GIAMBATTISTA MARINI

Algebraic cycles on abelian varieties and their decomposition

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2004_8_7B_1_231_0>
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and their Decomposition.

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Summary. – For an Abelian Variety $X$, the Künneth decomposition of the rational equivalence class of the diagonal $\Delta \subset X \times X$ gives rise to explicit formulas for the projectors associated to Beauville’s decomposition (1) of the Chow ring $CH^\bullet(X)$, in terms of push-forward and pull-back of $m$-multiplication. We obtain a few simplifications of such formulas, see theorem (4) below, and some related results, see proposition (9) below.

0. – Introduction.

Let $X$ be an abelian variety of dimension $n$ and denote by $CH_*(X)$ its Chow group of algebraic cycles modulo rational equivalence. In our notation, $CH_d(X)$ is the subgroup of $d$-dimensional cycles and $CH^p(X) := CH_{n-p}(X)$ is the subgroup of $p$-codimensional cycles. For $m \in \mathbb{Z}$, let $\text{mult}(m)$ denote the multiplication map $X \to X$, $x \mapsto mx$. By the use of Fourier-Mukai transform for abelian varieties (see [M] and [Be]), Beauville has established a decomposition

$$CH_d(X)_Q = \bigoplus_{s=-d}^{n-d} [CH_d(X)_{Q}]_s$$

where, by definition, $CH_d(X)_Q = CH_d(X) \otimes \mathbb{Q}$ is the Chow group with $\mathbb{Q}$-
coefficients and the right-hand-side subgroups are defined as follows:

\[ [CH_d(X)_\mathbb{Q}]_s := \{ W \in CH_d(X)_\mathbb{Q} \mid \text{mult}(m)_* W = m^{2d+s} W, \ \forall m \in \mathbb{Z} \} \]

\[ = \{ W \in CH^p(X)_\mathbb{Q} \mid \text{mult}(m)^* W = m^{2p-s} W, \ \forall m \in \mathbb{Z} \} , \]

where \( p = n - d \) is the codimension of \( W \).

This decomposition is a tool to understand cycles and rational equivalence on abelian varieties and it would give a beautiful answer to many questions concerning the Chow groups of abelian varieties (see [Be], [Bl], [J], [Ku] and [S]), provided that Beauville’s vanishing conjecture [Be] holds. This conjecture states that the factors of \( CH_d(X) \) with \( s < 0 \) vanish (see B.C. below). As pointed out in the abstract, by the use of Deninger-Murre projectors \( \delta_i \), (see [DM], [Ku]), the projections \( CH_d(X) \to [CH_d(X)]_s \) with respect to Beauville’s decomposition (1) can be written as linear forms of \( \text{mult}(m)_* \) and \( \text{mult}(m)^* \).

Theorem (4) simplifies such explicit descriptions. A further simplification is given for the case where one works modulo a piece of the decomposition, see proposition (9); see corollary (10) for a reformulation of Beauville’s conjecture.

1. – The algebraic set up.

We denote by \( \omega(z) \) the series expansion of \( \log(z + 1) \). Namely,

\[ \omega(z) := z - \frac{1}{2} z^2 + \frac{1}{3} z^3 \ldots . \]

Furthermore, for \( k \) and \( j \) non-negative integers we define constants \( a_{k,j} \) via the formal equality

\[ \sum_{j=0}^{\infty} a_{k,j} z^j = \frac{1}{k!} \omega(z)^k \]

Let \( A_r \in M_{r+1,r+1}(\mathbb{Q}) \) be the matrix \( (a_{k,j}) \), where \( k \) and \( j \) run in \( [0, \ldots, r] \). Let \( B_r \in M_{r+1,r+1}(\mathbb{Z}) \) be the matrix \( (b_{j,h}) \), where \( j \) and \( h \) run in \( [0, \ldots, r] \) and where, by definition, \( b_{j,h} = (-1)^{j-h} \binom{j}{h} \). It is understood that \( \binom{j}{h} = 0 \) provided that \( h > j \). For \( k = 0, 1, \ldots, r \) we define linear forms \( L_k^{(r)}(x_0, \ldots, x_r) \) by the following equality:

\[ \begin{pmatrix} L_0^{(r)} \\ \vdots \\ L_{r}^{(r)} \end{pmatrix} = A_r B_r \begin{pmatrix} x_0 \\ \vdots \\ x_r \end{pmatrix} , \]
namely we define (observe that $a_{k,j} = 0$, if $j < k$ and $b_{j,h} = 0$, if $h > j$)

$$L_k^{(r)}(x_0, \ldots, x_r) = \sum_{j=k}^{r} \sum_{h=0}^{j} a_{k,j} (-1)^{j-k} \binom{j}{h} x_h,$$

and for $k > r$ we define $L_k^{(r)} = 0$.

