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Some Quasivariational Problems with Memory.

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Sunto. – *Si considera una classe di problemi quasivariazionali astratti che possono descrivere effetti di memoria in vari contesti applicativi. In particolare, viene provata la loro risolubilità generalizzata sotto opportune ipotesi di monotonia e per mezzo di un risultato di punto fisso per applicazioni multivoche in spazi ordinati. Si sviluppa infine un'applicazione alla modellizzazione dei fenomeni di incrudimento in plasticità.*

Summary. – *This note deals with a class of abstract quasivariational evolution problems that may include some memory effects. Under a suitable monotonicity framework, we provide a generalized existence result by means of a fixed point technique in ordered spaces. Finally, an application to the modeling of generalized kinematic hardening in plasticity is discussed.*

1. – Introduction.

Let H be a separable Hilbert space and denote by $\mathcal{K}$ the set of non-empty, convex, and closed subsets of H. Assume we are given a reference time $T > 0$, a set valued function $K : [0, T] \to \mathcal{K}$, and a point $u_0 \in K(0)$. In his fundamental papers [19, 20] Moreau proved the existence of a global in time absolutely continuous solution (right continuous and of bounded variation solution, respectively) $u : [0, T] \to H$ of the abstract evolution problem

$$u'(t) + \partial I_{K(t)}(u(t)) \ni 0, \quad u(0) = u_0, \tag{1.1}$$

where the prime denotes derivation with respect to time, $\partial I_{K(t)}(u(t))$ is the *normal cone* to $K(t)$ at the point $u(t)$, and a suitable absolute continuity assumption (right continuity and bounded variation assumption, respectively) on K is imposed. The latter problem is generally referred to as *sweeping process* and stems in a variety of applications related to non-smooth mechanics, convex optimization, mathematical economics among others [16]. Moreover, it formal-

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ly includes as a special case the *evolution variational inequality*

$$v(t) \in K', \quad (v'(t), v(t)-w) \leq (f(t), v(t)-w) \quad \forall w \in K', \, v(0)=u_0,$$

where $(\cdot, \cdot)$ denotes the scalar product in H, $K' \in \mathcal{K}$, and $f \in L^1(0, T; H)$, through the positions $u(t) := v(t) - (1 * f)(t)$ and $K(t) := K' - (1 * f)(t)$, with the standard notation $(1 * f)(t) := \int_0^t f(s)\, ds$ for $t \in [0, T]$.

Let us now address a first generalization of problem (1.1) by considering the case of a function K depending on the solution u as well. In particular, we assume to be given $K : [0, T] \times H \to \mathcal{K}$ and a point $u_0 \in K(0, u_0)$ and look for a solution to the problem

$$u'(t) + \partial I_{K(t,\, u(t))}(u(t)) \ni 0, \quad u(0) = u_0. \tag{1.2}$$

The latter *quasivariational problem* arises in connection with the treatment of quasistatical evolution problems with friction, micro-mechanical damage models (see [13] and the references therein), and the evolution of shape memory alloys [2, 3]. Moreover, it formally includes the case of *quasivariational evolution inequalities*

$$v(t) \in K'(v(t)), \quad (v'(t), v(t)-w) \leq (f(t), v(t)-w) \quad \forall w \in K'(v(t)), \; v(0)=u_0,$$

where now $K' : H \to \mathcal{K}$, by means of the positions $u(t) := v(t) - (1 * f)(t)$, $K(t, u) := K'(u + (1 * f)(t)) - (1 * f)\, s(t)$ for $t \in (0, T)$. Let us mention the well-posedness analysis for problem (1.2) of the papers [6, 13, 14] which is indeed based on a Lipschitz regularity assumption on K with respect to the Hausdorff topology in $\mathcal{K}$. In particular, the Lipschitz constant λ related to the u dependence of K is asked to be less than 1. The latter requirement has a clear mathematical drawback since it entails the possibility of exploiting some contractivity techniques. On the other hand, this requirement is motivated by the fact that there exists some counterexample to global strong existence in the case $\lambda > 1$ (see the forthcoming Subsection 3.1). Using a different approach, the papers [3, 21] study (1.2) by replacing the Lipschitz regularity assumption with some monotonicity assumption and exploiting an order method.

The present analysis is concerned with a nonlocal extension of problem (1.2). In particular, we assume to be given $K : [0, T] \times L^2(0, T; H) \to \mathcal{K}$ (the reader is referred at once to [15] for definitions and properties of function spaces) and a suitable initial state u_0. Thus, we are interested in the following *nonlocal quasivariational problem*

$$u'(t) + \partial I_{K(t,\, u)}(u(t)) \ni 0, \quad u(0) = u_0. \tag{1.3}$$

In particular, we address the issue of a quasivariational dependence of a global functional type, possibly modeling some memory effect.

