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Homogeneous Carnot Groups Related to Sets of Vector Fields (*)

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Sunto. – *In questo articolo ci occupiamo del seguente problema: data una famiglia di campi vettoriali regolari $X_1, \dots, X_m$ su $\mathbb{R}^N$, ci chiediamo se esiste un gruppo omogeneo di Carnot $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ tale che $\sum_i X_i^2$ sia un sub-Laplaciano su $\mathbb{G}$. A tale proposito troviamo condizioni necessarie e sufficienti sugli assegnati campi vettoriali affinché la risposta alla suddetta domanda sia positiva. Inoltre esibiamo una costruzione esplicita della legge di gruppo $\circ$ che verifica i requisiti di cui sopra, fornendo dimostrazioni dirette. La prova è essenzialmente basata su una opportuna versione della formula di Campbell-Hausdorff. Per finire, mostriamo svariati esempi non banali del nostro metodo costruttivo.*

Summary. – *In this paper, we are concerned with the following problem: given a set of smooth vector fields $X_1, \dots, X_m$ on $\mathbb{R}^N$, we ask whether there exists a homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ such that $\sum_i X_i^2$ is a sub-Laplacian on $\mathbb{G}$. We find necessary and sufficient conditions on the given vector fields in order to give a positive answer to the question. Moreover, we explicitly construct the group law $\circ$ as above, providing direct proofs. Our main tool is a suitable version of the Campbell-Hausdorff formula. Finally, we exhibit several non-trivial examples of our construction.*

1. – Introduction and main results.

A Carnot group (or stratified group) is a connected and simply connected Lie group $(\mathbb{F}, *)$ whose Lie algebra $\mathfrak{f}$ admits a stratification, i.e., a direct sum decomposition (in the sense of vector spaces) $\mathfrak{f} = W_1 \oplus \dots \oplus W_r$ with $[W_1, W_i] = W_{i+1}$, $[W_1, W_r] = \{0\}$. A sub-Laplacian on $\mathbb{F}$ is any second order differential operator of the form $\sum_j X_j^2$, where the X_j's form a basis of W_1. The study of second order linear PDE's sum of squares of vector fields, started with Hörmander's paper [28], has significantly improved after the works by Folland [17] and by Folland&Stein [18] (who systematically developed Har-

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monic Analysis on stratified groups) and by Rothschild&Stein [37]. In this last paper, it was shown that any Hörmander operator can be locally approximated (in a suitable sense) by a sub-Laplacian on a free stratified group, a relevant result which increased the motivation in studying Carnot groups and their sub-Laplacians. Recently, many authors have investigated Carnot groups from different points of view. A part from the vaste literature on Heisenberg-type groups, we shall limit ourselves in mentioning only few very recent examples related to Carnot groups (referring the reader to the therein references for a more detailed bibliography): [4, 5, 6] (potential theory on Carnot groups), [20, 34] (geometric measure theory) [3, 22] (symmetry properties in Carnot groups), [12, 26] (quasiconformal mappings on Carnot groups), [24, 32] (geodesics on Carnot groups), [19, 21, 33] (boundary behavior of solutions to subelliptic equations), [1, 7, 8, 11] (parabolic-type equations for sub-Laplacians).

Usually, the Carnot group $\mathbb{F}$ and its composition law $*$ are part of the *data* of the problem and investigations are led with the aim of studying some specific properties and aspects of the *given* structure of $\mathbb{F}$. On the contrary, especially in the analysis of PDE's, an opposite situation sometimes occurs: given a linear second order operator $\mathcal{L} = \sum_j X_j^2$, where the X_j's are smooth vector fields on $\mathbb{R}^N$, it may be asked whether there exists a Lie group structure on $\mathbb{R}^N$ with respect to which $\mathcal{L}$ is a sub-Laplacian. Besides, in case the answer is affirmative, it could also be relevant to explicitly find the group law with the cited property. The aim of this paper is to answer to this question.

In order to describe the problem we are concerned with, we recall the definition of Carnot group that we shall be dealing with. Our definition may seem slightly different from the one given in literature, but it is indeed equivalent, as we observe below. We suppose that the Lie group $(\mathbb{R}^N, *)$ is endowed with a family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called *dilations*) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}), \tag{1.1}$$

where $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \dots + N_r = N$. We denote by $\mathfrak{f}$ the Lie algebra of $(\mathbb{R}^N, *)$. For $i = 1, \dots, N_1$, let Y_i be the vector field in $\mathfrak{f}$ that agrees at the origin with $\partial/\partial x_i^{(1)}$. We make the following assumption: the Lie algebra generated by $Y_1, \dots, Y_{N_1}$ is the whole $\mathfrak{f}$. With the above hypotheses, we call $\mathbb{F} = (\mathbb{R}^N, *, \delta_\lambda)$ a *homogeneous Carnot group*. We also say that $\mathbb{F}$ is of *step* r and has N_1 *generators*. If $X_1, \dots, X_{N_1}$ is any basis for $\operatorname{span}\{Y_1, \dots, Y_{N_1}\}$, the second order differential operator $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ will be called a *sub-Laplacian* on $\mathbb{F}$. It is not difficult to recognize that any homogeneous Carnot group is a Carnot group according to the usual definition. On the other hand, up to isomorphism, the opposite implication is also true (for the details, see e.g. [8]).

The following are the problems we are mainly concerned with: given a set of smooth vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^N$,

(i) find necessary and sufficient conditions (mutually independent and simple to check) on the X_j's which guarantee that $\sum_{j=1}^{m} X_j^2$ is a sub-Laplacian on a suitable homogeneous Carnot group $\mathbb{F} = (\mathbb{R}^N, *, \delta_\lambda)$;

(ii) when the above conditions are satisfied, construct the Carnot group $\mathbb{F}$.

We now give a short descriptive plan of the paper. In Section 2, we recall some simple properties which are necessarily satisfied by the vector fields of the Lie algebra of a homogeneous Carnot group. These properties suggest to make hypotheses **(H0)-(H1)-(H2)** on the vector fields $X_1, \ldots, X_m$ in order to accomplish (i). In Section 3, we prove that these hypotheses are sufficient in order to solve our first problem: we indeed turn to construct the group $\mathbb{F}$, as stated in (ii) (see Theorem 3.9). We define the group law on $\mathbb{R}^N$ by means of the solution of exponential-type to a certain system of ODE's, canonically related to the given vector fields $X_1, \ldots, X_m$. The proof of the associativity of this law is a non-trivial task. We remark that our proof is meant to be as direct as possible and only relies on a suitable version of the Campbell-Hausdorff formula (see Lemma 3.4). In particular, we explicitly avoid to use the associativity of the so-called Campbell-Hausdorff operation on a Lie algebra, which seems to be a profound result (for this topic, see the Appendix). In Section 4, we give several examples of our construction: in particular, we treat a class of Carnot groups arising from Control Theory (Example 4.5), a class of operators that we call of Kolmogorov-type (Example 4.4) and we also exhibit a new example of sub-Laplacian on a Carnot group (Example 4.3) inspired by a degenerate operator considered in [9]. Finally, in the Appendix we sketch a proof of the needed version of the Campbell-Hausdorff formula, by reducing to apply a general result by Djoković [15].

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2. – The hypotheses on the vector fields.

Let $\mathbb{F} = (\mathbb{R}^N, *, \delta_\lambda)$ be a given homogeneous Carnot group of step r and with N_1 generators, according to the definition in Section 1. Moreover, let $\mathfrak{f}$ denote the Lie algebra of $\mathbb{F}$, i.e., the set of $*$-left-invariant vector fields on $\mathbb{F}$.

We adopt the following notation: I is the identity map on $\mathbb{F}$ and if $X = \sum_{i=1}^{N} a_i \partial_i$ is a vector field on $\mathbb{R}^N$, $XI = (a_1, \dots, a_N)^T$ is the column vector of the component functions of X. Complete proofs of the results we are going to recall throughout the present section can be found, e.g., in [8].

If τ_x denotes the left-translation by x on $\mathbb{F}$, then a vector field X belongs to $\mathfrak{f}$ if and only if $XI(x) = \mathfrak{J}_{\tau_x}(0)\, XI(0)$, for every $x \in \mathbb{F}$ ($\mathfrak{J}_{\tau_x}$ denotes the Jacobian matrix of τ_x). The map $J : \mathbb{R}^N \to \mathfrak{f}$, $\eta \mapsto X$ defined by $XI(x) = \mathfrak{J}_{\tau_x}(0)\, \eta$ is an isomorphism of vector spaces. As a consequence, any basis for $\mathfrak{f}$ is the image via J of a basis of $\mathbb{R}^N$. We call *Jacobian basis* of $\mathfrak{f}$ the one corresponding to the canonical basis of $\mathbb{R}^N$, i.e., the vector fields in $\mathfrak{f}$ agreeing at the origin with the coordinate partial derivatives. It is useful to notice that the vector fields of the Jacobian basis are the column vectors of the matrix $\mathfrak{J}_{\tau_x}(0)$. Let $X_1, \dots, X_m \in \mathfrak{f}$. If there exists $x_0 \in \mathbb{F}$ such that $X_1 I(x_0), \dots, X_m I(x_0)$ are linearly independent in $\mathbb{R}^N$, then $X_1 I(x), \dots, X_m I(x)$ are linearly independent for all $x \in \mathbb{F}$. Viceversa, if there exist $x_0 \in \mathbb{F}$ and scalars $c_1, \dots, c_m$ such that $c_1 X_1 I(x_0) + \dots + c_m X_m I(x_0) = 0$, then $c_1 X_1 I(x) + \dots + c_m X_m I(x) = 0$ for all $x \in \mathbb{F}$. As a consequence, if $X_1, \dots, X_m$ belong to $\mathfrak{f}$, they are linearly independent iff they are linearly independent at every point or, equivalently, at one point at least.

We now recall the definition of the exponential map on $\mathfrak{f}$. If $X \in \mathfrak{f}$ then, for every fixed $x \in \mathbb{F}$, the system of ODE's $\dot\gamma(t) = (XI)(\gamma(t))$, $\gamma(0) = x$, has a unique C^∞ solution defined on the whole $\mathbb{R}$. If γ is such a solution, we set $\exp[X](x) := \gamma(1)$. The *exponential map* is defined as

$$\mathrm{Exp} : \mathfrak{f} \to \mathbb{F}, \qquad \mathrm{Exp}(X) := \exp[X](0).$$

Exp is an analytic diffeomorphism. We denote by Log the inverse map of Exp. There is a remarkable link between the group law and the exponential map.

REMARK 2.1. – For every $x, y \in \mathbb{F}$, we have $x * y = \exp[\mathrm{Log}(y)](x)$.

