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Determining Two Coefficients in Elliptic Operators via Boundary Spectral Data: a Uniqueness Result.

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Sunto. – *Sia Ω un dominio limitato e sufficientemente regolare di $\mathbb{R}^N$, $N \geq 2$, e siano $(\lambda_k)_{k=1}^\infty$ e $(\varphi_k)_{k=1}^\infty$ rispettivamente gli autovalori e le autofunzioni corrispondenti del problema (con condizioni al bordo di Neumann)*

$$-\operatorname{div}(a(x)\,\nabla\varphi_k) + q(x)\,\varphi_k = \lambda_k \varrho(x)\,\varphi_k \quad \text{in } \Omega, \qquad a\frac{\partial}{\partial \boldsymbol{n}}\varphi_k = 0 \quad \text{su } \partial\Omega\,.$$

Dimostriamo che i dati spetrali al bordo di Dirichlet $(\lambda_k)_{k=1}^\infty$, $(\varphi_{k|\partial\Omega})_{k=1}^\infty$ determinano in modo unico la mappa γ di Neumann-Dirichlet (o la mappa di Steklov-Poincaré) per un problema ellittico relativo. Sotto opportune ipotesi sui coefficienti a, q, ϱ proviamo in seguito la loro identificabilità. Dimostriamo risultati analoghi nel caso di condizioni al bordo di Dirichlet.

Summary. – *For a bounded and sufficiently smooth domain Ω in $\mathbb{R}^N$, $N \geq 2$, let $(\lambda_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ be respectively the eigenvalues and the corresponding eigenfunctions of the problem (with Neumann boundary conditions)*

$$-\operatorname{div}(a(x)\,\nabla\varphi_k) + q(x)\,\varphi_k = \lambda_k \varrho(x)\,\varphi_k \quad \text{in } \Omega, \qquad a\frac{\partial}{\partial \boldsymbol{n}}\varphi_k = 0 \quad \text{su } \partial\Omega\,.$$

We prove that knowledge of the Dirichlet boundary spectral data $(\lambda_k)_{k=1}^\infty$, $(\varphi_{k|\partial\Omega})_{k=1}^\infty$ determines uniquely the Neumann-to-Dirichlet (or the Steklov-Poincaré) map γ for a related elliptic problem. Under suitable hypothesis on the coefficients a, q, ϱ their identifiability is then proved. We prove also analogous results for Dirichlet boundary conditions.

1. – The main results.

Let Ω be a bounded and sufficiently smooth domain in $\mathbb{R}^N$, $N \geq 2$, and let a, ϱ be two strictly positive functions defined in Ω. We denote by $(\lambda_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ respectively the eigenvalues (in increasing order) and the corresponding eigenfunctions of the following problem (with Neumann boundary conditions):

$$(1.1)\qquad \begin{cases} -\operatorname{div}(a\nabla\varphi_k) + q\varphi_k = \lambda_k \varrho\varphi_k & \text{in } \Omega, \\ a\dfrac{\partial}{\partial \boldsymbol{n}}\varphi_k = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega |\varphi_k(x)|^2 \varrho(x)\, dx = 1. \end{cases}$$

We denote by

$$(1.2)\qquad \operatorname{Dbsd}(a, q, \varrho) := \{(\lambda_k, \varphi_{k|\partial\Omega});\ k \geqslant 1\},$$

the *Dirichlet boundary spectral data* of problem (1.1). Under suitable assumptions on the regularity of the coefficients a, q, ϱ, we ask the following question: does knowledge of the Dirichlet boundary spectral data Dbsd(a, q, ϱ) (1.2) determine the coefficients a, q, ϱ uniquely in Ω?