We now introduce a numerical lemma, the proof of which is very straightforward (and omitted).

**Lemma 3.** Let $j \geq 1$ and $\sigma \geq 0$ be integers. Then

$$\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} h^\sigma = \begin{cases} 0 & \text{if } \sigma < j \\ \sigma! & \text{if } \sigma = j. \end{cases}$$

2. – Projections of cycles.

Next, using linear forms $L_k^{(r)}$, we give a criterium to identify components (with respect to Beauville’s decomposition 1) of the algebraic cycles. In the sequel, $X$ denotes an abelian variety of dimension $n$; $W \in CH_d(X) \subset CH(X)$ denotes a rational algebraic cycle of dimension $d$ and $p = n - d$ its codimension; furthermore, $W_s$ denotes a component of $W$ with respect to Beauville’s decomposition (1), in particular $s$ is an integer in the range $[-d, n-d]$. We also consider linear forms $L_k^{(r)}$ as introduced in the previous section. The interpretation, in terms of push-forward and pull-back of multiplication maps, of the decomposition of the diagonal $\Delta \in CH_n(X \times X)$ (see [DM], [Ku]) gives

$$W_s = ([\log(\Delta)] \ast \text{rel}^{2d+s} \circ W)/(2d+s)! = ([\log(\Delta)] \ast \text{rel}^{2n-2d-s} \circ W)/(2n-2d-s)!,$$

where $\ast \text{rel}$ denotes the relative Pontryagin product on $CH_n(X \times X)$ with respect to projection on the first factor and where, for $\alpha \in CH_n(X \times X)$, $\alpha$ denotes its transpose. This equality in turn, in terms of our $L_k^{(r)}$ gives

$$W_s = L_{2d+s}(\text{mult}(0), \ldots, \text{mult}(r), \alpha) W = L_{2p-s}(\text{mult}(0)^*, \ldots, \text{mult}(r)^*), \forall r \geq 2n.$$

It is worthwhile to stress that the linear forms $L_k^{(r)}$ enter in a natural way (for $r=2n$) as an explicit version of Deninger-Murre-Künnemann projectors in terms of push-forward and pull-back of multiplication maps. The following theorem (4) goes further, it says that such equalities hold for $r$ that takes smaller values (see (4a) and (4b) below). We also want to stress that linear forms $L_k^{(r)}$ have an increasing length with respect to $r$ (see the list at the next page).
Theorem 4. – Let $X$, $W$ and $W_s$ be as above. Then

\[(4_a) \quad W_s = L_{2d+s}(\text{mult}(0)_*, \ldots, \text{mult}(n+d)_*) W, \quad \forall r \geq n + d;\]

\[(4_b) \quad W_s = L_{2p-s}(\text{mult}(0)^*, \ldots, \text{mult}(r)^*) W, \quad \forall r \geq n + p.\]

Formulas $(4_a)$ and $(4_b)$ are obtained by using lemma $(7)$ below. We shall also see that $(4_b)$ can be refined: the equality there also holds for $r \geq n + p - \min\{d, 2\}$. A similar achievement does not hold for $(4_a)$. As an explicit example we want to point out that for a 4-dimensional abelian variety and a 2-cycle $W$ the known formula for projectors would give

\[
W_1 = 8W - 14 \text{mult}(2)^* W + \frac{56}{3} \text{mult}(3)^* W - \frac{35}{2} \text{mult}(4)^* W + \]

\[
\frac{56}{5} \text{mult}(5)^* W - \frac{14}{3} \text{mult}(6)^* W + \frac{8}{7} \text{mult}(7)^* W - \frac{1}{8} \text{mult}(8)^* W
\]

meanwhile, by theorem $(4)$, or better by remark $(8)$, one has the simpler expression $W_1 = 4W - 3 \text{mult}(2)^* W + \frac{4}{3} \text{mult}(3)^* W - \frac{1}{4} \text{mult}(4)^* W$.