This remark shows that the composition law on $\mathbb{F}$ is completely determined by the algebra $\mathfrak{f}$. Whence, if two homogeneous Carnot groups $(\mathbb{R}^N, *_1)$, $(\mathbb{R}^N, *_2)$ have the same Lie algebra, then $*_1$ and $*_2$ coincide. This proves that, given smooth vector fields $X_1, \dots, X_m$ on $\mathbb{R}^N$, there exists at most one homogeneous Carnot group structure on $\mathbb{R}^N$ with Lie algebra $\mathrm{Lie}\{X_1, \dots, X_m\}$. Here, we have denoted by $\mathrm{Lie}\{X_1, \dots, X_m\}$ the least Lie sub-algebra of $C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ containing $X_1, \dots, X_m$. We have

$$\mathrm{Lie}\{X_1, \dots, X_m\} = \mathrm{span}\{X_J \mid J \in \{1, \dots, m\}^k,\ k \in \mathbb{N}\},$$

where we have set $X_J := [X_{j_1}, \dots [X_{j_{k-1}}, X_{j_k}] \dots]$ if $J = (j_1, \dots, j_k)$. We say that X_J is a commutator of length k of $X_1, \dots, X_m$. If $J = j_1$, we also say that $X_J := X_{j_1}$ is a commutator of length 1 of $X_1, \dots, X_m$.

Let $\{\delta_\lambda\}_\lambda$ denote the group of dilations on $\mathbb{F}$ as in (1.1). A real function $a(x)$ defined on $\mathbb{F}$ is called *δ_λ-homogeneous* of degree $\beta \in \mathbb{R}$ if, for every $x \in \mathbb{F}$ and $\lambda > 0$, it holds $a(\delta_\lambda(x)) = \lambda^\beta a(x)$. A linear differential operator X is called *δ_λ-homogeneous* of degree $\beta \in \mathbb{R}$ if, for every $\varphi \in C^\infty(\mathbb{F})$ and $\lambda > 0$, it holds $X(\varphi \circ \delta_\lambda) = \lambda^\beta (X\varphi) \circ \delta_\lambda$. With reference to the form (1.1) of the dilation δ_λ, we define a *homogeneous weight* of a multi-index $\gamma \in (\mathbb{N} \cup \{0\})^N$, $|\gamma|_{\mathbb{F}} := \sum_{i=1}^{r} \sum_{j=1}^{N_i} i\gamma_j^{(i)}$.

REMARK 2.2. – The only smooth δ_λ-homogeneous functions of degree β are the polynomial functions of the form $\sum_{|\gamma|_{\mathbb{F}} = \beta} c_\gamma x^\gamma$, $c_\gamma \in \mathbb{R}$. Consequently, a smooth vector field δ_λ-homogeneous of degree $k \leq r$ ($k \in \mathbb{N}$) has the following form

$$\sum_{i=k}^{r} \sum_{j=1}^{N_i} a_j^{(i)}(x^{(1)}, \ldots, x^{(i-k)}) \cdot (\partial/\partial x_j^{(i)}), \tag{2.1}$$

where $a_j^{(i)}$ is a δ_λ-homogeneous polynomial of degree $i - k$. In particular, a smooth vector field δ_λ-homogeneous of degree $k > r$ is the null operator.

As an application of Remark 2.2, we have

$$\mathfrak{J}_{\tau_x}(0) = \begin{bmatrix} \mathbb{I}_{N_1} & 0 & \cdots & 0 \\ J_2^{(1)} & \mathbb{I}_{N_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ J_r^{(1)}(x) & \cdots & J_r^{(r-1)}(x) & \mathbb{I}_{N_r} \end{bmatrix}$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix, whereas $J_i^{(k)}(x)$ is a $N_i \times N_k$ matrix whose entries are δ_λ-homogeneous polynomials of degree $i - k$. In particular, if we let $\mathfrak{J}_{\tau_x}(0) = \big(Z^{(1)}(x) \ldots Z^{(r)}(x)\big)$, where $Z^{(k)}(x)$ is a $N \times N_k$ matrix, then the column vectors of $Z^{(k)}(x)$ are δ_λ-homogeneous vector fields of degree k. From all the above remarks, it straightforwardly follows that the vector fields of the Jacobian basis $\{Z_i^{(k)} \mid k \leq r,\ i \leq N_k\}$ satisfy the following conditions:

(A0) $Z_1^{(1)}, \ldots, Z_{N_1}^{(1)}$ are linearly independent and δ_λ-homogeneous of degree 1;

(A1) the dimension of span$\{Z_1^{(k)} I(x), \ldots, Z_{N_k}^{(k)} I(x)\}$ is independent of x and equals N_k;

(A2) the dimension of span$\{Z_i^{(k)} I(x) \mid k \leq r,\ i \leq N_k\}$ is independent of x and equals N.

We now turn to problem (i) stated in Section 1. To this purpose, throughout the end of this section, $X_1, \ldots, X_m$ ($m \geq 2$) will be a given set of smooth vector fields on $\mathbb{R}^N$. We first fix some notation. Given $k \in \mathbb{N}$, we denote by $W^{(k)}$ the set spanned by the commutators of length k of $X_1, \ldots, X_m$ and, for $x \in \mathbb{R}^N$, we set

$$W^{(k)} I(x) := \operatorname{span}\{XI(x) \,|\, X \in W^{(k)}\} = \operatorname{span}\{X_J I(x) \,|\, J \in \{1, \ldots, m\}^k\}.$$

Finally, for $x \in \mathbb{R}^N$, we set

$$\operatorname{Lie}\{X_1, \ldots, X_m\}\, I(x) := \operatorname{span}\{XI(x) \,|\, X \in \operatorname{Lie}\{X_1, \ldots, X_m\}\}.$$

The properties (A0)-(A1)-(A2) suggest the following assumptions on the X_j's.

The hypotheses on the vector fields.

With the previous notation, we assume that the smooth vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^N$ satisfy the following hypotheses: there exists a family of dilations on $\mathbb{R}^N$ of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, x^{(2)}, \ldots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}) \tag{2.2}$$

where $r \geq 1$ is a given integer, $x^{(i)} \in \mathbb{R}^{N_i}$ $(i = 1, \ldots, r)$ and $N_1 + \ldots + N_r = N$, such that

(H0) $X_1, \ldots, X_m$ *are linearly independent and δ_λ-homogeneous of degree* 1;

(H1) $\dim(W^{(k)}) = \dim\big(W^{(k)} I(0)\big)$, *for every* $k = 1, \ldots, r$;

(H2) $\dim(\operatorname{Lie}\{X_1, \ldots, X_m\}\, I(0)) = N$.

For what has been recalled above, hypotheses **(H0)-(H1)-(H2)** are *necessary* to ensure that $\sum_{i=1}^m X_i^2$ is a sub-Laplacian on a suitable homogeneous Carnot group. The aim of Section 3 is to show that these hypotheses are also *sufficient* to this purpose. We explicitly remark that **(H0)-(H1)-(H2)** are independent. Indeed:

- the vector fields $\partial_{x_1}, \partial_{x_2}$ on $\mathbb{R}^3$, together with the Euclidean dilations, satisfy **(H0)** and **(H1)** but not **(H2)**;
- the vector fields $\partial_{x_1}, \partial_{x_2}, \partial_{x_1} + x_2\partial_{x_3}$ on $\mathbb{R}^3$, together with the group of dilations $(\lambda x_1, \lambda x_2, \lambda^2 x_3)$, satisfy **(H0)** and **(H2)** but not **(H1)**;
- the vector fields $\partial_{x_1} + x_1\partial_{x_2}, \partial_{x_2}$ on $\mathbb{R}^2$ satisfy **(H1)** and **(H2)** but do not satisfy **(H0)** for *any* dilation $(\lambda^\alpha x_1, \lambda^\beta x_2)$ on $\mathbb{R}^2$.

To end this section, we prove a simple result which relates the dimension of $W^{(k)}$ to the structure (2.2) of the dilation δ_λ. The proof of this proposition also contains some useful remarks.

PROPOSITION 2.3. – *If* $X_1, \ldots, X_m$ *satisfy hypotheses* **(H0)-(H1)-(H2)**, *then for every* $k = 1, \ldots, r$ *we have* $\dim(W^{(k)}) = N_k$, *where* $N_1, \ldots, N_r$ *are as in* (2.2).

PROOF. – To prove the assertion, we first observe that, by the hypothesis **(H0)** and by Remark 2.2, the commutators of the X_j's with length greater than r vanish identically. We then set, for $k = 1, \ldots, r$, $H_k := \dim(W^{(k)})$ and we fix a basis $\{Z_1^{(k)}, \ldots, Z_{H_k}^{(k)}\}$ for $W^{(k)}$. By the definitions, $Z_1^{(k)} I(0), \ldots, Z_{H_k}^{(k)} I(0)$ clearly span $W^{(k)} I(0)$. As a consequence, by **(H1)**, we infer that $Z_1^{(k)} I(0), \ldots, Z_{H_k}^{(k)} I(0)$ form a basis of $W^{(k)} I(0)$. We then prove that

$$Z_1^{(1)}, \ldots, Z_{H_1}^{(1)}; \ldots ; Z_1^{(r)}, \ldots, Z_{H_r}^{(r)} \text{ is a basis of } \mathrm{Lie}\{X_1, \ldots, X_m\}. \tag{2.3}$$

It suffices to prove the linear independence. Suppose $\sum_{k \leqslant r, j \leqslant H_k} \lambda_j^{(k)} Z_j^{(k)} = 0$, with $\lambda_j^{(k)} \in \mathbb{R}$. We then have $Z^{(1)} := \sum_{j \leqslant H_1} \lambda_j^{(1)} Z_j^{(1)} = -\sum_{k=2}^{r} \sum_{j \leqslant H_k} \lambda_j^{(k)} Z_j^{(k)} =: Z^{(2)}$. Since $Z^{(1)}$ is δ_λ-homogeneous of degree 1, whereas $Z^{(2)}$ is a sum of δ_λ-homogeneous operators of degree $\geqslant 2$, this is possible only if $Z^{(1)} = 0$ and $Z^{(2)} = 0$. From $Z^{(1)} = 0$, we derive that all the $\lambda_j^{(1)}$'s are zero. From $Z^{(2)} = 0$ and by repeating a similar argument finitely many times, it follows that all the $\lambda_j^{(k)}$'s vanish. From hypothesis **(H2)** and from (2.3), we infer that the column vectors of the matrix

$$A := \left(Z_1^{(1)} I(0) \cdots Z_{H_1}^{(1)} I(0) \cdots Z_1^{(r)} I(0) \cdots Z_{H_r}^{(r)} I(0)\right)$$

span $\mathbb{R}^N$. We shall show that they are also linearly independent. We fix $k \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, H_k\}$. From (2.1), we have

$$Z_j^{(k)} = \sum_{s=k}^{r} \sum_{r=1}^{N_s} a_{s,r}^{(k,j)} (\partial / \partial x_r^{(s)}), \tag{2.4}$$

where $a_{s,r}^{(k,j)}$ is a δ_λ-homogeneous polynomial of degree $s - k$. In particular, $a_{s,r}^{(k,j)}(0) = 0$ for every $s = k+1, \ldots, r$. As a consequence, the matrix A has the block-diagonal form

$$\begin{pmatrix} A^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^{(r)} \end{pmatrix} \quad \text{where } A^{(k)} = (a_{k,i}^{(k,j)}(0))_{1 \leqslant i \leqslant N_k, 1 \leqslant j \leqslant H_k}.$$

