(\sum_{i=1}^{n-1} a_{ui}x_i + a_{un}\right), \qquad \zeta^{uu}_{n+1,\,u} = 2a_{un},$$

$$\zeta^{uu}_{v} = -2a_{uu}a_{vn}, \qquad \zeta^{uu}_{n} = \frac{2a_{uu}}{x_n}\left(\sum_{i=1}^{n-1} a_{in}x_i + 1\right), \qquad \zeta^{vw}_{uv} = a_{uw}$$

$$\zeta^{vw}_{vw} = a_{vv}, \qquad \zeta^{vw}_{ww} = a_{vw}, \qquad \zeta^{vw}_{nv} = -\frac{1}{x_n}\left(\sum_{j=1}^{n-1} a_{wj}x_j + a_{wn}\right),$$

$$\zeta^{vn}_{nv} = -\frac{1}{x_n}\left(\sum_{j=1}^{n-1} a_{jn}x_j + 1\right), \qquad \zeta^{vw}_{nw} = -\frac{1}{x_n}\left(\sum_{j=1}^{n-1} a_{vj}x_j + a_{vn}\right), \qquad \zeta^{vw}_{n+1,\,v} = a_{wn},$$

$$\zeta^{vw}_{n+1,\,w} = a_{vn}\,, \qquad \zeta^{vw}_{u} = -2\,a_{vw}\,a_{un}\,, \qquad \zeta^{vw}_{n} = \frac{2\,a_{vw}}{x_n}\left(\sum_{i=1}^{n-1} a_{in}\,x_i + 1\right),$$

$$\zeta^{un}_{vu}\,a_{vn}\,, \qquad \zeta^{un}_{nu} = -\frac{1}{x_n}\left(\sum_{i=1}^{n-1} a_{in}\,x_i + 1\right), \qquad \zeta^{un}_{n+1,\,u} = 1\,,$$

$$\zeta^{un}_{v} = a_{uv} - 2\,a_{un}\,a_{vn}\,, \qquad \zeta^{un}_{n} = -\frac{1}{x_n}\left(\sum_{i=1}^{n-1} a_{ui}\,x_i + a_{un}\right) + \frac{2\,a_{un}}{x_n}\left(\sum_{i=1}^{n-1} a_{in}\,x_i + 1\right),$$

$$u,\, v,\, w = 1, \dots, n-1\,; \qquad v < w\,.$$

Using the above expressions of the functions ξ, η and ζ, we can write the Deltheil's system for the integral invariant function $\Phi \neq 0$ of the group $H_{n(n+2)}$:

$$-x_1\,\frac{\partial\Phi}{\partial x_1} + A_1\,\frac{\partial\Phi}{\partial A_1} + a_{11}\,\frac{\partial\Phi}{\partial a_{11}} + \sum_{h=1}^{n} a_{1h}\,\frac{\partial\Phi}{\partial a_{1h}} = -(n+1)\,\Phi\,,$$

$$-x_2\,\frac{\partial\Phi}{\partial x_1} + A_1\,\frac{\partial\Phi}{\partial A_2} + a_{12}\,\frac{\partial\Phi}{\partial a_{22}} + \sum_{h=1}^{n} a_{1h}\,\frac{\partial\Phi}{\partial a_{2h}} = 0\,,$$

....

$$-x_{n-1}\,\frac{\partial\Phi}{\partial x_1} + A_1\,\frac{\partial\Phi}{\partial A_{n-1}} + a_{1,\,n-1}\,\frac{\partial\Phi}{\partial a_{n-1,\,n-1}} + \sum_{h=1}^{n} a_{1h}\,\frac{\partial\Phi}{\partial a_{n-1,\,h}} = 0\,,$$

$$-x_n\,\frac{\partial\Phi}{\partial x_1} + A_1\,\frac{\partial\Phi}{\partial A_n} = 0\,,$$

$$-x_1\,\frac{\partial\Phi}{\partial x_2} + A_2\,\frac{\partial\Phi}{\partial A_1} + a_{12}\,\frac{\partial\Phi}{\partial a_{11}} + \sum_{h=1}^{n} a_{2h}\,\frac{\partial\Phi}{\partial a_{1h}} = 0\,,$$

$$-x_2\,\frac{\partial\Phi}{\partial x_2} + A_2\,\frac{\partial\Phi}{\partial A_2} + a_{22}\,\frac{\partial\Phi}{\partial a_{22}} + \sum_{h=1}^{n} a_{2h}\,\frac{\partial\Phi}{\partial a_{2h}} = -(n+1)\,\Phi\,,$$

....