As for to point out the possible applicative interest of this perspective, we shall briefly present a mechanical model that may be casted in terms of (1.3). Let us refer the reader at once to the monographs [1, 7, 9, 10, 22] for the discussion on the physical ground-bases and the mathematical modeling of *associative elastoplasticity* and start from the simplest possible one-dimensional kinematic hardening model. To this aim, let $H = \mathbb{R}$, and assume we are given the *strain rate* $\tau : [0, T] \to H$. We will denote by $\sigma, a : (0, T) \to H$ the *stress* and a *kinematic internal hardening parameter*, respectively, and we are interested in the corresponding dynamics form the initial configuration (σ_0, a_0). Following Han & Reddy [10], one has that the evolution of a body that undergoes *linear kinematic hardening* is governed by the constitutive relations in $(0, T)$,

$$(1.4) \qquad \begin{gathered} (\sigma, a) \in K_1, \quad (\sigma' - \tau)(\sigma - s) + a'(a - b) \leqslant 0, \quad \forall (s, b) \in K_1, \\ (\sigma(0), a(0)) = (\sigma_0, a_0), \end{gathered}$$

where $K_1 := \{(\sigma, a) \in H \times H : \varphi(\sigma - a) \leqslant 1\}$ with the *yield function* $\varphi : H \to [0, +\infty]$ convex, proper, and lower semicontinuous, $0 = \varphi(0) = \min \varphi$ and $(\sigma_0, a_0) \in K_1$. The general existence theory for evolution variational inequalities provides, for all $\tau \in L^2(0, T; H)$, a unique almost everywhere solution $(\sigma, a) \in W^{1,2}(0, T; H \times H)$ to (1.4) (in this regard, the reader shall be referred also to the analysis of the full PDE problem for associative elastoplasticity that is addressed by Johnson [11]). Our aim is now to recast the basic problem (1.4) in an equivalent form which is independent of the internal parameter a and generalizes to a wide class of hardening situations. To this end, one readily gets from (1.4) that, for all $t \in (0, T)$, the actual value $a(t)$ depends in a nonlocal in time fashion on the evolution of σ on $(0, t)$. In particular, letting $K_2 := \{\tau \in H : \varphi(\tau) \leqslant 1\}$, we simply check that (see also [10, Ex. 4.9, p. 90])

$$(1.5) \qquad a' \in \partial I_{K_2}(\sigma - a) \quad \text{in} \quad (0, T), \quad a(0) = a_0.$$

Namely, for all $\sigma \in W^{1,2}(0, T; H)$, one has that there exists a unique $a_\sigma \in W^{1,2}(0, T; H)$ fulfilling the above initial condition and inclusion almost everywhere. On the other hand, it is straightforward to compute that

$$\{\sigma \in H : (\sigma, a(t)) \in K_1\} \equiv K_2 + a(t).$$

By introducing the notation $K_3(t, \sigma) := K_2 + a_\sigma(t)$ problem (1.4) reads equivalently as follows

$$(1.6) \qquad \sigma'(t) + \partial I_{K_3(t, \sigma)}(\sigma(t)) \ni \tau(t) \quad \text{for} \quad t \in (0, T), \quad \sigma(0) = \sigma_0.$$

Finally (1.6) may be rewritten into the abstract framework of (1.3) with the choices $u = \sigma - 1 * \tau$ and $K(t, u) := K_3(t, u + 1 * \tau) - (1 * \tau)(t)$.

The existence of weak solutions to problem (1.3) has been obtained in [23] where indeed this author addresses an even more general problem by replacing the normal cone $\partial I_{K(t,\,u)}(u(t))$ with the (suitably generalized) gradient of a proper, convex, and lower semicontinuous potential. Indeed, the latter generalization is suitable of being applied to some parabolic PDE problem as well. The aim of note is to recast the results of [23] in the case of normal cones. In particular, we focus on the existence of generalized solutions to (1.3) by exploiting the so-called *variational section method*. Namely, we shall fix $\overline{u} \in L^2(0, T; H)$ and suitably solve the variational problem

$$u'(t) + \partial I_{K(t,\,\overline{u})}(u(t)) \ni 0, \quad u(0) = u_0,$$

that falls into the class of (1.1). Then, we have implicitly defined the so-called *variational selection* mapping $\mathcal{S}(\overline{u}) = u$. The key assumption of our analysis will clearly concern the functional dependence of K on $\overline{u}$ and shall be regarded as of *monotonicity type*. By introducing an *order structure* on the solution set, we claim that our key assumption of K entails the validity of an abstract *comparison tool* among weak solutions. In particular, the latter comparison principle asserts that, whenever we refer to ordered data $\overline{u}_1$ and $\overline{u}_2$, the corresponding solution sets $\mathcal{S}(\overline{u}_1)$ and $\mathcal{S}(\overline{u}_2)$ show some ordering property as well (see below). Finally, we shall present a suitable *fixed point device* for multivalued applications in ordered sets that entails, in particular, the existence of a fixed point for the variational selection $\mathcal{S}$. Before moving on, we shall remark that the above introduced method has some analogies with the theory of the solvability of equations in the *viscosity sense* [8]. Namely, it is remarkable that our fixed point lemma is indeed very close to the well-known Perron method for the construction of viscosity solutions. Indeed, as one shall see, our existence result will rely both on the above mentioned comparison result and on the existence of a pair of ordered sub and supersolutions, exactly as in the viscosity theory.