Collecting all the information on A, we infer that its columns form a basis for $\mathbb{R}^N$. In particular, we have

$$\sum_{k=1}^{r} H_k = N = \sum_{k=1}^{r} N_k. \tag{2.5}$$

We finally fix $k \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, N_k\}$. With the notation in (2.2), we set $\xi_j^{(k)} := (0^{(1)}, \ldots, e_j^{(k)}, \ldots, 0^{(r)})$, where $e_j^{(k)}$ is the j-th coordinate unit vector of $\mathbb{R}^{N_k}$. Since $\xi_j^{(k)}$ is a linear combination of the columns of A, this readily implies that the H_k columns of the sub-matrix $A^{(k)}$ span $\mathbb{R}^{N_k}$. As a consequence $H_k \geqslant N_k$, which yields, together with (2.5), $H_k = N_k$. ■

Following the notation in the proof of Proposition 2.3, for every $x \in \mathbb{R}^N$ we have

$$(Z_1^{(1)} I(x) \cdots Z_{N_1}^{(1)} I(x) \cdots Z_1^{(r)} I(x) \cdots Z_{N_r}^{(r)} I(x)) = \begin{pmatrix} A^{(1)} & 0 & \cdots & 0 \\ \star & A^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & A^{(r)} \end{pmatrix},$$

where $A^{(1)}, \ldots, A^{(r)}$ are square constant non-singular matrices. As a consequence, if the vector fields $X_1, \ldots, X_m$ satisfy hypotheses **(H0)-(H1)-(H2)**, then they also satisfy

$$\textbf{(H1)*} \quad \dim(W^{(k)} I(x)) = \dim(W^{(k)}), \quad \forall k \leqslant r, \ \forall x \in \mathbb{R}^N.$$

$$\textbf{(H2)*} \quad \dim(\mathrm{Lie}\{X_1, \ldots, X_m\}\, I(x)) = N, \quad \forall x \in \mathbb{R}^N.$$

Condition **(H2)*** is the well-known Hörmander's hypoellipticity condition for the vector fields $X_1, \ldots, X_m$. We explicitly remark that these last two properties hold true at *every* $x \in \mathbb{R}^N$ as a consequence of their being supposed to hold at the origin only.

3. – Construction of the group.

To begin with, we recall some basic results on the solution of «exponential-type» of an autonomous system of ODE's. We shall make use of notation similar to that used for the exponential map on a Lie group. However, we explicitly remark that it is not required any group structure here.

Let $X = \sum_{j=1}^{N} (XI)_j \partial_j$ be a given smooth vector field on $\mathbb{R}^N$. Let $x \in \mathbb{R}^N$ be fixed. Let $\gamma(t)$ be the solution with maximal domain $\mathcal{D}(X, x) \subseteq \mathbb{R}$ to the autonomous ordinary Cauchy problem

$$\text{(C)} \quad \dot{\gamma}(t) = XI(\gamma(t)), \quad \gamma(0) = x. \tag{3.1}$$

We shall also use the notation $\gamma(t) = \gamma(t; x) = \gamma_X(t; x)$. Whenever $1 \in \mathcal{D}(X, x)$, we set

$$\exp[X](x) := \gamma_X(1; x). \tag{3.2}$$

For any fixed compact subset K of $\mathbb{R}^N$, there exists $\varepsilon_K > 0$ such that $\gamma(t; x)$ is well-defined for every $(t; x) \in (-\varepsilon_K, \varepsilon_K) \times K$. From the uniqueness of the solution to (C), we infer

$$x \in K, \quad |s| + |t| < \varepsilon_K \Rightarrow \gamma_X(s; \gamma_X(t; x)) = \gamma_X(s+t; x), \tag{3.3}$$

$$x \in K, \quad |\lambda t| < \varepsilon_K \Rightarrow \gamma_X(\lambda t; x) = \gamma_{\lambda X}(t; x). \tag{3.4}$$

From (3.4) we derive that, if K is any compact subset of $\mathbb{R}^N$ and if $\lambda \in \mathbb{R}$ is small enough, then $\exp[\lambda X](x)$ is well-posed for every $x \in K$. Indeed, we have $\exp[\lambda X](x) = \gamma_{\lambda X}(1; x) = \gamma_X(\lambda; x)$ which is well-defined for any $(\lambda; x) \in (-\varepsilon_K, \varepsilon_K) \times K$. Moreover, if $\exp[-X](\exp[X](x))$ is well-posed, it holds

$$\exp[-X](\exp[X](x)) = x. \tag{3.5}$$

Indeed, combining (3.3) and (3.4), $\exp[-X](\exp[X](x)) = \gamma_X(-1; \gamma_X(1; x)) = x$. Furthermore, by (3.4), we also obtain

$$\exp[tX](x) = \gamma(t), \qquad \text{where } \gamma \text{ solves (C)}, \tag{3.6}$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ such that both sides are defined. By the smoothness of X, $\gamma(t)$ is infinitely differentiable on a neighborhood of 0 and its Taylor expansion is

$$\gamma(t; x) = \sum_{k=0}^{n} \frac{1}{k!} X^k I(x) \cdot t^k + \mathcal{O}_x(t^{n+1}), \quad \text{as } t \to 0. \tag{3.7}$$

Here $X^0 := I$ and X^k is the k-th power of the operator X. To prove (3.7), we first observe that (since γ solves problem (C) in (3.1))

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t))) = (Xf)(\gamma(t)), \quad \forall f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N). \tag{3.8}$$

By induction on k, we prove that $(\mathrm{d}/\mathrm{d}t)^k \gamma = (X^k I)(\gamma)$. When $k = 1$, this follows from (3.8) with $f = I$. Then we have $(\mathrm{d}/\mathrm{d}t)^{k+1} \gamma = (\mathrm{d}/\mathrm{d}t)((X^k I)(\gamma)) = (X(X^k I))(\gamma) = (X^{k+1} I)(\gamma)$. The second equality follows from (3.8) with $f = X^k I$. Now, (3.7) readily holds.

Let $Z_1, \dots, Z_N$ be given smooth vector fields on $\mathbb{R}^N$. If $\xi \in \mathbb{R}^N$, we consider the vector field $\xi \cdot Z := \sum_{j=1}^{N} \xi_j Z_j$. If $x_0 \in \mathbb{R}^N$ is fixed and if $|\xi|$ is small enough, $\Psi(\xi) := \exp[\xi \cdot Z](x_0)$ is well-posed. Moreover, Ψ is a smooth function defined on a neighborhood of the origin. We claim that $(\partial/\partial \xi_k)\, \Psi(0) = Z_k I(x_0)$. To prove this we recall that, by definition, $\Psi(\xi) = \gamma(1; \xi)$ where $\dot\gamma(t) =$

$\sum_{j=1}^{N} \xi_j Z_j I(\gamma(t))$ and $\gamma(0) = x_0$. As a consequence, we have

$$\Psi(\xi) = \gamma(0) + \int_0^1 \dot{\gamma}(t)\, \mathrm{d}t = x_0 + \sum_{j=1}^{N} \xi_j \int_0^1 Z_j I(\gamma(t;\, \xi))\, \mathrm{d}t\, .$$

This gives

$$(\partial/\partial \xi_k)\,|_{\xi=0}\, \Psi(\xi) = \left(\int_0^1 Z_k I(\gamma(t;\, \xi))\, \mathrm{d}t + \sum_{j=1}^{N} \xi_j \cdot (\partial/\partial \xi_k) \int_0^1 Z_j I(\gamma(t;\, \xi))\, \mathrm{d}t \right) \Bigg|_{\xi=0}$$

$$= \int_0^1 Z_k I(\gamma(t;\, 0))\, \mathrm{d}t = Z_k I(x_0).$$

In particular, if $Z_1 I(x_0), \ldots, Z_N I(x_0)$ are linearly independent, then Ψ is a diffeomorphism from a neighborhood of $\xi = 0$ onto a neighborhood of $\Psi(0) = x_0$. Its inverse function defines the so-called *logarithmic coordinates* around x_0.

Throughout the end of this section, $X_1, \ldots, X_m$ will be a given set of smooth vector fields satisfying hypotheses **(H0)-(H1)-(H2)** in Section 2, whereas $\{\delta_\lambda\}_\lambda$ will be a fixed family of dilations in $\mathbb{R}^N$ as in (2.2). We let $\mathfrak{g} := \mathrm{Lie}\,\{X_1, \ldots, X_m\}$. We explicitly remark that, even if the notation $\mathfrak{g}$ is usually referred to the Lie algebra of a Lie group, no underlying Lie group structure is yet assumed here. Finally, for every $k = 1, \ldots, r$, $Z_1^{(k)}, \ldots, Z_{N_k}^{(k)}$ will be a fixed basis for $W^{(k)}$ (for the definition of $W^{(k)}$, see Section 2). With the notation introduced above, we set $\xi \cdot Z = \sum_{k=1}^{r} \sum_{j=1}^{N_k} \xi_j^{(k)} Z_j^{(k)}$. The next result will be crucial in the sequel.

PROPOSITION 3.1. – *Following the above notation, the map*

$$\mathrm{Exp} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}, \qquad \xi \mapsto \mathrm{Exp}\,(\xi) := \exp\,[\xi \cdot Z](0)$$

is everywhere defined on $\mathbb{R}^N$ *and is a global diffeomorphism of* $\mathbb{R}^N$ *onto itself. Moreover, the component functions of* Exp *are polynomials. The inverse function of* Exp, *which we shall denote by* Log, *has polynomial component functions too.*

We explicitly note that we used a notation slightly different from the usual one for the exponential map related to a Lie group: indeed Exp is here defined on $\mathbb{R}^N$ instead of on an algebra of vector fields.