$$-x_{n-1}\,\frac{\partial\Phi}{\partial x_2} + A_2\,\frac{\partial\Phi}{\partial A_{n-1}} + a_{2,\,n-1}\,\frac{\partial\Phi}{\partial a_{n-1,\,n-1}} + \sum_{h=1}^{n} a_{2h}\,\frac{\partial\Phi}{\partial a_{n-1,\,h}} = 0\,,$$

$$-x_n\,\frac{\partial\Phi}{\partial x_2} + A_2\,\frac{\partial\Phi}{\partial A_n} = 0\,,$$

....

$$-x_1 \frac{\partial \Phi}{\partial x_{n-1}} + A_{n-1} \frac{\partial \Phi}{\partial A_1} + a_{1,\,n-1} \frac{\partial \Phi}{\partial a_{11}} + \sum_{h=1}^{n} a_{h,\,n-1} \frac{\partial \Phi}{\partial a_{1h}} = 0\,,$$

$$-x_2 \frac{\partial \Phi}{\partial x_{n-1}} + A_{n-1} \frac{\partial \Phi}{\partial A_2} + a_{2,\,n-1} \frac{\partial \Phi}{\partial a_{22}} + \sum_{h=1}^{n} a_{h,\,n-1} \frac{\partial \Phi}{\partial a_{2h}} = 0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-x_{n-1} \frac{\partial \Phi}{\partial x_{n-1}} + A_{n-1} \frac{\partial \Phi}{\partial A_{n-1}} + a_{n-1,\,n-1} \frac{\partial \Phi}{\partial a_{n-1,\,n-1}} + \sum_{h=1}^{n} a_{h,\,n-1} \frac{\partial \Phi}{\partial a_{n-1,\,h}} = -(n+1)\,\Phi\,,$$

$$-x_n \frac{\partial \Phi}{\partial x_{n-1}} + A_{n-1} \frac{\partial \Phi}{\partial A_n} = 0\,,$$

$$-x_1 \frac{\partial \Phi}{\partial x_n} + A_n \frac{\partial \Phi}{\partial A_1} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{1i} x_i + a_{1n} \right) \frac{\partial \Phi}{\partial a_{11}} - \frac{1}{x_n} \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} a_{ji} x_i + a_{jn} \right) \frac{\partial \Phi}{\partial a_{1j}} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \frac{\partial \Phi}{\partial a_{1n}} = \frac{(n+1)\,x_1}{x_n}\,\Phi\,,$$

$$-x_2 \frac{\partial \Phi}{\partial x_n} + A_n \frac{\partial \Phi}{\partial A_2} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{2i} x_i + a_{2n} \right) \frac{\partial \Phi}{\partial a_{22}} - \frac{1}{x_n} \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} a_{ji} x_i + a_{jn} \right) \frac{\partial \Phi}{\partial a_{2j}} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \frac{\partial \Phi}{\partial a_{2n}} = \frac{(n+1)\,x_2}{x_n}\,\Phi\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-x_{n-1} \frac{\partial \Phi}{\partial x_n} + A_n \frac{\partial \Phi}{\partial A_{n-1}} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{n-1,\,i} x_i + a_{n-1,\,n} \right) \frac{\partial \Phi}{\partial a_{n-1,\,n-1}} - \frac{1}{x_n} \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} a_{ji} x_j + a_{jn} \right) \frac{\partial \Phi}{\partial a_{n-1,j}} - \frac{1}{x_n} \left(\sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \frac{\partial \Phi}{\partial a_{n-1,n}} = \frac{(n+1)\,x_{n-1}}{x_n}\,\Phi\,,$$

$$-x_n \frac{\partial \Phi}{\partial x_n} + A_n \frac{\partial \Phi}{\partial A_n} = 0\,,$$

$$\sum_{h=1}^{n} x_1 x_h \frac{\partial \Phi}{\partial x_h} + \frac{\partial \Phi}{\partial A_1} + a_{1n} \frac{\partial \Phi}{\partial a_{11}} + \sum_{i=1}^{n-1} a_{in} \frac{\partial \Phi}{\partial a_{1i}} + \frac{\partial \Phi}{\partial a_{1n}} = -(n+1)\,x_1\,\Phi\,,$$

$$\sum_{h=1}^{n} x_2 x_h \frac{\partial \Phi}{\partial x_h} + \frac{\partial \Phi}{\partial A_2} + a_{2n} \frac{\partial \Phi}{\partial a_{22}} + \sum_{i=1}^{n-1} a_{in} \frac{\partial \Phi}{\partial a_{2i}} + \frac{\partial \Phi}{\partial a_{2n}} = -(n+1)\, x_2 \Phi ,$$