We shall provide a self-contained exposition by discussing some preliminary material on orders in Hilbert spaces and fixed points of non-decreasing set-valued applications (Section 2). Then, a (suitably revisited) version of the well-known counterexample [14] to global strong existence is presented and a notion of weak solution is discussed (Section 3). Finally, we state the existence result and sketch the main lines of its proof (Section 4). The above introduced plasticity model will serve us throughout the forthcoming analysis as a primer application of the abstract theory.

2. – Preliminaries.

Let us start by fixing some notation. In particular, we will use the symbol $(\cdot, \cdot)$ for the scalar product in H and $|\cdot|$ for the related norm.

For all $K \in \mathcal{K}$ we denote by $I_K \colon H \to [0, +\infty]$ the *indicator function* defined by $I_K(u) = 0$ if $u \in K$ and $I_K(u) = +\infty$ otherwise. It may be clearly checked that the latter function is proper, convex, and lower semicontinuous and one may define its *subdifferential* $\partial I_K \colon H \to 2^H$ as

$$w \in \partial I_K(u) \quad \text{iff} \quad u \in K \text{ and } (w, v-u) \leqslant 0 \ \ \forall v \in K .$$

Of course, for all $u \in K$, the set $\partial I_K(u)$ is nothing but the non-empty and closed *normal cone* to K at the point u. Moreover, ∂I_K turns out to be a maximal monotone graph. The reader is referred to [5] for definitions and details.

For later reference, let us recall that the *Hausdorff distance* d_H between two non-empty sets $F, G \subset H$ is defined by

$$d_H(F, G) := \max \left\{ \sup_{f \in F} \inf_{g \in G} |f - g| , \sup_{g \in G} \inf_{f \in F} |f - g| \right\} .$$

2.1. *Orders in Hilbert spaces.*

This introductory discussion mainly follows the presentation in Baiocchi & Capelo [4]. Let us assume we are given a non-empty set $C \subset H$ such that

$$C = \{ u \in H : (u, v) \geqslant 0 \ \ \forall v \in C \} .$$

Hence, C turns out to be a closed strict cone with vertex at the origin and one easily checks that the relation

$$u \leqslant v \quad \text{iff} \quad v - u \in C ,$$

defines an order in H. Moreover, for all $u, v \in H$, we will introduce the notations

$$u^+ := \pi_C(u), \qquad u^- := \pi_C(-u) ,$$

$$u \vee v := u + (v-u)^+ , \qquad u \wedge v := u - (u-v)^+ ,$$

where π_C stands for the projection on C. Of course, π_C is well defined and one checks that $u = u^+ - u^-$ and $(u^+, u^-) = 0$ for all $u \in H$. Hence, in particular, $u \vee v = v + (u-v)^+$ and $u \wedge v = v - (v-u)^+$ as well. Let us stress that the symbols $\wedge$ and $\vee$ are chosen just for the sake of notational simplicity. The datum (H, C) of the latter construction is said to be a *Hilbert pseudo-lattice* (see [4, Sec. 19.5, p. 399] for details).

Letting $F \subset H$, we recall that $f \in F$ is a *maximal element* of F iff, for all $f' \in F$, $f \leqslant f'$ implies $f = f'$. Then, f is the *maximum (minimum)* of F iff $f' \leqslant f$ ($f \leqslant f'$, respectively) for all $f' \in F$. Moreover, $u \in H$ is an *upper bound* of F iff $f \leqslant u$ for all $f \in F$ and $u \in H$ is the *supremum* or *least upper bound* iff u is the minimum of the set of upper bounds of F. Moreover, we say that F is a *chain* if it is totally ordered and that F is an *interval* iff there exist $u_*, u^* \in H$ such that $F \equiv \{ u \in H : \ u_* \leqslant u \leqslant u^* \}$. In the latter case we use the notation $F = [u_*, u^*]$. The set F is said to be *s-inductive* iff every chain of F is bounded above and it is said to be *completely s-inductive* iff every chain of F has a supremum. The well-known Zorn lemma reads as follows.

LEMMA 2.1. – *Let F be s-inductive. Then F has a maximal element.*

Whenever (H, C) is a Hilbert pseudo-lattice, it is straightforward to check that the same holds for $(L^2(0, T; H), C')$ with $C' := \{u \in L^2(0, T; H) : u \in C$ a.e. in $(0, T)\}$. Namely, the space $L^2(0, T; H)$ may be endowed with the order $\leqslant'$ defined, for all $u, v \in L^2(0, T; H)$, as

$$u \leqslant' v \quad \text{iff} \quad v \leqslant v \text{ a.e in } (0, T).$$

For the sake of notational simplicity we will use the same symbol $\leqslant$ for the two orders in H and in $L^2(0, T; H)$ throughout the remainder of the paper.