Remark. – Beauville’s conjecture (see [Be]) states that

\[(B.C.) \quad [CH_d(X)_Q]_s = 0, \quad \text{if } s < 0.\]

As a consequence of theorem $(4)$, proving the conjecture is equivalent to proving that either

$L_{2d+s}^{(n+d)}(\text{mult}(0)_*, \ldots, \text{mult}(n+d)_*)$ or $L_{2p-s}^{(n+p)}(\text{mult}(0)^*, \ldots, \text{mult}(n+p)^*)$ acts trivially on $CH_d(X)_Q$, for $s < 0$. Another equivalent formulation for Beauville’s conjecture (B.C.) is that the property $(4_b)$ holds also for $r \geq 2p$ (this is trivial: since $L_{2p-s}^{(2p)} = 0$ for $s < 0$, if $(4_b)$ holds for $r = 2p$, B.C. holds as well; it is straightforward to check that the converse implication follows from the proof of theorem $(4)$).

Remark. – Let us look at $(4_a)$ and $(4_b)$. The operators

$L_{2d+s}^{(r)}(\text{mult}(0)_*, \ldots, \text{mult}(r)_*)$

are non-trivial for $r \geq n + d$ and the operators $L_{2p-s}^{(r)}(\text{mult}(0)^*, \ldots, \text{mult}(r)^*)$ are non-trivial for $r \geq n + p$. Infact, since $-d \leq s \leq n - d$, then $2d + s \leq n + d$ as well as $2p - s \leq n + p$. 
Clearly, one has
\[ \text{mult}(0) \, W = \begin{cases} 0 & \text{if } d = \dim W > 0; \\ \deg W \cdot o & \text{if } W \text{ is a 0-cycle, where } o \text{ is the origin of } X. \end{cases} \]

\[ \text{mult}(1) \, W = W \]

For \( n + d \) that takes the indicated value, the operators \( L_k = L_k^{(n + d)}(\ldots, \text{mult}(i)_* \ldots) \) act as follows.

**n+d=1**
\[
\begin{align*}
L_0 W &= \text{mult}(0)_* W \\
L_1 W &= -\text{mult}(0)_* W + W
\end{align*}
\]

**n+d=2**
\[
\begin{align*}
L_0 W &= \text{mult}(0)_* W \\
L_1 W &= -\frac{3}{2} \text{mult}(0)_* W + 2W - \frac{1}{2} \text{mult}(2)_* W \\
L_2 W &= \frac{1}{2} \text{mult}(0)_* W - W + \frac{1}{2} \text{mult}(2)_* W
\end{align*}
\]

**n+d=3**
\[
\begin{align*}
L_0 W &= \text{mult}(0)_* W \\
L_1 W &= -\frac{11}{6} \text{mult}(0)_* W + 3W - \frac{3}{2} \text{mult}(2)_* W + \frac{1}{3} \text{mult}(3)_* W \\
L_2 W &= \text{mult}(0)_* W - \frac{5}{2} W + 2 \text{mult}(2)_* W - \frac{1}{2} \text{mult}(3)_* W \\
L_3 W &= -\frac{1}{6} \text{mult}(0)_* W + \frac{1}{2} W - \frac{1}{2} \text{mult}(2)_* W + \frac{1}{6} \text{mult}(3)_* W
\end{align*}
\]

**n+d=4**
\[
\begin{align*}
L_0 W &= \text{mult}(0)_* W \\
L_1 W &= -\frac{25}{12} \text{mult}(0)_* W + 4W - 3 \text{mult}(2)_* W + \frac{4}{3} \text{mult}(3)_* W - \frac{1}{4} \text{mult}(4)_* W \\
L_2 W &= \frac{35}{24} \text{mult}(0)_* W - \frac{13}{3} W + \frac{19}{4} \text{mult}(2)_* W - \frac{7}{3} \text{mult}(3)_* W + \frac{11}{4} \text{mult}(4)_* W \\
L_3 W &= -\frac{5}{12} \text{mult}(0)_* W + \frac{3}{2} W - 2 \text{mult}(2)_* W + \frac{7}{6} \text{mult}(3)_* W - \frac{1}{4} \text{mult}(4)_* W \\
L_4 W &= \frac{1}{24} \text{mult}(0)_* W - \frac{1}{6} W + \frac{1}{4} \text{mult}(2)_* W - \frac{1}{6} \text{mult}(3)_* W + \frac{1}{24} \text{mult}(4)_* W
\end{align*}
\]