Many authors have focused their attention to study similar problems. In 1946 G. Borg [1] and N. Levinson [9] asked the question whether knowledge of the eigenvalues $(\lambda_k)_{k=1}^\infty$ of the Sturm-Liouville problem

$$(1.3)\qquad \begin{cases} -\varphi_k'' + q(x)\,\varphi_k = \lambda_k\varphi_k & \text{in } (0, l), \\ \varphi_k(0) = \varphi_k(l)0, \end{cases}$$

determine $q \in L^\infty(0, l)$ uniquely. It is clear that the operators associated with the potentials $q(x)$ and $q(l-x)$ have the same eigenvalues, therefore the spectrum alone, in general, is not sufficient to determine the potential q uniquely. They proved the identifiability of q in (1.3) from knowledge of eigenvalues $(\lambda_k)_{k=1}^\infty$ and of normalizing constants

$$c_k := \int_0^l |\varphi_k|^2\, dx,$$

by supposing $\varphi_k'(0) = 1$. Later on I.M. Gel'fand & B.M. Levitan [7] have given a reconstruction formula of the potential q from the sequence of eigenvalues $(\lambda_k)_{k=1}^\infty$ and of normalizing constants $(c_k)_{k=1}^\infty$ (under the hypothesis that $\varphi_k'(0) = 1$). More recently A.I. Nachman, J. Sylvester & G. Uhlmann [12] have studied a similar problem in the multidimensional setting. More precisely, let Ω be a bounded and sufficiently smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, and let $(\lambda_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ be respectively the eigenvalues and the eigenfunctions of the problem (with Dirichlet boundary conditions):

$$
(1.4) \qquad \begin{cases} -\Delta\varphi_k + q\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_k(x)|^2\,dx = 1. \end{cases}
$$

They show that the boundary spectral data $(\lambda_k)_{k=1}^\infty$, $\left(\frac{\partial}{\partial \boldsymbol{n}}\varphi_{k|\partial\Omega}\right)_{k=1}^\infty$ determine $q \in C^\infty(\overline{\Omega})$ uniquely. The idea of the proof is the following. For $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, and $\lambda \in \mathbb{C}$, $\lambda \notin \{-\lambda_k;\, k \geq 1\}$, let $u_\lambda \in H^1(\Omega)$ solve

$$
(1.5) \qquad \begin{cases} -\Delta u_\lambda + (q+\lambda)\,u_\lambda = 0 & \text{in } \Omega, \\ u_\lambda = \varphi & \text{on } \partial\Omega. \end{cases}
$$

Let us denote by $\widetilde{\gamma}_\lambda$ the Dirichlet-to-Neumann map related to problem (1.5), that is

$$
\widetilde{\gamma}_\lambda : \varphi \mapsto \frac{\partial}{\partial \boldsymbol{n}} u_{\lambda|\partial\Omega}.
$$

A.I. Nachman, J. Sylvester & G. Uhlmann prove, in a first step, that the boundary spectral data $(\lambda_k)_{k=1}^\infty$, $\left(\frac{\partial}{\partial \boldsymbol{n}}\varphi_{k|\partial\Omega}\right)_{k=1}^\infty$ determine $\widetilde{\gamma}_\lambda$ uniquely, for all $\lambda \in \mathbb{C}$, $\lambda \notin \{-\lambda_k;\, k \geq 1\}$ (see [12], Lemmas 3.1, 3.2). Then, in a second step, they show that the coefficient q in (1.5) is uniquely determined by the Dirichlet-to-Neumann map $\widetilde{\gamma}_\lambda$ (see [12], Theorem 1.5). Using a similar method B. Canuto & O. Kavian [5] have proved the same result by supposing only $q \in L^\infty(\Omega)$.

Now assume that the functions a, q, ϱ in (1.1) satisfy the following assumptions:

$$
(1.6) \qquad \Omega \subset \mathbb{R}^N \text{ is a bounded Lipschitz domain, and } N \geq 2,
$$

$$
(1.7) \qquad a,\, q,\, \varrho \in L^\infty(\Omega),
$$

$$
(1.8) \qquad a \geq \alpha \text{ a.e. in } \Omega, \text{ for some constant } \alpha > 0,
$$

$$
(1.9) \qquad \varrho \geq \beta \text{ a.e. in } \Omega, \text{ for some constant } \beta > 0.
$$

We shall denote by λ^* a real number satisfying

$$
(1.10) \qquad \lambda^* > \lambda_* := \inf\{\lambda \in \mathbb{R};\, q + \lambda\varrho \geq 0 \text{ a.e. in } \Omega\}.
$$

Observe also that if q_j, ϱ_j satisfy (1.7)-(1.9) for $j = 1, 2$, then we shall suppose that $\lambda^* > \max(\lambda_{*1}, \lambda_{*2})$.