Let us stress from the very beginning that the concrete choices for (H, C) that we mainly have in mind are $H = \mathbb{R}$ and $C = [0, +\infty)$, $H = \mathbb{R}^m$, $m \in \mathbb{N}$, with $C = \{u = (u_1, \dots, u_m) \in \mathbb{R}^m : u_i \geqslant 0, i = 1, \dots, m\}$, $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open and $C = \{u \in L^2(\Omega) : u \geqslant 0$ a.e. in $\Omega\}$, or $H = (L^2(\Omega))^m$ and $C = \{u = (u_1, \dots, u_m) \in (L^2(\Omega))^m : u_i \geqslant 0$ a.e. in Ω, $i = 1, \dots, m\}$.

We shall now introduce on the parts of H ($L^2(0, T; H)$) the relation $\sim$ as

$$F \sim G \quad \text{iff} \quad (f \in F, g \in G \Rightarrow f \wedge g \in F, f \vee g \in G),$$

for all F, G non-empty subsets of H ($L^2(0, T; H)$, respectively). In particular, relation $\sim$ turns out to be an order on the non-empty closed intervals.

Assume now we are given $K_1, K_2 \in \mathcal{K}$ such that $K_1 \sim K_2$. It is a standard matter to exploit the definition of subdifferential and deduce that, for all $v_1 \in \partial I_{K_1}(u_1)$ and $v_2 \in \partial I_{K_2}(u_2)$, one has

$$(v_1, u_1 - w_1) \geqslant 0 \quad \forall w_1 \in K_1 \quad \text{and} \quad (v_2, u_2 - w_2) \geqslant 0 \quad \forall w_2 \in K_2.$$

Now, we may choose $w_1 := u_1 \wedge u_2$, $w_2 := u_1 \vee u_2$, take the sum in the corresponding inequalities, and deduce that

$$(v_1 - v_2, (u_1 - u_2)^+) \geqslant 0\,. \tag{2.1}$$

More precisely the following lemma [23, Lemma 4.1] holds true.

LEMMA 2.2. – *Let $K_1, K_2 \in \mathcal{K}$. The following are equivalent:*

i) $K_1 \sim K_2$,

ii) $(v_1 - v_2, (u_1 - u_2)^+) \geqslant 0$ for all u_i, v_i such that $v_i \in \partial I_{K_i}(u_i)$ for $i = 1, 2$.

2.2. *Fixed point lemma.*

Let us now come to our fixed point device, namely Lemma 2.4. The latter is nothing but an extension to the case of set-valued mappings between Hilbert pseudo-lattices of a former fixed point lemma for non-decreasing single-valued functions on ordered sets. Of course the current literature on fixed point results for multivalued applications is quite rich. Nevertheless, let us stress that we could not find a reference for the forthcoming Lemma 2.4. Hence, we aim to provide here a direct proof together with some comments. We shall start from the following result.

LEMMA 2.3. – *Let (H, C) be a Hilbert pseudo-lattice and $I := [u_*, u^*] \subset H$. Assume that $S : I \to I$ is non-decreasing. Then, the set $\{u \in I : u = S(u)\}$ is non-empty and has a minimum and a maximum.*

An order set version of the latter result was announced by Kolodner [12] and a proof is to be found, for instance, in [4, Thm. 9.26, p. 223]. Let us now introduce some notations. Namely, letting F, G denote non-empty subsets of H ($L^2(0, T; H)$, respectively) we define the relation $\preccurlyeq$ as

$$F \preccurlyeq G \quad \text{iff} \quad \forall f \in F \quad \exists g \in G \text{ such that } f \leqslant g . \tag{2.2}$$

Of course $F \sim G$ implies that $F \preccurlyeq G$ while the opposite implication does not hold. For the sake of notational simplicity, in the following we write, for instance, $f \preccurlyeq F$ instead of $\{f\} \preccurlyeq F$, etc. We are in the position of proving the following lemma.

LEMMA 2.4. – *Let (H, C) be a Hilbert pseudo-lattice and $I := [u_*, u^*] \subset H$. Assume that $S : (I, \leqslant) \to (2^I, \preccurlyeq)$ is non-decreasing and has non-empty and weakly compact values. Then, there exists $u \in I$ such that $u \in S(u)$.*

PROOF. – Let $U := \{v \in I : v \preccurlyeq S(v)\}$. We will prove that: (i) U is non-empty, (ii) U with the induced order is completely s-inductive, (iii) U has a maximal element u, (iv) u is a fixed point for S (namely $u \in S(u)$).