....

$$\sum_{h=1}^{n} x_{n-1} x_h \frac{\partial \Phi}{\partial x_h} + \frac{\partial \Phi}{\partial A_{n-1}} + a_{n-1,\,n} \frac{\partial \Phi}{\partial a_{n-1,\,n-1}} + \sum_{i=1}^{n-1} a_{in} \frac{\partial \Phi}{\partial a_{n-1,\,i}} + \frac{\partial \Phi}{\partial a_{n-1,\,n}} = -(n+1)\, x_{n-1} \Phi ,$$

$$\sum_{h=1}^{n} x_n x_h \frac{\partial \Phi}{\partial x_h} + \frac{\partial \Phi}{\partial A_n} = -(n+1)\, x_n \Phi ,$$

$$-\frac{\partial \Phi}{\partial x_1} - \sum_{h=1}^{n} A_1 A_h \frac{\partial \Phi}{\partial A_h} - 2 a_{1n} \left(\sum_{i=1}^{n-1} a_{ii} \frac{\partial \Phi}{\partial a_{ii}} + \sum_{i<j}^{1,\,n-1} a_{ij} \frac{\partial \Phi}{\partial a_{ij}} \right) + \sum_{i=1}^{n-1} (a_{1i} - 2 a_{1n} a_{in}) \frac{\partial \Phi}{\partial a_{in}} = (n+1)(n a_{1n} + A_1)\, \Phi ,$$

$$-\frac{\partial \Phi}{\partial x_2} - \sum_{h=1}^{n} A_2 A_h \frac{\partial \Phi}{\partial A_h} - 2 a_{2n} \left(\sum_{i=1}^{n-1} a_{ii} \frac{\partial \Phi}{\partial a_{ii}} + \sum_{i<j}^{1,\,n-1} a_{ij} \frac{\partial \Phi}{\partial a_{ij}} \right) + \sum_{i=1}^{n-1} (a_{2i} - 2 a_{2n} a_{in}) \frac{\partial \Phi}{\partial a_{in}} = (n+1)(n a_{2n} + A_2)\, \Phi ,$$

....

$$-\frac{\partial \Phi}{\partial x_{n-1}} - \sum_{h=1}^{n} A_{n-1} A_h \frac{\partial \Phi}{\partial A_h} - 2 a_{n-1,\,n} \left(\sum_{i=1}^{n-1} a_{ii} \frac{\partial \Phi}{\partial a_{ii}} + \sum_{i<j}^{1,\,n-1} a_{ij} \frac{\partial \Phi}{\partial a_{ij}} \right) + \sum_{i=1}^{n-1} (a_{n-1,\,i} - 2 a_{2n} a_{in}) \frac{\partial \Phi}{\partial a_{in}} = (n+1)(n a_{n-1,\,n} + A_{n-1})\, \Phi ,$$

$$-\frac{\partial \Phi}{\partial x_n} - \sum_{h=1}^{n} A_n A_h \frac{\partial \Phi}{\partial A_h} + \frac{2}{x_n} \left(\sum_{i=1}^{n-1} a_{in} x_i + 1 \right) \left(\sum_{u=1}^{n-1} a_{uu} \frac{\partial \Phi}{\partial a_{uu}} + \sum_{v<w}^{1,\,n-1} a_{vw} \frac{\partial \Phi}{\partial a_{vw}} + \sum_{u=1}^{n-1} a_{un} \frac{\partial \Phi}{\partial a_{un}} \right) - \frac{1}{x_n} \sum_{u=1}^{n-1} \left(\sum_{i=1}^{n-1} a_{ui} x_i + a_{un} \right) \frac{\partial \Phi}{\partial a_{un}} = -\frac{n+1}{x_n} \left[n \left(\sum_{i=1}^{n-1} a_{in} x_i \right) - A_n + n - 1 \right] \Phi .$$

Solving the system

$$\begin{cases} -x_n \dfrac{\partial \Phi}{\partial x_h} + A_h \dfrac{\partial \Phi}{\partial A_n} = 0\,, \qquad (h = 1, \ldots, n) \\ \sum\limits_{h=1}^{n} x_n x_h \dfrac{\partial \Phi}{\partial x_h} + \dfrac{\partial \Phi}{\partial A_n} = -(n+1)\, x_n \Phi \end{cases}$$

we find

$$\Phi = \frac{f(a_{11}, a_{12}, \ldots, a_{n-1,\, n})}{(A_1 x_1 + \cdots\cdots + A_n x_n + 1)^{n+1}}\,, \qquad (f \neq 0)\,.$$