Then for $\lambda \geq \lambda^*$, $\psi \in L^2(\partial\Omega)$, $\psi \not\equiv 0$, there exists a unique $u_\lambda \in H^1(\Omega)$ solving

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u_\lambda) + (q(x) + \lambda\varrho(x))\,u_\lambda = 0 & \text{in } \Omega, \\ a\dfrac{\partial}{\partial \boldsymbol{n}}u_\lambda = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

Let us denote by γ_λ the *Neumann-to-Dirichlet* map related to problem (1.11), that is

$$\gamma_\lambda \colon \psi \mapsto u_{\lambda|\partial\Omega}.$$

We ask the question: do the Dirichlet boundary spectral data $\mathrm{Dbsd}(a, q, \varrho)$ (1.2) determine the Neumann-to-Dirichlet map γ_λ *uniquely*?

Our first result in the present paper is the following

THEOREM 1.1. – *Assume that Ω satisfies* (1.6). *For $j = 1, 2$, let a_j, q_j, ϱ_j satisfy assumption* (1.7), *and a_j, ϱ_j assumptions* (1.8), (1.9) *respectively. Let us denote by $u_{j\lambda}$, $(\lambda_{jk})_{k=1}^\infty$ and $(\varphi_{jk})_{k=1}^\infty$ respectively the solution of* (1.11) *and the eigenvalues and the eigenfunctions of* (1.1) *when $a := a_j$, $q := q_j$, $\varrho := \varrho_j$. With the notations introduced in* (1.2), *suppose that*

$$\mathrm{Dbsd}(a_1, q_1, \varrho_1) = \mathrm{Dbsd}(a_2, q_2, \varrho_2). \tag{1.12}$$

Then

$$\gamma_{1\lambda} = \gamma_{2\lambda} \tag{1.13}$$

for all $\lambda \geq \lambda^*$, where $\gamma_{j\lambda}(\psi) := u_{j\lambda|\partial\Omega}$.

We note that, due to the analycity of $\lambda \mapsto \gamma_{j\lambda}$ on $\mathbb{C}\setminus(-\lambda_k;\, k \geq 1\}$, the conclusion of Theorem 1.1 holds true for all $\lambda \in \mathbb{C}$, $\lambda \notin \{-\lambda_k;\, k \geq 1\}$, where we denote $\lambda_k := \lambda_{1k} = \lambda_{2k}$.

A similar result remains valid if we replace in (1.1) the Neumann boundary conditions with the corresponding Dirichlet boundary conditions. More precisely let $(\lambda_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ be respectively the eigenvalues (in increasing order) and the corresponding eigenfunctions of the following problem (with Dirichlet boundary conditions):

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla\varphi_k) + q(x)\,\varphi_k = \lambda_k\varrho(x)\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega |\varphi_k(x)|^2\varrho(x)\,dx = 1. \end{cases} \tag{1.14}$$

We denote by

$$\mathrm{Nbsd}(a, q, \varrho) := \left\{ \left(\lambda_k, a \frac{\partial}{\partial \boldsymbol{n}} \varphi_{k|\partial\Omega} \right); k \geq 1 \right\}, \tag{1.15}$$

the *Neumann boundary spectral data* of problem (1.14). For $\lambda \geq \lambda^*$, λ^* as in (1.10), and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, $\varphi \not\equiv 0$, let $u_\lambda \in H^1(\Omega)$ solve

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u_\lambda) + (q(x) + \lambda\varrho(x))\, u_\lambda = 0 & \text{in } \Omega, \\ u_\lambda = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

Let us denote by $\widetilde{\gamma_\lambda}$ the *Dirichlet-to-Neumann* map related to problem (1.16), that is

$$\widetilde{\gamma_\lambda} : \varphi \mapsto a \frac{\partial}{\partial \boldsymbol{n}} u_{\lambda|\partial\Omega}.$$

In what follows we shall denote by

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) \tag{1.17}$$

the euclidean distance of the point $x \in \Omega$ from the boundary $\partial\Omega$, and by Ω^ε an ε-neighborhood of $\partial\Omega$ in $\overline{\Omega}$, that is

$$\Omega^\varepsilon := \{x \in \overline{\Omega} \text{ s.t. } \delta(x) < \varepsilon\} \tag{1.18}$$

for some $\varepsilon > 0$ given.