Proof of (i): since $S(u_*) \subset I$, we readily check that $u_* \in U$.

Proof of (ii): let $L = \{\lambda_\alpha\}_{\alpha \in A}$ be a chain in U, where $(A, <)$ is a totally ordered set of indices. From the assumptions on S we readily deduce that $u_* \preccurlyeq L \preccurlyeq u^*$ and that the sequence $(\lambda_\alpha, u^* - u_*)$ is non-decreasing with α and bounded. Hence, it has a finite limit. Then, for all β, $\alpha \in A$ with $\alpha < \beta$, we have that

$$|\lambda_\beta - \lambda_\alpha|^2 \leqslant (\lambda_\beta - \lambda_\alpha, u^* - u_*).$$

Namely L is a Cauchy sequence and monotonically converges to $\lambda = \sup\limits_{\alpha \in A} \lambda_\alpha$ as α increases. Of course, any subsequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of L is converging to the same limit.

Since $\lambda_n \leqslant \lambda$ and $\lambda_n \in U$, we have that $\lambda_n \preccurlyeq S(\lambda_n) \preccurlyeq S(\lambda)$. Namely, there exists $s_n \in S(\lambda)$ such that $\lambda_n \leqslant s_n$. Being $S(\lambda)$ weakly compact, one can extract a (not relabeled) subsequence such that s_n weakly converges to $s \in S(\lambda)$. Then, we prove that, for all $c \in C$,

$$(s - \lambda, c) = \lim_{n \to +\infty} (s_n - \lambda_n, c) \geq 0 .$$

Finally $\lambda \leq s \in S(\lambda)$ which amounts to say that $\lambda \in U$.

Proof of (iii): one applies Lemma 2.1.

Proof of (iv): the maximal element u belongs to U, thus there exists $v \in S(u)$ such that $u \leq v$. Hence $S(u) \preccurlyeq S(v)$ and, in particular $v \preccurlyeq S(v)$. Finally, $v \in U$ and, since u is maximal, one has that $u \equiv v \in S(u)$.

A few comments on the latter lemma are in order. First of all, one observes that, since any non-decreasing function $S : I \to I$ may be regarded as a non-decreasing multivalued application $S : (I, \leq) \to (2^I, \preccurlyeq)$ with non-empty and weakly compact values, Lemma 2.3 actually extends the existence result of the former Lemma 2.4. On the other hand, nothing can be said in general on the existence of a minimum or a maximum for the set of fixed points of the application S in the framework of Lemma 2.4. Indeed, let us consider $I := [0, 1]$ endowed with the usual order and the map $S_1(0) := \{1\}$ and $S_1(u) := \{u, 1\}$ for all $u \in (0, 1]$. We readily check that $\{u \in I : u \in S_1(u)\} \equiv (0, 1]$. The construction of a counterexample for the maximum case is just slightly more involved and it not reported, for the sake of simplicity. Moreover, it is clear that the weak compactness of the values of the mapping S is not necessary in order to have fixed points. Nevertheless, we cannot remove this assumption from the statement of Lemma 2.4 as it is shown by the counterexample $I := [0, 1]$ and $S_2(u) := (u + 1)/2$ for all $u \in [0, 1)$, $S_2(1) := [0, 1)$.

3. – Formulation.

3.1. *Counterexample.*

We start by describing a suitably modified version of the well-known counterexample [14] to the global strong solvability of (1.2). Assume that $K : [0, T] \times H$ is uniformly Lipschitz continuous with respect to the Hausdorff metric. Namely, we ask for

$$d_H(K(t, u), K(s, v)) \leq \mu |t - s| + \lambda |u - v| ,$$

for some positive constants μ, λ, and any $t, s \in [0, T]$, $u, v \in H$. We shall provide an example of K with $\lambda > 1$ for which no global absolutely continuous solution exists. To this aim let $H = \mathbb{R}$ and define $K'(w) := [\psi(w), +\infty)$ for $w \in \mathbb{R}$ and $\psi(w) := (2w - 1/2)^+$. It is straightforward to check that the problem

$$w' + \partial I_{K'(w)}(w) \ni 1, \quad w(0) = 0,$$

has the unique strong solution $w(t) = t$ on $(0, 1/2)$ (note that $\psi(1/2) = 1/2$). On the other hand, there is no absolutely continuous solution to the latter problem for $t > 1/2$. Indeed, the variational inequality entails $w' \geqslant 1$ and the region $\{1/2 < w < 1\}$ is not accessible for w since $1/2 < w < 1$ implies $w < \psi(w)$ and $w \notin K'(w)$. Starting from the latter example, one easily checks that the choice $K(t, u) := K'(u+t) - t$ is uniformly Lipschitz continuous of constant $\lambda = 2$ and problem (1.2) admits no absolutely continuous solutions on $(0, 1)$.