By substituting the above expression in the system:

$$\begin{cases} -x_1 \dfrac{\partial \Phi}{\partial x_1} + A_1 \dfrac{\partial \Phi}{\partial A_1} + a_{11} \dfrac{\partial \Phi}{\partial a_{11}} + \sum\limits_{h=1}^{n} a_{1h} \dfrac{\partial \Phi}{\partial a_{1h}} = -(n+1)\, \Phi \\ -x_2 \dfrac{\partial \Phi}{\partial x_2} + A_2 \dfrac{\partial \Phi}{\partial A_1} + a_{12} \dfrac{\partial \Phi}{\partial a_{12}} + \sum\limits_{h=1}^{n} a_{2h} \dfrac{\partial \Phi}{\partial a_{1h}} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -x_1 \dfrac{\partial \Phi}{\partial x_{n-1}} + A_{n-1} \dfrac{\partial \Phi}{\partial A_1} + a_{1,\, n-1} \dfrac{\partial \Phi}{\partial a_{1,\, n-1}} + \sum\limits_{h=1}^{n} a_{h,\, n-1} \dfrac{\partial \Phi}{\partial a_{h,\, n-1}} = 0 \\ \sum\limits_{h=1}^{n} x_1 x_h \dfrac{\partial \Phi}{\partial x_h} + \dfrac{\partial \Phi}{\partial A_1} + a_{1n} \dfrac{\partial \Phi}{\partial a_{11}} + \sum\limits_{i=1}^{n-1} a_{in} \dfrac{\partial \Phi}{\partial a_{1i}} + \dfrac{\partial \Phi}{\partial a_{1n}} = -(n+1)\, x_1 \Phi \end{cases}$$

we get the following system:

$$\begin{cases} 2a_{11} \dfrac{\partial \log f}{\partial a_{11}} + a_{12} \dfrac{\partial \log f}{\partial a_{12}} + \cdots\cdots + a_{1n} \dfrac{\partial \log f}{\partial a_{1n}} = -(n+1) \\ 2a_{12} \dfrac{\partial \log f}{\partial a_{11}} + a_{22} \dfrac{\partial \log f}{\partial a_{12}} + \cdots\cdots + a_{2n} \dfrac{\partial \log f}{\partial a_{1n}} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 2a_{1,\, n-1} \dfrac{\partial \log f}{\partial a_{11}} + a_{2,\, n-1} \dfrac{\partial \log f}{\partial a_{12}} + \cdots\cdots + a_{n-1,\, n} \dfrac{\partial \log f}{\partial a_{1n}} = 0 \\ 2a_{1n} \dfrac{\partial \log f}{\partial a_{11}} + a_{2n} \dfrac{\partial \log f}{\partial a_{12}} + \cdots\cdots + a_{n-1,\, n} \dfrac{\partial \log f}{\partial a_{1,\, n-1}} + \dfrac{\partial \log f}{\partial a_{1n}} = 0 \end{cases}$$

whose (unique) solution is:

$$\frac{\partial \log f}{\partial a_{11}} = -\frac{(n+1)\,\Delta_{11}}{2\Delta}, \qquad \frac{\partial \log f}{\partial a_{1l}} = -\frac{2(n+1)\,\Delta_{1l}}{2\Delta}, \qquad (l = 2, \dots, n),$$

where Δ_{hk} is the cofactor of a_{hk} in the determinant $\Delta \neq 0$. Hence we find:

$$f = \frac{K(a_{22}, a_{23}, \dots, a_{n-1,n})}{|\Delta|^{(n+1)/2}} \qquad (K \neq 0),$$

and therefore

$$\Phi = \frac{K(a_{22}, a_{23}, \dots, a_{n-1,n})}{(A_1 x_1 + \dots + A_n x_n + 1)^{n+1}\,|\Delta|^{(n+1)/2}}.$$

If we substitute the above expression in the other equations of the Deltheil's system we get

$$K = \text{constant}.$$

Hence the Deltheil's system has the unique solution (up to a constant factor):

$$\Phi = \frac{1}{\left(\sum_{h=1}^{n} A_h x_h + 1\right)^{n+1} |\Delta|^{(n+1)/2}}.$$

Consequently the non-degenerate conic sections' family in the projective space $\mathcal{P}_n$ is measurable and its elementary measure is:

$$\frac{dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \wedge dA_1 \wedge dA_2 \wedge \dots \wedge dA_n \wedge da_{11} \wedge da_{12} \wedge \dots \wedge da_{n-1,n}}{\left(\sum_{h=1}^{n} A_h x_h + 1\right)^{n+1} \Delta^{(n+1)/2}}.$$

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