If we suppose that $|a_1 - a_2|$ and $|\varrho_1 - \varrho_2|$ vanish on the boundary $\partial\Omega$ at the same order as δ^2 (see below for a more precise statement), we can prove that if the Neumann boundary spectral data $\mathrm{Nbsd}(a_j, q_j, \varrho_j)$ coincide then the Dirichlet-to-Neumann map $\widetilde{\gamma_{j\lambda}}$ are the same. As a matter of fact we show the following

THEOREM 1.2. – *Assume that Ω satisfies* (1.6). *For $j = 1, 2$, let us denote by $u_{j\lambda}$, $(\lambda_{jk})_{k=1}^\infty$ and $(\varphi_{jk})_{k=1}^\infty$ respectively the solution of* (1.16) *and the eigenvalues and the eigenfunctions of* (1.14) *when $a := a_j$, $q := q_j$, $\varrho := \varrho_j$. With the notations introduced in* (1.15), *suppose that*

$$\mathrm{Nbsd}(a_1, q_1, \varrho_1) = \mathrm{Nbsd}(a_2, q_2, \varrho_2), \tag{1.19}$$

and, for some constant $c > 0$,

$$|a_1 - a_2| \leq c\delta^2 \quad \text{in } \Omega, \tag{1.20}$$

$$|\varrho_1 - \varrho_2| \leq c\delta^2 \quad \text{in } \Omega. \tag{1.21}$$

Then

$$\widetilde{\gamma_{1\lambda}} = \widetilde{\gamma_{2\lambda}} \tag{1.22}$$

for all $\lambda \geqslant \lambda^*$, *where* $\widetilde{\gamma_{j\lambda}}(\varphi) := a_j \dfrac{\partial}{\partial \boldsymbol{n}} u_{j\lambda|\partial\Omega}$.

We observe that, as in Theorem 1.1, the conclusion of Theorem 1.2 holds true for all $\lambda \in \mathbb{C} \setminus \{-\lambda_k;\, k \geqslant 1\}$, where we denote by $\lambda_k := \lambda_{1k} = \lambda_{2k}$.

We point out that if in Theorem 1.2 we assume that $a_1 - a_2$ and $\varrho_1 - \varrho_2$ are in $C^1(\Omega^\varepsilon)$, for some $\varepsilon > 0$, then assumptions (1.20), (1.21) can be respectively expressed and somewhat weakened into the assumptions

$$(a_1 - a_2)_{|\partial\Omega} = |\nabla(a_1 - a_2)|_{|\partial\Omega} = 0 \quad \text{on } \partial\Omega,$$

and

$$(\varrho_1 - \varrho_2)_{|\partial\Omega} = |\nabla(\varrho_1 - \varrho_2)|_{|\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

Concerning the identifiability of the coefficients a, q, ϱ in (1.1) from the Dirichlet boundary spectral data $\mathrm{Dbsd}(a, q, \varrho)$ we may state the following results. If we suppose that the coefficients $q_1 \equiv q_2$ in Ω, then we prove the identifiability of the coefficients a in $\overline{\Omega}$ and ϱ in Ω. More precisely we show the following

COROLLARY 1.3. – *Under the assumptions of Theorem* 1.1, *for* $j = 1, 2$ *let* $a_j \in W^{2,p}(\Omega)$, $p > \dfrac{N}{2}$ *for* $N \geqslant 3$, $p = \infty$ *for* $N = 2$, *and let* $q_1 = q_2 =: q$ *in* Ω. *Assume that*

$$\mathrm{Dbsd}(a_1, q, \varrho_1) = \mathrm{Dbsd}(a_2, q, \varrho_2).$$

Then

$$a_1 = a_2 \ \text{ in } \overline{\Omega}, \quad \text{and} \quad \varrho_1 = \varrho_2 \ \text{ in } \Omega.$$