3.2. *Problem formulation.*

The latter counterexample motivates the introduction of a suitable concept of weak solution. To this aim let us recall that

(H) (H, C) is a sep. Hilbert pseudo-lattice, $u_0 \in H$, and $K : [0, T] \times L^2(0, T; H) \to \mathcal{K}$.

For any $u \in L^2(0, T; H)$, let us define

$$A(u) := \{v \in L^2(0, T; H) : v(t) \in K(t, u) \text{ for a.e. } t \in (0, T)\}$$

and state the following.

Problem Q. – To find $u \in A(u)$ such that

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T (v', v - u) \geqslant 0 \quad \forall v \in W^{1,2}(0, T; H) \cap A(u). \tag{3.1}$$

Relation (3.1) is easily deduced from (1.3) and the definition of subdifferential by integration on $(0, T)$ whenever a regular test function $v \in W^{1,2}(0, T; H)$ is considered. The latter weak formulation was already discussed in [17, 18] where indeed the authors focus on a somehow related local in time quasivariational problem.

Of course we will need to assume that there exists at least one function u belonging to $A(u)$ (otherwise the latter Problem Q has trivially no solutions) and that, at least for such u, the set $W^{1,2}(0, T; H) \cap A(u)$ is non-empty (otherwise relation (3.1) is automatically fulfilled and u reduces to the solution of a suitable *viability problem*). Let us ask for

(A) for any $u \in L^2(0, T; H)$ the set $W^{1,2}(0, T; H) \cap A(u)$ is non-empty.

Namely, we impose some regularity to the multi-mapping $t \mapsto K(t, u)$. On the other hand we do not explicitly require to have some $u \in A(u)$ since this will be obtained as a by-product of our overall assumptions (see below).

3.3. *Monotonicity assumption.*

Let us now come to the key assumption of this analysis. We will ask for

(M) for any $u_1, u_2 \in L^2(0, T; H)$ one has that $u_1 \leqslant u_2 \Rightarrow A(u_1) \sim A(u_2)$.

Let us stress that, in the framework of assumptions (H) and (A), for all $u \in L^2(0, T; H)$ the set $A(u)$ is non-empty, convex, and closed. Moreover, the latter monotonicity assumption (M) may be equivalently rewritten as

$$u_1 \leqslant u_2 \Rightarrow K(t, u_1) \sim K(t, u_2) \quad \text{for a.e.} \quad t \in (0, T),$$

for all $u_1, u_2 \in L^2(0, T; H)$. We shall check that this is exactly the situation of the kinematic hardening model (1.4). Indeed, referring to Section 1 for notations, assume we are given $(\sigma_i, a_i) \in W^{1,2}(0, T; H \times H)$, $i = 1, 2$, two solutions to (1.4) such that $\sigma_1 \leqslant \sigma_2$. Then we readily check that

$$(\sigma_1, a_1), (\sigma_2, a_2) \in K_1, \ \sigma_1 \leqslant \sigma_2 \Rightarrow (\sigma_1, a_1 \wedge a_2), (\sigma_2, a_1 \vee a_2) \in K_1.$$

Hence, we write (1.4) for $i = 1, 2$, choose $(s_1, b_1) := (\sigma_1, a_1 \wedge a_2)$ and $(s_2, b_2) := (\sigma_2, a_1 \vee a_2)$ and deduce that $a_1 \leqslant a_2$ as well. Eventually, the definition of K_3 implies that

$$\sigma_1 \leqslant \sigma_2 \Rightarrow K_3(t, \sigma_1) \sim K_3(t, \sigma_2) \quad \text{for a.e.} \quad t \in (0, T). \tag{3.2}$$

In particular, the monotonicity condition (M) can be easily inferred.

4. – Result.

According to Section 1, in order to solve the above quasivariational Problem Q we shall be concerned with its *variational section*. Namely, we are interested in the following problem.

PROBLEM V. – Given $\overline{u} \in L^2(0, T; H)$, to find $u \in A(\overline{u})$ such that

$$\frac{1}{2} |v(0) - u_0|^2 + \int_0^T (v', v - u) \geqslant 0 \quad \forall v \in W^{1,2}(0, T; H) \cap A(\overline{u}). \tag{4.1}$$

Indeed, we first prove that Problem V admits at least a solution for all data $\overline{u} \in L^2(0, T; H)$. Hence, we will define the *variational selection* mapping $\mathcal{S}: L^2(0, T; H) \to 2^{L^2(0, T; H)}$ carrying the datum $\overline{u}$ into the solutions of Problem V. We will call $\overline{u} \in L^2(0, T; H)$ a *subsolution* of Problem Q if $\overline{u} \leqslant u$ for all $u \in \mathcal{S}(\overline{u})$. Analogously $\overline{u} \in L^2(0, T; H)$ is called *supersolution* of Problem Q if $u \leqslant \overline{u}$ for all $u \in \mathcal{S}(\overline{u})$. Our main result reads as follows.