PROOF. – By definition, we have $\mathrm{Exp}\,(\xi) = \gamma(1)$, where γ solves the Cauchy problem

$$(\mathrm{P})_1 \quad \dot{\gamma}(t) = \sum_{k=1}^{r} \sum_{j=1}^{N_k} \xi_j^{(k)} Z_j^{(k)} I(\gamma(t)), \quad \gamma(0) = 0.$$

We now fix $k \in \{1, \dots, r\}$ and $j \in \{1, \dots, N_k\}$. Following the notation of the stratification in (2.2), the system $(\mathrm{P})_1$ can be rewritten in the form

$$(\mathrm{P})_2 \begin{cases} \dot{\gamma}^{(1)}(t) = A_1^{(1)} \cdot \xi^{(1)}, & \gamma^{(1)}(0) = 0 \\ \dot{\gamma}^{(2)}(t) = A_2^{(2)} \cdot \xi^{(2)} + A_2^{(1)}(\gamma^{(1)}(t)) \cdot \xi^{(1)}, & \gamma^{(2)}(0) = 0 \\ \vdots & \vdots \\ \dot{\gamma}^{(r)}(t) = A_r^{(r)} \cdot \xi^{(r)} + \sum_{k=1}^{r-1} A_r^{(k)}(\gamma^{(1)}(t), \dots, \gamma^{(r-k)}(t)) \cdot \xi^{(k)}, & \gamma^{(r)}(0) = 0. \end{cases}$$

Here, for every $s \leq r$ and $k \leq s$, we have introduced the $N_s \times N_k$ matrix

$$A_s^{(k)} = A_s^{(k)}(x^{(1)}, \dots, x^{(s-k)}) := (a_{s,\,i}^{(k,\,j)}(x))_{1 \leq i \leq N_s,\, 1 \leq j \leq N_k}$$

where $a_{s,\,r}^{(k,\,j)}$ is the δ_λ-homogeneous polynomial of degree $s-k$ as in (2.4). The system $(\mathrm{P})_2$ can be directly integrated starting from the first N_1 scalar equations and then proceeding downwards. The solution $\gamma(t) = \int_0^t \dot{\gamma}(s)\,\mathrm{d}s$ of $(\mathrm{P})_2$ is then given by

$$\begin{cases} \gamma^{(1)}(t) = tA_1^{(1)} \cdot \xi^{(1)}, \\ \gamma^{(2)}(t) = tA_2^{(2)} \cdot \xi^{(2)} + \int_0^t A_2^{(1)}(\gamma^{(1)}(s))\,\mathrm{d}s \cdot \xi^{(1)}, \\ \vdots \\ \gamma^{(r)}(t) = tA_r^{(r)} \cdot \xi^{(r)} + \sum_{k=1}^{r-1} \int_0^t A_r^{(k)}(\gamma^{(1)}(s), \dots, \gamma^{(r-k)}(s))\,\mathrm{d}s \cdot \xi^{(k)}. \end{cases}$$

In particular, we see that $\gamma^{(1)}$ only depends on $\xi^{(1)}$ (polynomially), $\gamma^{(2)}$ depends on $\xi^{(1)}$ and $\xi^{(2)}$ (polynomially), and so on. It is then immediate to recognize that $\mathrm{Exp}(\xi)$ has polynomial component functions. Finally, for any given $\eta \in \mathbb{R}^N$, the equation $\eta = \mathrm{Exp}(\xi)$ can be rewritten as

$$\eta^{(1)} = A_1^{(1)} \cdot \xi^{(1)}, \dots, \eta^{(r)} = A_r^{(r)} \cdot \xi^{(r)} + f^{(r)}(\xi^{(1)}, \dots, \xi^{(r-1)}), \tag{3.9}$$

where $f^{(2)}, \dots, f^{(r)}$ are polynomial functions and the $A_k^{(k)}$'s are constant nonsingular matrices (see also the proof of Proposition 2.3). As a consequence, Exp is bijective and its inverse function has polynomial component functions (as it appears by considering 3.9). ■

We are now in the position to define the group law related to the vector fields $X_1, \dots, X_m$. The following definition is suggested by Remark 2.1.

DEFINITION 3.2. – *If* $X_1, \dots, X_m$ *satisfy hypotheses* **(H0)-(H1)-(H2)**, *we set*

$$x,\ y \in \mathbb{R}^N, \quad x \circ y := \exp[\mathrm{Log}(y)\cdot Z](x). \tag{3.10}$$

At first sight, $\circ$ may seem to depend on the choice of the basis Z for $\mathfrak{g}$. This is not the case, as it will follow when we prove that $\circ$ defines a homogeneous Carnot group structure on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$ (which is fixed and completely determined by the X_j's). Since $X_1, \dots, X_m$ are smooth vector fields, by standard arguments of regular dependence for ODE's, we recognize that the map $\mathbb{R}^N \times \mathbb{R}^N \ni (x, y) \mapsto x \circ y \in \mathbb{R}^N$ is of class C^∞. The following result, together with Remark 2, proves that $\circ$ has in fact polynomial component functions.

THEOREM 3.3. – *With the notation of Definition* 3.2, *we have*

$$\delta_\lambda(x \circ y) = (\delta_\lambda(x)) \circ (\delta_\lambda(y)), \quad \forall \lambda > 0,\ \forall x,\ y \in \mathbb{R}^N. \tag{3.11}$$

PROOF. – First of all, we fix $x \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$, and we show that, if γ is the solution to $\dot\gamma = (\eta\cdot Z)\, I(\gamma)$, $\gamma(0) = x$, then $\mu(t) := \delta_\lambda(\gamma(t))$ is the solution to $\dot\mu = (\delta_\lambda(\eta)\cdot Z)\, I(\mu)$, $\mu(0) = \delta_\lambda(x)$. Indeed,

$$\begin{aligned}\dot\mu = \delta_\lambda(\dot\gamma) &= \sum_{k \le r}\sum_{j \le N_k} \eta_j^{(k)}\, \delta_\lambda(Z_j^{(k)} I(\gamma)) = \sum_{k \le r}\sum_{j \le N_k} \eta_j^{(k)} \lambda^k (Z_j^{(k)} I)(\delta_\lambda(\gamma)) \\ &= \sum_{k \le r}\sum_{j \le N_k} (\delta_\lambda(\eta))_j^{(k)} (Z_j^{(k)} I)(\mu).\end{aligned}$$

Here we used the fact that, for $k \le r$ and $j \le N_k$, $Z_j^{(k)}$ is (as a consequence of hypothesis **(H0)**) a δ_λ-homogeneous vector field of degree k. The above yields

$$\exp[\delta_\lambda(\eta)\cdot Z](\delta_\lambda(x)) = \mu(1) = \delta_\lambda(\gamma(1)) = \delta_\lambda(\exp[\eta\cdot Z](x)). \tag{3.12}$$

If in (3.12), we choose $x = 0$, we obtain

$$\mathrm{Exp}(\delta_\lambda(\eta)) = \delta_\lambda(\mathrm{Exp}(\eta)), \quad \forall \lambda > 0,\ \forall \eta \in \mathbb{R}^N. \tag{3.13}$$

By taking Log at both sides of (3.13), we also get

$$\delta_\lambda(\mathrm{Log}(y)) = \mathrm{Log}(\delta_\lambda(y)), \quad \forall \lambda > 0,\ \forall y \in \mathbb{R}^N. \tag{3.14}$$

Finally, (3.12) and (3.14) give

$$\begin{aligned}\delta_\lambda(x \circ y) = \delta_\lambda(\exp[(\mathrm{Log}\, y)\cdot Z](x)) &= \exp[\delta_\lambda(\mathrm{Log}(y))\cdot Z](\delta_\lambda(x)) \\ &= \exp[\mathrm{Log}(\delta_\lambda(y))\cdot Z](\delta_\lambda(x)) = (\delta_\lambda(x)) \circ (\delta_\lambda(y)).\end{aligned}$$

This completes the proof. ■

By the equalities (3.13) and (3.14), we infer that for every $k \leqslant r$ and $j \leqslant N_k$, the component functions $(\mathrm{Exp})_j^{(k)}$ and $(\mathrm{Log})_j^{(k)}$ are precisely δ_λ-homogeneous polynomials of degree k. Next crucial step is to prove that $\circ$ endows $\mathbb{R}^N$ with a group structure. The main task is to show that $\circ$ is associative. To this end, we need the following result, which is a consequence of the Campbell-Hausdorff formula.

LEMMA 3.4. – *For every $X, Y \in \mathfrak{g}$ there exists $Z \in \mathfrak{g}$ uniquely determined by X and Y such that* (*see also the definition of* $\exp[\cdot]$ *in* (3.2))

$$\exp[Y](\exp[X](x)) = \exp[Z](x), \qquad \forall\, x \in \mathbb{R}^N. \tag{3.15}$$

PROOF. – This is a straightforward corollary of Theorem 5.2 in the Appendix. Indeed, we can choose Z as in (5.2). ∎

We explicitly remark that Z in (3.15) depends only on X and Y and by the structure of $\mathfrak{g}$ and in particular it does not depend on $x \in \mathbb{R}^N$. Lemma 3.4 enables us to endow $\mathfrak{g} = \mathrm{Lie}\{X_1, \ldots, X_m\}$ with a binary operation $\diamond$, as we hereafter describe. Let $X, Y \in \mathfrak{g}$ be arbitrarily given. Let $Z = Z(X, Y)$ be the vector field in $\mathfrak{g}$ uniquely determined by X and Y via formula (5.2) and satisfying (3.15). We set $X \diamond Y := Z(X, Y)$. This defines a binary operation $(X, Y) \mapsto X \diamond Y$ on $\mathfrak{g}$. Since the map

$$\mathbb{R}^N \to \mathfrak{g}, \qquad \xi \mapsto \xi \cdot Z := \sum_{k=1}^{r} \sum_{j=1}^{N_k} \xi_j^{(k)} Z_j^{(k)}$$

is a bijection, we can also define a binary operation on $\mathbb{R}^N$ as follows: for every $\xi, \eta \in \mathbb{R}^N$ we set $\xi * \eta := \zeta$ where ζ is the only vector of $\mathbb{R}^N$ such that $(\xi \cdot Z) \diamond (\eta \cdot Z) = \zeta \cdot Z$. From the above definitions and by (3.15), we have

$$\exp[\eta \cdot Z](\exp[\xi \cdot Z](x)) = \exp[(\xi * \eta) \cdot Z](x), \qquad \forall\, x \in \mathbb{R}^N. \tag{3.16}$$

This has an important consequence: for every fixed $x, y \in \mathbb{R}^N$, by Definition 3.2, we derive

$$x \circ y = \exp[\mathrm{Log}(y) \cdot Z](x) = \exp[\mathrm{Log}(y) \cdot Z](\exp[\mathrm{Log}(x) \cdot Z](0))$$

$$(\text{see (3.16)}) = \exp[(\mathrm{Log}(x) * \mathrm{Log}(y)) \cdot Z](0) = \mathrm{Exp}(\mathrm{Log}(x) * \mathrm{Log}(y)).$$