From Beauville’s conjecture point of view the first interesting case is
\[ W_{-1} = L_5^{(8)}(\ldots, \text{mult}(i)_* \ldots) = L_5^{(7)}(\ldots, \text{mult}(i)^* \ldots), \]
for \( W \in CH^2(X)_0 \) and \( \dim X = 5 \), see [Be]. Indeed, we have also \( W_{-1} = L_5^{(r)}(\ldots, \text{mult}(i)^* \ldots), \) for \( r \geq 5 = n + p - \min \{d, 2\} \).

Next we prove theorem (4) and some related results. First, we recall that
the Chow group of an abelian variety has two ring structures: the first one is
given by the intersection product, the second one is given by the Pontryagin
product, which we shall always denote by \( \star \). Consider the ring \( CH_\bullet(X \times X) \)
with the natural sum of cycles and the relative Pontryagin product with re-
spect to projection on the first factor \( X \times X \rightarrow X \) (in other terms, we consider Pontryagin product on \( X \times X \) regarded as an abelian scheme over \( X \) via the first-factor-projection). Let \( \Delta \in CH_n(X \times X) \) be the diagonal and let \( E = X \times \{0\} \in CH_n(X \times X) \) be the unit of \( CH_\bullet(X \times X) \) with respect to the product above, where \( o \) is the origin of \( X \). The projectors \( \delta_0, \ldots, \delta_{2n} \) are defined by
(see [Ku], pag. 200)

\[
\delta_j = \frac{1}{(2n-j)!} ([\log(\Delta)]^{\star \text{rel}2n-j} - \frac{1}{2} (\Delta - E)^{\star \text{rel}2} + \frac{1}{3} (\Delta - E)^{\star \text{rel}3} \ldots) \text{rel}2n-j.
\]

Since \( (\Delta - E)^{\star \text{rel}2n+1} = 0 \) (see [Ku]), the series above are infact finite sums. Now let \( \Delta_m \) denote the graph of \( \text{mult}(m) \). By Deninger, Murre and Künnemann theorem (see [DM], [Ku]) we have

\[
[t \Delta_m] \circ \delta_j = m^j \delta_j, \quad \forall m \in \mathbb{Z}, \ 0 \leq j \leq 2n;

\]

\[
q \delta_j = \delta_{2n-j}, \quad \forall 0 \leq j \leq 2n;
\]

where the composition above is the composition of correspondences and where, for \( \sigma \in \text{Corr}(A, B) \), \( t \sigma \in \text{Corr}(B, A) \) denotes its transpose. As a consequence, for \( W \in CH_d(X)_\mathbb{Q} \) and \( 0 \leq j \leq 2n \), one has

\[
\text{mult}(m)^*(\delta_j \circ W) = [t \Delta_m] \circ (\delta_j \circ W) = m^j(\delta_j \circ W), \quad \forall m \in \mathbb{Z}.
\]

Clearly, one identifies \( CH_\bullet(X) \) with \( \text{Corr} (\text{Spec } \mathbb{C}, X) = CH_\bullet (\text{Spec } \mathbb{C} \times X) \). Thus, by the definition (2) one has

\[
(5') \quad \delta_j \circ W \in [CH_d(X)_\mathbb{Q}]_s, \quad s := 2n - 2d - j.
\]

Since \( \sum \delta_j = \Delta \) acts as the identity map, (5) and \( (5') \) give

\[
(5'') \quad W_s = \delta_{2n-2d-s} \circ W = t \delta_{2d+s} \circ W
\]

where, as usual, \( W_s \) denotes the component of \( W \) with respect to Beauville’s decomposition (1).