THEOREM 4.1. – *Assume* $(H), (A), (M)$, *and that there exists a subsolution* u_* *and a supersolution* u^* *to Problem Q such that* $u_* \leq u^*$. *Then, the set of solutions* u *to Problem Q such that* $u_* \leq u \leq u^*$ *is non-empty.*

As far as uniqueness is concerned, let us stress that nothing can be said in general for Problem Q. Indeed, even Problem V fails to have a unique solution as it is shown in [18, Ex. 1.2]. Namely, the variational selection $\mathcal{S}(\overline{u})$ is in general a set.

4.1. *Sketch of the proof.*

Let us briefly outline the proof of Theorem 4.1 (the reader shall refer to [23] for the details).

As a first step we shall fix $\overline{u} \in L^2(0, T; H)$ and solve Problem V. To this aim, one addresses the regularized problem of finding $u_\varepsilon \in W^{1,2}(0, T; H)$ fulfilling, for almost every $t \in (0, T)$,

$$u_\varepsilon'(t) + \partial I^\varepsilon_{K(t, \overline{u})}(u_\varepsilon(t)) = 0, \quad u_\varepsilon(0) = u_0. \tag{4.2}$$

Here $\partial I^\varepsilon_{K(t, \overline{u})}$ represent the well-known Yosida approximation of $\partial I_{K(t, \overline{u})}$ (see [5] for definitions and properties). We easily check that the latter problem is uniquely solvable. Hence, it is straightforward to verify that the solutions u_ε are bounded in $L^\infty(0, T; H)$, uniformly with respect to ε. Let us denote by $S(\overline{u})$ the set of weakstar limits in $L^\infty(0, T; H)$ of subsequences of u_ε as ε goes to 0. Moving from (4.2) and choosing any $v \in W^{1,2}(0, T; H) \cap A(\overline{u})$ we readily check that,

$$\frac{1}{2}\,|v(0) - u_0|^2 + \int_0^T ((v', v - u_\varepsilon) + I^\varepsilon_{K(t, \overline{u})}(v)) = \frac{1}{2}\,|(u_\varepsilon - v)(t)|^2 + \int_0^T I^\varepsilon_{K(t, \overline{u})}(u_\varepsilon).$$

Next, we exploit the properties of the Yosida approximation, pass to the lim inf as $\varepsilon \to 0$ in both sides of the latter relation, and deduce that u solves Problem V. Namely, we have proved that

$$\emptyset \neq S(\overline{u}) \subset \mathcal{S}(\overline{u}) \quad \forall \overline{u} \in L^2(0, T; H).$$

We shall turn our attention to the map S instead of $\mathcal{S}$ and show that some monotonicity property in $[u_*, u^*]$ may be proved. To this aim, we will make use of the following technical lemma, which goes in the same direction of Lemma 2.2 and whose proof is reported in [23].

LEMMA 4.2. – *Assume* (H) *and* (M). *Moreover let* $\overline{u}_1, \overline{u}_2, u_1, u_2, v_1, v_2 \in L^2(0, T; H)$ *with* $\overline{u}_1 \leq \overline{u}_2$ *and* $v_i(t) = \partial I^\varepsilon_{K(t, \overline{u}_i)}(u_i(t))$ *for almost every* $t \in (0, T)$

and $i = 1, 2$. *Then*

$$(v_1 - v_2, (u_1 - u_2)^+) \geqslant 0 \quad a.e.\ in\ (0, T).$$

Let us fix $\overline{u}_1, \overline{u}_2 \in [u_*, u^*]$ such that $\overline{u}_1 \leqslant \overline{u}_2$ and denote by $u_{1\varepsilon}$ and $u_{2\varepsilon}$ the solutions to (4.2) with data $\overline{u}_1$ and $\overline{u}_2$, respectively. By taking the difference in the respective equations (4.2), testing on $(u_{1\varepsilon} - u_{2\varepsilon})^+$, and integrating on $(0, t)$ for $t \in (0, T)$, we get that

$$\frac{1}{2} \left| (u_{1\varepsilon} - u_{2\varepsilon})^+ (t) \right|^2 + \int_0^t (v_{1\varepsilon} - v_{2\varepsilon}, (u_{1\varepsilon} - u_{2\varepsilon})^+) = 0 ,$$

where $v_{i\varepsilon}(t) := \partial I^{\varepsilon}_{K(t, \overline{u}_i)}(u_{i\varepsilon}(t))$ for almost every $t \in (0, T)$ and $i = 1, 2$. Finally, it is a standard matter to apply Lemma 4.2 and deduce that

$$\overline{u}_1 \leqslant \overline{u}_2 \Rightarrow u_{1\varepsilon} \leqslant u_{2\varepsilon} \text{ for all } \varepsilon > 0. \tag{4.3}$$