Then for every $x, y \in \mathbb{R}^N$ and for every $\xi, \eta \in \mathbb{R}^N$

$$\mathrm{Log}(x \circ y) = \mathrm{Log}(x) * \mathrm{Log}(y), \qquad \mathrm{Exp}(\xi * \eta) = \mathrm{Exp}(\xi) \circ \mathrm{Exp}(\eta). \tag{3.17}$$

THEOREM 3.5. – *Let $\circ$ be the composition law in Definition* 3.2. *Then* $(\mathbb{R}^N, \circ)$ *is a group.*

PROOF. – *Identity element.* We prove that $0 \in \mathbb{R}^N$ is such that $x \circ 0 = 0 \circ x = x$ for every $x \in \mathbb{R}^N$. Indeed, $\mathrm{Exp}(0) = \mathrm{Log}(0) = 0$ yields

$$x \circ 0 = \exp[\mathrm{Log}(0)\cdot Z](x) = \exp[0\cdot Z](x) = x\,;$$

$$0 \circ x = \exp[\mathrm{Log}(x)\cdot Z](0) = \mathrm{Exp}(\mathrm{Log}\, x) = x.$$

Inverse element. We show that

$$x \circ \mathrm{Exp}(-\mathrm{Log}(x)) = 0 = \mathrm{Exp}(-\mathrm{Log}(x)) \circ x, \qquad \forall\, x \in \mathbb{R}^N. \tag{3.18}$$

We only prove the first equality, since the second one is analogous. From (3.5), we get

$$\begin{aligned} x \circ \mathrm{Exp}(-\mathrm{Log}(x)) &= \exp[\mathrm{Log}(\mathrm{Exp}(-\mathrm{Log}(x)))\cdot Z](x) = \exp[-\mathrm{Log}(x)\cdot Z](x) \\ &= \exp[-\mathrm{Log}(x)\cdot Z](\exp[\mathrm{Log}(x)\cdot Z](0)) = 0. \end{aligned}$$

In particular, the inverse of $x \in \mathbb{R}^N$ is given by $x^{-1} = \mathrm{Exp}(-\mathrm{Log}(x))$.

Associativity. Let $x, y, z \in \mathbb{R}^N$ be fixed. We have to prove that $(x \circ y) \circ z = x \circ (y \circ z)$. By (3.16) we infer

$$(x \circ y) \circ z = \exp[\mathrm{Log}(z)\cdot Z](\exp[\mathrm{Log}(y)\cdot Z](x)) = \exp[(\mathrm{Log}(y) * \mathrm{Log}(z))\cdot Z](x),$$

whereas, from (3.17) we derive

$$x \circ (y \circ z) = \exp[\mathrm{Log}(y \circ z)\cdot Z](x) = \exp[(\mathrm{Log}(y) * \mathrm{Log}(z))\cdot Z](x).$$

This completes the proof. ∎

REMARK 3.6. – The associativity of $\circ$ could also be deduced once it is known that $\diamond$ is an associative operation. Since the associativity of $\diamond$, the so-called Campbell-Hausdorff operation, is a profound result (see also the Appendix for related references), we preferred to provide an argument *only relying on the existence* of the operation $\diamond$ rather than on its non-trivial property of being associative. On the contrary, we explicitly remark that we are now able to derive that $\diamond$ is associative *as a consequence* of the associativity of $\circ$. Indeed, the identities in (3.17) show that the binary operation $*$ on $\mathbb{R}^N$ (and consequently the binary operation $\diamond$ on $\mathfrak{g}$) defines a Lie group structure isomorphic to $(\mathbb{R}^N, \circ)$.

By Theorem 3.5 and the smoothness of $\circ$, $\mathbb{G} := (\mathbb{R}^N, \circ)$ is a Lie group. It is natural to ask if $\mathrm{Lie}\{X_1, \ldots, X_m\}$ coincides with the Lie algebra of $\mathbb{G}$.

THEOREM 3.7. – *Each of the fields* $Z_j^{(k)}$ $(k \leq r,\, j \leq N_k)$ *is left-invariant on* $(\mathbb{R}^N, \circ)$.

PROOF. – Let $\alpha \in \mathbb{R}^N$ and $u \in C^\infty(\mathbb{R}^N)$ be fixed. We have to prove that for every $k \leq r$ and $j \leq N_k$ it holds $Z_j^{(k)}(u(\alpha \circ x)) = (Z_j^{(k)} u)(\alpha \circ x)$. From (3.6) and

(3.8), for every smooth f we infer

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0}(f(\exp[tZ_j^{(k)}](x))) = (Z_j^{(k)}f)(x). \tag{3.19}$$

On the other side, if $e_j^{(k)}$ denotes the j-th coordinate unit vector of $\mathbb{R}^{N_k}$ and if we set $\xi_j^{(k)} := (0^{(1)}, \ldots, e_j^{(k)}, \ldots, 0^{(r)})$, we have (noting that $tZ_j^{(k)} = t\xi_j^{(k)}\cdot Z$)

$$(Z_j^{(k)}f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0}(f(\exp[t\xi_j^{(k)}\cdot Z](x))) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0}(f(x\circ \operatorname{Exp}(t\xi_j^{(k)}))).$$

In particular, if we choose $f = u(\alpha\circ\cdot)$, we obtain (by the associativity of $\circ$)

$$\begin{aligned} Z_j^{(k)}(u(\alpha\circ x)) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0}\{u(\alpha\circ(x\circ \operatorname{Exp}(t\xi_j^{(k)})))\} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0}\{u((\alpha\circ x)\circ \operatorname{Exp}(t\xi_j^{(k)}))\} = (Z_j^{(k)}u)(\alpha\circ x). \end{aligned}$$

This completes the proof. ∎

COROLLARY 3.8. – *The Lie algebra of* $(\mathbb{R}^N, \circ)$ *coincides with* $\mathrm{Lie}\{X_1, \ldots, X_m\}$

PROOF. – By Theorem 3.7, it follows that $\mathfrak{g} = \mathrm{Lie}\{X_1, \ldots, X_m\}$ is contained in the Lie algebra of $\mathbb{G}$, which is N-dimensional since the underlying manifold of $\mathbb{G}$ is $\mathbb{R}^N$. On the other side, $\mathfrak{g}$ is also N-dimensional (see Proposition 2.3), whence the Lie algebra of $\mathbb{G}$ coincides with $\mathfrak{g}$. ∎

We are now in the position to state and complete the proof of our main result.

THEOREM 3.9. – *Let* $X_1, \ldots, X_m$ *be smooth vector fields satisfying hypotheses* **(H0)-(H1)-(H2)** *of Section* 2. *Let* $\{\delta_\lambda\}_\lambda$ *be the family of dilations defined in* (2.2). *Finally, let* $\circ$ *be the operation on* $\mathbb{R}^N$ *introduced in Definition* 3.2. *Then* $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ *is a homogeneous Carnot group of step* r *and with* m *generators. Moreover, the Lie algebra of* $\mathbb{G}$ *coincides with* $\mathrm{Lie}\{X_1, \ldots, X_m\}$ *and* $\sum_{j=1}^m X_j^2$ *is a sub-Laplacian on* $\mathbb{G}$.

PROOF. – By Theorem 3.5, $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a Lie group. By Theorem 3.3, $\{\delta_\lambda\}_\lambda$ is a family of Lie group automorphisms of $\mathbb{G}$. From Proposition 2.3, we directly infer that $m = N_1$ (where N_1 is as in the definition of δ_λ). By Remark 3.8, the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ coincides with $\mathrm{Lie}\{X_1, \ldots, X_m\}$. Let $Y_1, \ldots, Y_m$ be the vector fields of $\mathfrak{g}$ that agree with the first m partial derivatives at the origin. Since $\mathfrak{g} = \mathrm{Lie}\{X_1, \ldots, X_m\}$, every Y_j is a linear combination of commutators of $X_1, \dots, X_m$. Since all the X_j's and all the Y_j's are δ_λ-homogeneous of degree 1, it is not difficult to show that the Y_j are necessarily linear combinations only of $X_1, \dots, X_m$. Since $Y_1, \dots, Y_m$ are linearly independent, this proves that $\operatorname{span}\{Y_1, \dots, Y_m\} = \operatorname{span}\{X_1, \dots, X_m\}$. As a consequence, we have $\operatorname{Lie}\{Y_1, \dots, Y_m\} = \operatorname{Lie}\{X_1, \dots, X_m\} = \mathfrak{g}$. This completes the proof. ■

4. – Examples.

In this section, we show several examples of sets of vector fields satisfying the hypotheses **(H0)-(H1)-(H2)** in Section 2. We then construct the related homogeneous Carnot groups, as described in Section 3. Given $n \in \mathbb{N}$, $\mathbb{B}_n$ will denote the following $n \times n$ (nilpotent of step n) matrix

$$\mathbb{B}_n := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \tag{4.1}$$

4.1. *The Laplace operator.*

The simplest examples of sub-Laplacians are the constant coefficient elliptic operators. The only homogeneous Carnot group on $\mathbb{R}^N$ of step 1 is the usual additive group on $\mathbb{R}^N$. If δ_λ denotes the Euclidean dilation on $\mathbb{R}^N$, a set of vector fields satisfying the hypotheses **(H0)-(H1)-(H2)** is necessarily given by $\{X_j\}_{j \leq N}$, where $X_j = \sum_{i=1}^N a_{i,j}\,\partial_i$ and $A = (a_{i,j})_{i,j}$ is a non-singular $N \times N$ matrix, whence $\sum_{j=1}^N X_j^2$ is a constant coefficient elliptic operator. Given $\xi, x \in \mathbb{R}^N$, we have $\exp[\xi](x) := \exp\left[\sum_{j=1}^N \xi_j X_j\right](x) = \gamma(1)$, where $\dot{\gamma}(r) = A \cdot \xi$ and $\gamma(0) = x$. This yields $\exp[\xi](x) = x + A \cdot \xi$, $\operatorname{Exp}(\xi) = A \cdot \xi$, $\operatorname{Log}(y) = A^{-1} \cdot y$ and consequently $x \circ y = \exp[\operatorname{Log}(y)](x) = x + y$.