For the proof of theorem (4) we need the following.
Lemma 6. – Let $W$ be as in the theorem. Then

$$[(\Delta - E)^*_{\text{nil}}] \circ W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)_* W$$

$$t[(\Delta - E)^*_{\text{nil}}] \circ W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^* W$$

Proof. – Since $E$ is the unit for relative Pontryagin product and since $\Delta^*_{\text{nil}} \circ W = \operatorname{mult}(h)_* W$ as well as $t[\Delta^*_{\text{nil}}] \circ W = \operatorname{mult}(h)^* W$, the two equalities follow by a straightforward computation.

Lemma 7. – Let $W$ be as in the theorem. Then

$$(7a) \quad [(\Delta - E)^*_{\text{nil}}] \circ W = 0, \quad \forall j \geq n + d + 1;$$

$$(7b) \quad t[(\Delta - E)^*_{\text{nil}}] \circ W = 0, \quad \forall j \geq n + p + 1.$$ 

Proof. – We prove $(7b)$, the proof of $(7a)$ is very similar. By lemma (6), we have to show that for $j \geq n + p + 1$ one has

$$\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^* W = 0.$$ 

By linearity of the left-hand-side operator we are free to assume that $W$ belongs to one of the factors from Beauville decomposition (1), namely we are free to assume that $W \in [CH_d(X)\mathbb{Q}]_s$ for some $s \in [-d, n - d]$. Thus (see 2), we assume that $\operatorname{mult}(m)^* W = m^{2p-s} W$, $\forall m \in \mathbb{Z}$. It follows

$$\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^* W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} h^{2p-s} W.$$ 

For $s$ in the range above, the range for $2p-s$ is $[p, n+p]$; in particular, we have $2p-s < j$. By lemma (3), the coefficient $\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} h^{2p-s}$ vanishes. Then we are done.

Proof (of theorem 4). – We start with formula $(4a)$. Let $k = 2d + s$. Then, we have

$$W_s = \delta_{2n-2d-s} \circ W = \frac{1}{(2d+s)!} [\log(\Delta)^*_{\text{rel}} 2d+s] \circ W$$

$$= \sum_{j=k}^{2n} a_{k,j} (\Delta - E)^*_{\text{nil}} \circ W.$$ 

Now observe that by lemma (7), we have $(\Delta - E)^*_{\text{nil}} \circ W = 0$ for $j \geq n + d + 1$. 


Thus, the summation above can be taken up to \( r \), provided that \( r \geq n + d \). It follows that
\[
W_s = \sum_{j = k}^{r} a_{k, j} (\Delta - E)^{*}\text{rel}^{j} \circ W, \quad \forall r \geq n + d.
\]
Looking at the definition of the operators \( L_{k}^{(r)} \) it is then clear that (4a) follows from the first equality from lemma (6),
\[
(\Delta - E)^{*}\text{rel}^{j} \circ W = \sum_{h = 0}^{j} (-1)^{j - h} \binom{j}{h} \text{mult}(h)^{*} W.
\]
The proof of formula (4b) is similar. For \( r \geq n + p \) we have
\[
W_s = \delta_{2d + s} \circ W = \frac{1}{(2p - s)!} \cdot [\log((\Delta)^{*}\text{rel}^{2p - s})] \circ W
\]
\[
= \sum_{j = 2p - s}^{2n} a_{2p - s, j} \cdot [(\Delta - E)^{*}\text{rel}^{j}] \circ W
\]
\[
= \sum_{j = 2p - s}^{r} a_{2p - s, j} \cdot \sum_{h = 0}^{j} (-1)^{j - h} \binom{j}{h} \text{mult}(h)^{*} W
\]
\[
= L_{2p - s}^{(r)} (\text{mult}(0)^{*}, \ldots, \text{mult}(r)^{*}) W
\]
where the 4\th equality follows by lemma (7), the 5\th equality follows by lemma (6) and the 6\th equality follows by the definition of the operators \( L_{k}^{(r)} \).

**Remark 8.** – The equality (7b) can be improved. We have,
\[
[(\Delta - E)^{*}\text{rel}^{j}] \circ W = 0, \quad \forall j \geq n + p + 1 - \delta
\]
where \( \delta = \min\{d, 2\} \). Infact, since \([CH_{d}(X)_{\mathbb{Q}}]_{s} = 0\) provided that \( s \leq \min\{-d + 1, -1\} \) (see [Be]), the actual range for \( s \) can be shrinked to \( \min\{-d + 2, 0\} \leq s \leq n - d \). Thus in turn, one obtains (8’) by the same proof of (7b). As a consequence, (4b) can be refined: the equality there also holds for all \( r \geq n + p - \delta \) (where \( \delta \) is as above).