Unfortunately, moving from the latter position we cannot infer that $\overline{u}_1 \leqslant \overline{u}_2$ implies that $u_1 \leqslant u_2$ for all $u_i \in S(\overline{u}_i)$, $i = 1, 2$, since the extracted subsequences converging to u_1 and u_2 need not have the same indices. However, in the framework of Theorem 4.1, it is straightforward to check that, for all $\overline{u} \in [u_*, u^*] =: I$, one has that $S(\overline{u}) \subset I$ as well. Namely, relation (4.3) ensures that, for all $v \in S(\overline{u})$, there exists $w \in S(u_*)$ such that $u_* \leqslant w \leqslant v$. This is easily obtained by successively extracting subsequences. An analogous argument entails that $v \leqslant u^*$ as well. On the other hand, owing for instance to the metrizability of the weak topology of $L^2(0, T; H)$ on bounded sets, we readily check that $S(\overline{u})$ is weakly compact. We may now check that $S : (I, \leqslant) \to (2^I, \preccurlyeq)$ is non-decreasing. As above, we exploit (4.3) and deduce that, if $\overline{u}_1 \leqslant \overline{u}_2$, for all $u_1 \in S(\overline{u}_1)$ there exists $u_2 \in S(\overline{u}_2)$ such that $u_1 \leqslant u_2$. We are now in the position of applying Lemma 2.4 and deduce that the set $\{u \in I : u \in S(u)\}$ is non-empty, whence Theorem 4.1 is completely proved.

By carefully analyzing the latter proof one readily checks that the existence of sub and supersolutions to Problem Q assumed in the statement of Theorem 4.1 may be substantially weakened. Indeed, one needs just the existence of points u_*, u^* in $L^2(0, T; H)$ such that $S([u_*, u^*]) \subset [u_*, u^*]$. This is especially interesting with respect to applications where it is in general useful to exploit the approximation properties of the points in the image of S. According to these considerations we stress that we actually proved the following stronger existence result.

THEOREM 4.3. – *Assume* (H), (A), (M), *and that there exists* $u_*, u^* \in L^2(0, T; H)$ *such that* $S([u_*, u^*]) \subset [u_*, u^*]$. *Then, the set of solutions* u *to Problem Q such that* $u_* \leqslant u \leqslant u^*$ *is non-empty.*

4.2. *Generalized kinematic hardening.*

Theorem 4.1 leaves open the question whether a subsolution u_* and a supersolution u^* to Problem Q such that $u_* \leqslant u^*$ actually exist. Indeed this does not follow from our general assumptions and we must explicitly require it. Instead of discussing some abstract conditions for the existence of such sub and supersolutions, we prefer to present the example of a concrete construction in our kinematic hardening situation. Referring again to Section 1 for the notations, let us consider the quantity $\sigma_* := \sigma_0 - 1 * \tau^-$, and solve for σ such that (see (1.6))

$$\sigma'(t) + \partial I_{K_3(t,\,\sigma_*)}(\sigma(t)) \ni \tau(t) \quad \text{for a.e.} \quad t \in (0, T), \qquad \sigma(0) = \sigma_0. \tag{4.4}$$

Is is straightforward to check that the definitions of K_2 and K_3 imply that $\sigma_*(t) \in K_3(t, \sigma_*)$ for almost every $t \in (0, T)$. Hence, we also have that $(\sigma \vee \sigma_*)(t) \in K_3(t, \sigma_*)$ for almost every $t \in (0, T)$ and (4.4) entails that

$$(\sigma' - \tau)(\sigma - \sigma \vee \sigma_*) \leqslant 0 \quad \text{a. e.} \quad \text{in } (0, T).$$

Next, one multiplies relation $\sigma'_* + \tau^- = 0$ by $(\sigma - \sigma_*)^-$ and takes the sum with the above inequality obtaining

$$-(\sigma - \sigma_*)'(\sigma - \sigma_*)^- + \tau^+(\sigma - \sigma_*)^- \leqslant 0 \quad \text{a.e.} \quad \text{in } (0, T).$$

Since $\sigma(0) = \sigma_*(0) = \sigma_0$, we readily check that $\sigma \geqslant \sigma_*$ almost everywhere in $(0, T)$. Hence, by defining $u_* := \sigma_* - 1 * \tau$, one has that $u = \sigma - 1 * \tau \geqslant \sigma_* - 1 * \tau = u_*$. Namely, u_* is a subsolution in the above sense. An analogous argument with the choices $\sigma^* := \sigma_0 + 1 * \tau^+$ and $u^* := \sigma^* - 1 * \tau$ brings to the construction of a suitable supersolution.

Although we developed here the theory for the case of linear kinematic hardening, it is clear that the constitutive relation (1.6) is suitable of describing quite more general hardening situations. In particular, it is noteworthy to point out that all the physics of the phenomenon is translated into the monotonicity condition (M). Namely, as soon as (M) is fulfilled, we are entitled to address by the same techniques some generalized hardening effects such as suitable classes of nonlinear kinematic or combined isotropic-kinematic hardening.

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