4.2. *The Kohn Laplacians.*

We now consider the group $\mathbb{H}^n := \mathbb{R}^{2n+1}$ whose points will be denoted by $z = (x, y, t)$, $x, y \in \mathbb{R}^N$ and $t \in \mathbb{R}$. We also set $X_j := \partial_{x_j} + 2y_j\,\partial_t$, $Y_j := \partial_{y_j} - 2x_j\,\partial_t$ $(j = 1, \dots, n)$. If we equip $\mathbb{H}^n$ with the dilations $\delta_\lambda(z) = (\lambda x, \lambda y, \lambda^2 t)$, it is easily verified that the above $2n$ vector fields fulfill the hypotheses **(H0)-(H1)-(H2)**. We let $T := [X_j, Y_j] = -4\,\partial_t$. Fixed $\zeta = (\xi, \eta, \tau)$, $z = (x, y, t) \in \mathbb{H}^n$, we have $\exp[\zeta](z) := \exp\left[\sum_{j=1}^n (\xi_j X_j + \eta_j Y_j) + \tau T\right](z) = (\mu(1), \nu(1), \varrho(1))$, where

$$(\dot{\mu}, \dot{\nu}, \dot{\varrho})(r) = (\xi, \eta, -4\tau + 2\langle \nu(r), \xi \rangle - 2\langle \mu(r), \eta \rangle), \quad (\mu, \nu, \varrho)(0) = (x, y, t).$$

This yields $\exp[\zeta](z) = (x+\xi,\, y+\eta,\, t-4\tau+2\langle y, \xi\rangle - 2\langle x, \eta\rangle)$, whence $\mathrm{Exp}(\zeta) = (\xi, \eta, -4\tau)$, $\mathrm{Log}(z') = (x', y', -t'/4)$. If we fix $z, z' \in \mathbb{H}^n$, we obtain

$$z \circ z' = \exp[\mathrm{Log}(z')](z) = (x+x',\, y+y',\, t+t'+2\langle y, x'\rangle - 2\langle x, y'\rangle). \tag{4.2}$$

The group $(\mathbb{H}^n, \circ)$ is the well-known *Heisenberg group*, and the sub-Laplacian $\Delta_{\mathbb{H}^n} := \sum_{j=1}^{n} (X_j^2 + Y_j^2)$ is the *Kohn Laplacian.*

4.3. *Bony-type sub-Laplacians.*

In [9, Remarque 3.1], J.M. Bony refers to the following operator L on $\mathbb{R}^{1+N}$ (whose points are denoted by $(t, x_1, \dots, x_N)$)

$$L = \left(\frac{\partial}{\partial t}\right)^2 + \left(t\frac{\partial}{\partial x_1} + t^2\frac{\partial}{\partial x_2} + \dots + t^N\frac{\partial}{\partial x_N}\right)^2$$

as an example of a sum of squares satisfying Hörmander condition but nevertheless with a quadratic form «very degenerate». Clearly L is not a sub-Laplacian on any Carnot group, since the vector field $\sum_{j=1}^{N} t^j\, \partial/\partial x_j$ vanishes on the hyperplane $t=0$. It is however sufficient to add a new coordinate in order to *lift* L to a sub-Laplacian. We indeed consider on $\mathbb{R}^{2+N}$ (whose points are denoted by (t, s, x), $t, s \in \mathbb{R}$, $x \in \mathbb{R}^N$) the following operator $\mathcal{L} := T^2 + S^2$ where

$$T := \partial_t, \qquad S := \partial_s + t\partial_{x_1} + \frac{t^2}{2!}\partial_{x_2} + \dots + \frac{t^N}{N!}\partial_{x_N}.$$

If we equip $\mathbb{R}^{2+N}$ with the family of dilations defined by

$$\delta_\lambda(t, s, x) := (\lambda t, \lambda s, \lambda^2 x_1, \lambda^3 x_2, \dots, \lambda^{N+1} x_N),$$

it is readily verified that T and S are δ_λ-homogeneous of degree 1 and linearly independent, whence hypothesis **(H0)** is fulfilled. For every $k = 1, \dots, N$, we then consider the vector field

$$X_k := [\underbrace{T, [T, \dots [T}_{k \text{ times}}, S]\dots]] = \partial_{x_k} + t\partial_{x_{k+1}} + \dots + \frac{t^{N-k}}{(N-k)!}\partial_{x_N}.$$

With the notation of Section 2, we have $W^{(1)} = \mathrm{span}\{T, S\}$ and (for $k = 1, \dots, N$) $W^{(k+1)} = \mathrm{span}\{X_k\}$. It is then easy to recognize that the hypothesis **(H1)** is satisfied. Finally, we have $\dim(\mathrm{Lie}\{T, S\}\, I(0)) = 2+N$, whence also the hypothesis **(H2)** holds. As a consequence $\mathcal{L}$ is a sub-Laplacian on a suitable homogeneous Carnot group $(\mathbb{G}, \circ)$ on $\mathbb{R}^{2+N}$, with step $1+N$ and with 2 generators. We now turn to construct the group multiplication $\circ$ on $\mathbb{G}$, as described in Section 3. Let $\alpha, \beta \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ be fixed. We have

$$\alpha T + \beta S + \sum_{k=1}^{N} \xi_k X_k = \left(\alpha, \beta, \left(\beta \quad t^j/j! + \sum_{k=1}^{j} \xi_k t^{j-k}/(j-k)! \right)_{j=1,\dots,N} \right).$$

This yields $\exp[\alpha, \beta, \xi](t, s, x) := \exp\left[\alpha T + \beta S + \sum_{k=1}^{N} \xi_k X_k \right](t, s, x) = (\tau, \sigma, \gamma)(1)$, where

$$\begin{cases} \dot\tau(r) = \alpha, \ \tau(0) = t, \quad \dot\sigma(r) = \beta, \ \sigma(0) = s, \\ \dot\gamma_j(r) = \beta \tau^j(r)/j! + \sum_{k=1}^{j} \xi_k \tau^{j-k}(r)/(j-k)!, \ \gamma_j(0) = x_j \quad (j = 1, \dots, N). \end{cases}$$

From a direct integration, it follows that $\exp[\alpha, \beta, \xi](t, s, x)$ is given by

$$\left(\alpha + t, \beta + s, x_j + \beta \frac{(\alpha+t)^{j+1} - t^{j+1}}{(j+1)!\, \alpha} + \sum_{k=1}^{j} \xi_k \frac{(\alpha+t)^{j-k+1} - t^{j-k+1}}{(j-k+1)!\, \alpha} \quad (j = 1, \dots, N) \right).$$

We remark that the case $\alpha = 0$ is also contained, since all the incremental ratios in the above expression are defined in a natural way even when $\alpha = 0$. We now define the following matrices

$$F(\alpha) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\alpha}{2!} & 1 & 0 & \cdots & 0 \\ \frac{\alpha^2}{3!} & \frac{\alpha}{2!} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{\alpha^{N-1}}{N!} & \frac{\alpha^{N-2}}{(N-1)!} & \cdots & \frac{\alpha}{2!} & 1 \end{pmatrix}, \quad V(\alpha) := \begin{pmatrix} \frac{\alpha}{2!} \\ \frac{\alpha^2}{3!} \\ \frac{\alpha^3}{4!} \\ \vdots \\ \frac{\alpha^N}{(N+1)!} \end{pmatrix}, \quad U(\alpha) := \begin{pmatrix} \alpha \\ \frac{\alpha^2}{2!} \\ \frac{\alpha^3}{3!} \\ \vdots \\ \frac{\alpha^N}{N!} \end{pmatrix}$$

$\widetilde{F}(\alpha, t) := \alpha^{-1}((\alpha + t)\, F(\alpha + t) - tF(t))$, $\widetilde{V}(\alpha, t) := \alpha^{-1}((\alpha + t)\, V(\alpha + t) - tV(t))$. It then holds

$$\exp[\alpha, \beta, \xi](t, s, x) = (\alpha + t, \beta + s, x + \widetilde{F}(\alpha, t) \cdot \xi + \beta \widetilde{V}(\alpha, t)),$$

$$\operatorname{Exp}(\alpha, \beta, \xi) = (\alpha, \beta, F(\alpha) \cdot \xi + \beta V(\alpha)),$$

$$\operatorname{Log}(\tau, \sigma, y) = (\tau, \sigma, F^{-1}(\tau) \cdot (y - \sigma V(\tau)).$$

Let (t, s, x) and $(\tau, \sigma, y) \in \mathbb{R}^{2+N}$ be given. Then, we have

$$(t, s, x) \circ (\tau, \sigma, y) = (\tau + t, \sigma + s, x + \widetilde{F}(\tau, t) \cdot F^{-1}(\tau) \cdot (y - \sigma V(\tau)) + \sigma \widetilde{V}(\tau, t)).$$

If we now prove that the following identities hold

$$\widetilde{F}(\tau,\, t) = \exp(t\mathbb{B}_N)\cdot F(\tau), \qquad \widetilde{V}(\tau,\, t) - \exp(t\mathbb{B}_N)\cdot V(\tau) = U(t), \tag{4.3}$$

then the explicit form of the multiplication $\circ$ turns out to be:

$$\begin{aligned}(t, s, x)\circ(\tau, \sigma, y) &= (\tau + t, \sigma + s, x + \exp(t\mathbb{B}_N)\cdot y + \sigma U(t)) \\ &= \left(\tau + t, \sigma + s, x_1 + y_1 + \sigma t, \dots, x_N + \sum_{k=1}^{N} y_k \frac{t^{N-k}}{(N-k)!} + \sigma\frac{t^N}{N!}\right).\end{aligned} \tag{4.4}$$

The first identity in (4.3) follows by proving that, for every $i, j \in \{1, \dots, N\}$ with $i \geqslant j$, one has

$$\frac{(\tau + t)^{i-j+1} - t^{i-j+1}}{(i-j+1)!\,\tau} = \sum_{k=j}^{i} \frac{t^{i-k}}{(i-k)!}\cdot\frac{\tau^{k-j}}{(k-j+1)!},$$

which readily follows by applying Newton binomial formula to the left-hand side; the second identity is equivalent to

$$\frac{t^i}{i!} = \frac{(\tau + t)^{i+1} - t^{i+1}}{(i+1)!\,\tau} - \sum_{k=1}^{i} \frac{\tau^k}{(k+1)!}\cdot\frac{t^{i-k}}{(i-k)!}, \qquad i = 1\dots, N,$$

which can be proved analogously.