Furthermore, for the same reason, if Beauville’s conjecture (B.C.) mentioned above holds, then
\[
[(\Delta - E)^{*}\text{rel}^{j}] \circ W = 0, \quad \forall j \geq 2p + 1
\]
In particular, if \( W \) is a divisor (hence it satisfies B.C.), then \( [(\Delta - E)^{*}\text{rel}^{j}] \circ W = 0, \quad \forall j \geq 2p + 1 \)
$W = 0$, namely $3W - 3 \text{mult}(2)^*W + \text{mult}(3)^*W = 0$, which is obvious (in the case of divisors, this kind of computations provide trivial results).

Now fix $s$, working modulo $\bigoplus_{l \geq s+1} [CH_d(X)_{\mathbb{Q}}]_l$, or rather modulo $\bigoplus_{l \leq s-1} [CH^p(X)_{\mathbb{Q}}]$, yields simpler formulas than the ones from theorem (4); furthermore, it can be used to provide a reformulation for Beauville’s conjecture (B.C.), see corollary (10) and the example below.

**Proposition 9.** – Let $W$ and $W_s$ be as in the theorem. Then

$$(9_a) \quad W_s = \frac{1}{(2d+s)!} \sum_{h=0}^{2d+s} (-1)^{2d+s-h} \binom{2d+s}{h} \text{mult}(h)^*W,$$

modulo $\bigoplus_{l \geq s+1} [CH_d(X)_{\mathbb{Q}}]_l$.

Furthermore,

$$(9_b) \quad W_s = \frac{1}{(2p-s)!} \sum_{h=0}^{2p-s} (-1)^{2p-s-h} \binom{2p-s}{h} \text{mult}(h)^*W,$$

modulo $\bigoplus_{l \leq s-1} [CH^p(X)_{\mathbb{Q}}]_l$.

**Proof.** – We prove $(9_b)$. Let $K = \frac{1}{(2p-s)!} \sum_{h=0}^{2p-s} (-1)^{2p-s-h} \binom{2p-s}{h} \text{mult}(h)^*$. It suffices to prove that

$$KW = \begin{cases} 0 & \text{if } W \in [CH_d(X)_{\mathbb{Q}}]_l, \ l \geq s+1 \\ W & \text{if } W \in [CH_d(X)_{\mathbb{Q}}]_s. \end{cases}$$

This is clear by the proof of $(7_b)$; as for the case $W \in [CH_d(X)_{\mathbb{Q}}]_s$, the equality $KW = W$ follows since, by lemma (3), the coefficient $\sum_{h=0}^{\sigma} (-1)^{\sigma-h} \binom{\sigma}{h} h^\sigma$ equals $\sigma!$ (here $\sigma = 2p - s$). The proof of $(9_a)$ is similar. ■

A straightforward consequence of $(9_b)$ is the following.

**Corollary 10.** – Let $X$ be as in the theorem. Then, it satisfies Beauville’s conjecture for $d$-dimensional cycles if and only if

$$\sum_{k=0}^{s} (-1)^{k-h} \binom{k}{h} \text{mult}(h)^*$$

acts trivially on $CH_d(X)_{\mathbb{Q}}$ for $k \geq 2p + 1$, where $p = n - d$ as usual.

For 5-dimensional abelian varieties the only bad component that might exist is $[CH_3(X)_{\mathbb{Q}}]_{-1}$. Then, by the corollary above it follows that a 5-dimen-
sional abelian variety $X$ satisfies Beauville’s conjecture (B.C.) if and only if
\[ 5W - 10 \operatorname{mult}(2)^* W + 10 \operatorname{mult}(3)^* W - 5 \operatorname{mult}(4)^* W + \operatorname{mult}(5)^* W = 0, \]
for all $W \in CH_3(X)_{\mathbb{Q}}$.

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Dipartimento di Matematica, Università degli studi di Roma II «Tor Vergata», Via della Ricerca Scientifica, I-00133 Roma (Italy)
E-mail: marini@axp.mat.uniroma2.it

Pervenuta in Redazione il 7 dicembre 2002 e in forma rivista il 31 marzo 2003