4.4. *Kolmogorov-type sub-Laplacians.*

We describe an example of sub-Laplacians on Carnot groups arising in the theory of diffusion and Brownian motions. These groups have been introduced in [31] for the study of a class of hypoelliptic ultraparabolic operators including the classical model operators of Kolmogorov-Fokker-Planck. The group law in [31] was suggested by the structure of the fundamental solution of the operator in $\mathbb{R}^3$ $\partial_{x_1}^2 + x_1\partial_{x_2} - \partial_{x_3}$ given by Kolmogorov in [30]. For this reason, we shall call our sub-Laplacians *of Kolmogorov-type*. We consider the following sum of squares in $\mathbb{R}^{N+1}$

$$\mathcal{L} = \sum_{j=1}^{p_1} (\partial_{x_j})^2 + (\partial_t - \langle x,\, B\cdot\nabla\rangle)^2 = \sum_{j=1}^{p_1} X_j^2 + Y^2,$$

where $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})^T$, whereas B is a $N \times N$ matrix of the type

$$B = \begin{pmatrix} 0 & B^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & B^{(2)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & B^{(r)} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Here, for $j = 1 \ldots, r$, $B^{(j)}$ is a matrix of order $p_j \times p_{j+1}$ and with rank p_{j+1}, $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_{r+1} \geqslant 1$ and $p_1 + p_2 + \ldots + p_{r+1} = N$. We claim that (by a suitable choice of a basis for Lie $\{X_1, \ldots, X_{p_1}, Y\}$) the group law introduced in [31] can be easily obtained as we showed in Section 3. To this purpose, we equip $\mathbb{R}^{N+1}$ with the following family of dilations

$$\delta_\lambda(t, x) := (\lambda t, \lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^{r+1} x^{(r+1)}),$$

where, for every $j = 1, \ldots, r+1$, $x^{(j)} \in \mathbb{R}^{p_j}$. Furthermore, we consider the following linearly independent vector fields (for $h = 1, \ldots, p_1$)

$$X_h^{(1)} := \partial / \partial x_h^{(1)},$$

$$Y = \partial_t - \langle x, B \cdot \nabla \rangle = \partial_t - \sum_{j \leqslant r} \sum_{i \leqslant p_{j+1}} \sum_{k \leqslant p_j} B_{k,i}^{(j)} x_k^{(j)} \partial / \partial x_i^{(j+1)}.$$

It is easily checked that these fields are δ_λ-homogeneous of degree 1. In particular hypothesis **(H0)** holds. For $h = 1, \ldots, p_1$, we also have

$$X_h^{(2)} := [Y, X_h^{(1)}] = \sum_{i=1}^{p_2} B_{h,i}^{(1)} \partial / \partial x_i^{(2)}.$$

As a consequence, since, by assumption, the matrix $B^{(1)}$ has rank p_2, this implies that $W^{(2)} = \operatorname{span}\{X_h^{(2)} \mid h = 1, \ldots, p_1\}$ contains exactly p_2 linearly independent vector fields. In particular, since the fields in $W^{(2)}$ have constant coefficients, for every $j = 1, \ldots, p_2$, we have $\partial / \partial x_j^{(2)} \in W^{(2)}$. Let now $k \in \{3, \ldots, r+1\}$ be fixed. For $h = 1, \ldots, p_1$, we have

$$X_h^{(k)} := [Y, X_h^{(k-1)}] = \sum_{i=1}^{p_k} (B^{(1)} \cdot B^{(2)} \cdots B^{(k-1)})_{h,i} \partial / \partial x_i^{(k)}.$$

Since the matrix $B^{(1)} \cdot B^{(2)} \cdots B^{(k-1)}$ has rank p_k, $W^{(k)} = \operatorname{span}\{X_h^{(k)} \mid h = 1, \ldots, p_1\}$ contains exactly p_k linearly independent vector fields and, as previously observed, this implies that for every $j = 1, \ldots, p_k$, we have $\partial / \partial x_j^{(k)} \in W^{(k)}$.

From what has just been proved, it follows

$$\left.\begin{array}{l}\dim(W^{(k)})=\dim(\operatorname{span}\{X_1,\dots,X_{p_1},Y\})=1+p_1\\ \dim(W^{(k)})=p_k,\forall\, k=2,\dots,r+1\end{array}\right\}\Rightarrow \textbf{(H1)}\text{ holds;}$$
$$\dim(\operatorname{Lie}\{X_1,\dots,X_{p_1},Y\})=1+p_1+\ldots+p_{r+1}=1+N\Rightarrow \textbf{(H2)}\text{ holds.}$$

For $k=1,\dots,r+1$ and $j=1,\dots,p_k$, we set $Z_j^{(k)}:=\partial/\partial x_j^{(k)}$. If $t,\tau\in\mathbb{R}$ and $x,\xi\in\mathbb{R}^N$ are fixed, we have $\exp[\tau,\xi](t,x):=\exp\left[\tau Y+\sum_{k=1}^{r+1}\sum_{j=1}^{p_k}\xi_j^{(k)}Z_j^{(k)}\right](t,x)=(\mu(1),\gamma(1))$, where

$$\dot\mu(r)=\tau,\ \ \mu(0)=t;\qquad \dot\gamma(r)=\xi-\tau B^T\cdot\gamma(r),\ \ \gamma(0)=x.$$

This yields

$$\exp[\tau,\xi](t,x)=\left(\tau+t,\ \exp(-\tau B^T)\cdot x+\int_0^1\exp(-\tau(1-r)\,B^T)\cdot\xi\,\mathrm{d}r\right),$$

$$\operatorname{Exp}(\tau,\xi)=\left(\tau,\ \int_0^1\exp(-\tau(1-r)\,B^T)\cdot\xi\,\mathrm{d}r\right),$$

$$\operatorname{Log}(s,y)=\left(s,\left(\int_0^1\exp(-s(1-r)\,B^T)\,\mathrm{d}r\right)^{-1}\cdot y\right).$$

As a consequence,

$$(t,x)\circ(s,y)=(t+s,\ y+\exp(-sB^T)\cdot x). \tag{4.5}$$

We explicitly remark that this is the same group multiplication found in [31]. $(\mathbb{R}^{N+1},\circ)$ is a homogeneous Carnot group of step $r+1$ and with $1+p_1$ generators. We finally observe that, if $r=N-1$, $p_{r+1}=p_j=1$ and $B^{(j)}=(1)$ for every $j=1,\dots,r$, the second-order differential operator

$$(-\partial_t+x_1\,\partial_{x_2}+x_2\,\partial_{x_3}+\ldots+x_{N-1}\,\partial_{x_N})^2+(\partial_{x_1})^2$$

is a sub-Laplacian of Kolmogorov-type on $\mathbb{R}^{1+N}$ analogous to the one studied in [2] (see the following Example 4.5).

4.5. *Sub-Laplacians related to Carnot groups arising in Control Theory.*

In this subsection, we discuss an example of homogeneous Carnot group arising from Control Theory, while referring to [2] for a description of the relevance of this example in that context. In $\mathbb{R}^N$ we consider the following vector fields

$$X_1:=\partial_1+x_2\,\partial_3+x_3\,\partial_4+\ldots+x_{N-1}\,\partial_N,\qquad X_2:=\partial_2.$$

For every $k = 3, \ldots, N$, we have $X_k := [X_{k-1}, X_1] = \partial_k$ whence it is readily verified that X_1 and X_2 fulfill hypotheses **(H0)-(H1)-(H2)**, with respect to the family of dilations

$$\delta_\lambda(x_1, x_2, x_3, \ldots, x_N) := (\lambda x_1, \lambda x_2, \lambda^2 x_3, \ldots, \lambda^{N-1} x_N).$$

As a consequence, $\mathcal{L} = X_1^2 + X_2^2$ is a sub-Laplacian on a suitable homogeneous Carnot group $(\mathbb{G}, \circ)$ on $\mathbb{R}^N$, with step $N-1$ and with 2 generators. In [2] it is given a representation of $\mathbb{G}$ by means of matrices of the following form

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & \cdots & x_N \\ 0 & 1 & x_1 & \dfrac{x_1^2}{2!} & \cdots & \dfrac{x_1^{N-2}}{(N-2)!} \\ 0 & 0 & 1 & x_1 & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \dfrac{x_1^2}{2!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & x_1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \equiv (x_1, x_2, \ldots, x_N) \in \mathbb{G},$$

whereas the Lie group law is given by the matrix product. We hereafter show how to obtain the composition law following the lines described in Section 3. Let $\xi \in \mathbb{R}^N$ be fixed. We have

$$\sum_{k=1}^N \xi_k X_k = (\xi_1, \xi_2, \xi_3 + \xi_1 x_2, \ldots, \xi_N + \xi_1 x_{N-1}) = \xi + \xi_1 H \cdot x,$$

where H is the following $N \times N$ matrix

$$H := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{B}_{N-2} \end{pmatrix}.$$

This gives $\exp[\xi](x) := \exp\left[\sum_{k=1}^N \xi_k X_k\right](x) = \gamma(1)$ where $\dot\gamma(r) = \xi + \xi_1 H \cdot \gamma(r)$, $\gamma(0) = x$, whence

$$\gamma(r) = \exp(\xi_1 r H) \cdot x + \int_0^r \exp(\xi_1 (r-t) H) \cdot \xi \, \mathrm{d}t.$$

In particular,

$$\exp[\xi](x) = \exp(\xi_1 H) \cdot x + \int_0^1 \exp(\xi_1 (1-t) H) \cdot \xi \, \mathrm{d}t,$$

$$\mathrm{Exp}(\xi) = \int_0^1 \exp(\xi_1 (1-t) H) \cdot \xi \, \mathrm{d}t.$$

It is straightforward to recognize that for every $\varrho \in \mathbb{R}$, we have

$$\exp(\varrho H) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(\varrho \mathbb{B}_{N-1}) \end{pmatrix}.$$

Given $y = (y_1, \widehat{y}) \in \mathbb{R}^N$, the equation $y = \mathrm{Exp}(\xi)$ is equivalent to the following system (setting $\xi = (\xi_1, \widehat{\xi}) \in \mathbb{R}^N$)

$$y_1 = \xi_1, \qquad \widehat{y} = \int_0^1 \exp(\xi_1(1-t)\,\mathbb{B}_{N-1})\cdot\widehat{\xi}\,\mathrm{d}t\,.$$

As a consequence,

$$\mathrm{Log}(y) = \left(y_1, \left(\int_0^1 \exp(y_1(1-t)\,\mathbb{B}_{N-1})\,\mathrm{d}t \right)^{-1} \cdot \widehat{y} \right).$$

For any fixed $x,\ y \in \mathbb{R}^N$ this gives

(4.6) $$x \circ y = \exp[\mathrm{Log}(y)](x) =$$

$$y + \exp(y_1 H)\cdot x = (y_1 + x_1, \widehat{y} + \exp(y_1\,\mathbb{B}_{N-1})\cdot\widehat{x}) =$$

$$\left(y_1 + x_1,\, y_2 + x_2,\, y_3 + x_3 + y_1 x_2,\, \ldots,\, y_N + \sum_{j=2}^{N} \frac{y_1^{N-j}}{(N-j)!} x_j \right).$$

5. – Appendix. The Campbell-Hausdorff formula.

The main aim of the Appendix is to sketch a proof of Lemma 3.4. We show in details how to reduce to apply a general version of the Campbell-Hausdorff formula for formal power series, proved in [15]. Before doing this, we recall some references on the Campbell-Hausdorff formula. Roughly speaking, if X and Y are two non-commuting indeterminates, the (Baker-)Campbell-(Dynkin-)Hausdorff formula states that «$\log(\exp(X)\exp(Y))$» can be expressed in terms of an infinite sum of iterated commutators of X and Y. This statement can be made precise in many contexts, such as for formal power series, for matrix algebras, for general normed Banach algebras, for finite-dimensional Lie groups, for solutions of differential equations, etc. Classical references on this formula are Bourbaki [10], Hausner-Schwartz [25], Hochschild [27], Jacobson [29], Varadarajan [40]. The applications of this tool to Analysis are discussed for example in [28, 37, 41]. We would also like to cite a few papers concerning with remarkable applications of the Campbell-Hausdorff formula, referring the reader to the therein references for further details: [14, 16, 35, 36, 38, 39]. We finally recall the paper by Grayson&Grossman [23]: this paper, besides being concerned with the Campbell-Hausdorff formula, also contains a remarkable algorithm which permits to construct explicit models for every free nilpotent Lie algebra. It is interesting for the aim of our paper to notice that the models in [23] are given by the Lie algebras generated by certain vector fields on $\mathbb{R}^N$; these fields are homogeneous of degree 1 with respect to a suitable family of dilations as in (2.2) and they also satisfy hypotheses **(H0)-(H1)-(H2)** of Section 2.

Since in the previous sections we were mainly concerned with the Lie groups and Lie algebras setting, we briefly recall how the Campbell-Hausdorff formula naturally arises in that context. Let $(\mathfrak{f}, [\cdot, \cdot])$ be an abstract nilpotent finite-dimensional Lie algebra. For $X, Y \in \mathfrak{f}$ we set

$$X \diamond Y := \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_i + q_i \geq 1 \\ 1 \leq i \leq n}} \frac{(\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \ldots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n - 1} Y}{\left(\sum_{j=1}^{n} (p_j + q_j) \right) p_1! \, q_1! \ldots p_n! \, q_n!}$$

$$(5.1) \qquad = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X,[X, Y]] - \frac{1}{12}[Y,[X, Y]]$$

$$- \frac{1}{48}[Y,[X,[X, Y]]] - \frac{1}{48}[X,[Y,[X, Y]]] + \{\text{brackets of height} \geq 5\}.$$

Since $\mathfrak{f}$ is nilpotent, (5.1) is a finite sum and $\diamond$ determines a binary operation on $\mathfrak{f}$, which is defined by a universal sum of Lie monomials. We call $\diamond$ the *Campbell-Hausdorff operation* on $\mathfrak{f}$. The most relevant property of $\diamond$ is its associativity. As stated in Section 3, the associativity of $\diamond$ is a non-trivial result and is a consequence of abstract profound results which we now recall. Since $\mathfrak{f}$ is finite-dimensional, then (by the Third Fundamental Theorem of Lie, see [40, Theorem 3.15.1]) there exists a connected and simply connected Lie group $(\mathbb{F}, *)$ whose Lie algebra is isomorphic to $\mathfrak{f}$. The following theorem then shows that $\diamond$ defines a Lie group structure on $\mathfrak{f}$.

THEOREM 5.1 ([13], Theorem 1.2.1). – *Let* $(\mathbb{F}, *)$ *be a connected and simply connected Lie group. Suppose that the Lie algebra* $\mathfrak{f}$ *of* $\mathbb{F}$ *is nilpotent. Then* $\diamond$ *defines a Lie group structure on* $\mathfrak{f}$ *and* $\operatorname{Exp} : (\mathfrak{f}, \diamond) \to (\mathbb{F}, *)$ *is a Lie group isomorphism. In particular, if* Log *is the inverse function of* Exp, *the following* Campbell-Hausdorff formula for Lie groups *holds*

$$\operatorname{Log}(\operatorname{Exp}(X) * \operatorname{Exp}(Y)) = X \diamond Y, \qquad \forall X, Y \in \mathfrak{f}.$$

To end this Appendix, we turn to the proof of Lemma 3.4. Throughout the sequel, $X_1, \ldots, X_m$ will be smooth vector fields satisfying hypotheses **(H0)-(H1)-(H2)** of Section 2. We set $\mathfrak{g} = \operatorname{Lie}\{X_1, \ldots, X_m\}$. We recall that r is the step of nilpotence of $\mathfrak{g}$. The notation in Section 3 will also be used.

THEOREM 5.2. – *Let $X, Y \in \mathfrak{g}$ be fixed. Let Z be the differential operator defined by the formal expansion*

$$\sum_{j=1}^{r} \frac{(-1)^{j+1}}{j} \left(\sum_{k_1+k_2=1}^{r} \frac{X^{k_2} Y^{k_1}}{k_1 ! k_2 !} \right)^{j} = Z + \tag{5.2}$$

$$\left\{ \text{summands of the type } c \cdot Y^{y_n} X^{x_n} \cdots Y^{y_1} X^{x_1} \text{ with } \sum_{i=1}^{n} (y_i + x_i) > r \right\}.$$

Then, Z turns out to be a vector field belonging to Lie$\{X, Y\}$ (*hence in* $\mathfrak{g}$) *such that*

$$\exp[Y](\exp[X](x)) = \exp[Z](x), \qquad \forall x \in \mathbb{R}^N. \tag{5.3}$$

PROOF. – Throughout the end of the proof, $X, Y \in \mathfrak{g}$ are fixed. We begin by noticing that, arguing as in the proof of Proposition 3.1, we easily recognize that the map $\mathbb{R} \times \mathbb{R}^N \ni (t, x) \mapsto \exp[tX](x)$ has polynomial component functions. Moreover we have

$$\sum_{i=1}^{n} (y_i + x_i) > r \;\Rightarrow\; Y^{y_n} X^{x_n} \cdots Y^{y_1} X^{x_1} I \equiv 0. \tag{5.4}$$

This follows by recalling that any field in $\mathfrak{g}$ is a sum of vector fields δ_λ-homogeneous of degree at least 1 and by observing that the component functions of the identity map I are δ_λ-homogeneous monomials of degree at most r. Since $\exp[tX](x)$ is an analytic function of t, (3.7) gives $\exp[X] \equiv \sum_{k=0}^{r} \frac{1}{k!} X^k I$. We henceforth fix $x \in \mathbb{R}^N$ and we set

$$\Phi(t_1, t_2) := \exp[t_1 Y](\exp[t_2 X](x)), \qquad t_1, t_2 \in \mathbb{R}.$$

Clearly, any component function of Φ is a polynomial in t_1, t_2. From (3.6) and (3.8), we have

$$\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} \right)_{t=0} (f(\exp[tX](x))) = (X^k f)(x), \qquad f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N),\ k \in \mathbb{N},\ x \in \mathbb{R}^N.$$

This gives, for every $k_1, k_2 \in \mathbb{N}$,

$$\left(\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1} \partial t_2^{k_2}} \right) \Bigg|_{(t_1,t_2)=(0,0)} \Phi(t_1, t_2) = \left(\frac{\partial^{k_2}}{\partial t_2^{k_2}} \right)_{t_2=0} \left(\frac{\partial^{k_1}}{\partial t_1^{k_1}} \right)_{t_1=0} I(\exp[t_1 Y](\exp[t_2 X](x)))$$

$$= \left(\frac{\partial^{k_2}}{\partial t_2^{k_2}} \right)_{t_2=0} (Y^{k_1} I)(\exp[t_2 X](x)) = (X^{k_2} Y^{k_1} I)(x).$$

Since Φ is a polynomial and by exploiting (5.4), we derive

$$\exp[Y](\exp[X](x)) = \Phi(1, 1) = \sum_{k_1+k_2=0}^{r} \frac{1}{k_1!\,k_2!}(X^{k_2}Y^{k_1}I)(x). \tag{5.5}$$

We now introduce the following higher order differential operator

$$W(t, X, Y) := \sum_{j=1}^{r} \frac{(-1)^{j+1}}{j}\left(\sum_{k_1+k_2=1}^{r} \frac{t^{k_1+k_2}}{k_1!\,k_2!}X^{k_2}Y^{k_1}\right)^{j}.$$

We formally expand $W(t, X, Y)$ and we order it as a polynomial in t, setting

$$W(t, X, Y) = \sum_{k=1}^{r} t^k Z_k(X, Y) + \sum_{k=r+1}^{r^2} t^k Z_k(X, Y) =: Z(t, X, Y) + R(t, X, Y).$$

We explicitly remark that the differential operator Z appearing in (5.2) in the statement of the theorem is simply given by $Z(t, X, Y)$ with $t = 1$. It is easy to recognize that, by (5.4), any power of $R(t, X, Y)$ vanishes the identity map, i.e.,

$$(R(t, X, Y))^k I \equiv 0, \qquad \text{for every } k \geqslant 0. \tag{5.6}$$

We claim that

$$Z_k(X, Y) \in \operatorname{Lie}\{X, Y\}, \qquad \text{for every } k \geqslant r. \tag{5.7}$$

We now show that from (5.7) the theorem is proved. Indeed, if we set $Z := Z(1, X, Y)$, from the definition of $Z(t, X, Y)$ and from (5.7) we derive that $Z \in \operatorname{Lie}\{X, Y\}$. Then we turn to prove that (5.3) is satisfied. Indeed, we have

$$\begin{aligned}
\exp[Z](x) &= \sum_{k=0}^{r} \frac{1}{k!} Z^k I(x) \\
&= \sum_{k=0}^{r} \frac{1}{k!}(Z(1, X, Y) + R(1, X, Y))^k I(x) \\
&= \sum_{k=0}^{r} \frac{1}{k!}\left\{\sum_{j=1}^{r} \frac{(-1)^{j+1}}{j}\left(\sum_{k_1+k_2=1}^{r} \frac{1}{k_1!\,k_2!}X^{k_2}Y^{k_1}\right)^{j}\right\}^{k} I(x) \\
&= I(x) + \sum_{k_1+k_2=1}^{r} \frac{1}{k_1!\,k_2!}X^{k_2}Y^{k_1}I(x) = \exp[Y](\exp[X](x)).
\end{aligned}$$

The first equality follows from the fact that $Z \in \mathfrak{g}$ and from (3.7); the second one follows from (5.6) and homogeneity arguments; the third one is the definition of $W(1, X, Y)$; the fourth equality is a consequence of the formal power series expansion of the identity $1 + x = \exp(\log(1 + x))$, jointly with (5.4) (see below); the last equality follows from (5.5). More precisely, the fourth equality is a consequence of the identity

$$\forall H \in \mathfrak{g}, \qquad \sum_{k=0}^{r} \frac{1}{k!} \left\{ \sum_{j=1}^{r} \frac{(-1)^{j+1}}{j} H^j \right\}^k = I + H + \sum_{j=r+1}^{r^2} c_j H^j,$$

jointly with (5.4).

Finally we are left with the proof of the claim (5.7). A simple proof can be found in [15], where an analogous formula is derived in the more general context of the formal power series in two non-commuting indeterminates X and Y. The arguments used in [15] within the formal power series setting can be adapted also to the present case: indeed, the identities between formal power series therein found readily reduce, in our context, to identities between finite sums, by making use of arguments such as (5.4). This completes the proof of the theorem. ∎

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