If we suppose that the coefficient a is known in $\overline{\Omega}$, then we prove the identifiability of the coefficients q and ϱ in Ω, i.e the following result holds:

COROLLARY 1.4. – *Under the assumptions of Theorem* 1.2, *for* $j = 1, 2$ *let* $a := a_1 = a_2 \in W^{2,p}(\Omega)$, $p > \dfrac{N}{2}$ *for* $N \geqslant 2$. *Assume that*

$$\mathrm{Dbsd}(a, q_1, \varrho_1) = \mathrm{Dbsd}(a, q_2, \varrho_2).$$

Then

$$q_1 = q_2 \ \text{ in } \Omega, \quad \text{and} \quad \varrho_1 = \varrho_2 \ \text{ in } \Omega.$$

Concerning the identification of the coefficients a, q, ϱ in (1.14) from the Neumann boundary spectral data $\mathrm{Nbsd}(a, q, \varrho)$, we can prove similar results to that of Corollaries 1.3, 1.4. We begin by proving the following

COROLLARY 1.5. – *Under the assumptions of Theorem* 1.2, *for* $j = 1, 2$ *let* $a_j \in W^{2,p}(\Omega)$, $p > \frac{N}{2}$ *for* $N \geq 3$, $p = \infty$ *for* $N = 2$, *and let* $q := q_1 = q_2$ *in* Ω. *Assume that*

$$\mathrm{Nbsd}(a_1, q, \varrho_1) = \mathrm{Nbsd}(a_2, q, \varrho_2),$$

and

$$\nabla a_1 = \nabla a_2 \quad on \ \partial\Omega, \tag{1.23}$$

and $\varrho_1 - \varrho_2$ *satisfies* (1.21). *Then*

$$a_1 = a_2 \ \ in \ \ \overline{\Omega}, \quad and \quad \varrho_1 = \varrho_2 \ \ in \ \ \Omega.$$

We observe that under the assumption that the coefficients $a_j \in W^{2,p}(\Omega)$, $p > \frac{N}{2}$ hypothesis (1.23) is weaker than hypothesis (1.20) of Theorem 1.2. (Actually we are able to prove that the assumptions of the Corollary imply indeed $a_1 = a_2$.

In the following corollary we assume that the coefficient a is known in $\overline{\Omega}$ and we prove the identifiability of the coefficients q and ϱ in Ω (under the hypothesis (1.21)).

COROLLARY 1.6. – *Under the assumptions of Theorem* 1.2, *for* $j = 1, 2$ *let* $a := a_1 = a_2 \in W^{2,p}(\Omega)$, $p > \frac{N}{2}$ *for* $N \geq 2$. *Assume that*

$$\mathrm{Nbsd}(a, q_1, \varrho_1) = \mathrm{Nbsd}(a, q_2, \varrho_2),$$

and $\varrho_1 - \varrho_2$ *satisfies* (1.21). *Then*

$$q_1 = q_2 \ \ in \ \ \Omega, \quad and \quad \varrho_1 = \varrho_2 \ \ in \ \ \Omega.$$

Finally, if we assume that the coefficient ϱ is known in Ω, then we prove the identifiability of the coefficients a in $\overline{\Omega}$ and q in Ω (under the hypotesis (1.23) and (1.21)).

COROLLARY 1.7. – *Under the assumptions of Theorem* 1.2, *for* $j = 1, 2$ *let* $a_j \in W^{2,p}(\Omega)$, $p > \frac{N}{2}$ *for* $N \geq 3$, $p = \infty$ *for* $N = 2$ *and let* $\varrho := \varrho_1 = \varrho_2$ *a.e. in* Ω. *Assume that*

$$\mathrm{Nbsd}(a_1, q_1, \varrho) = \mathrm{Nbsd}(a_2, q_2, \varrho),$$

and

$$\nabla a_1 = \nabla a_2 \quad on \ \partial\Omega.$$

Then

$$a_1 = a_2 \ \text{in} \ \overline{\Omega}, \quad \text{and} \quad q_1 = q_2 \ \text{in} \ \Omega.$$

The paper is organized as follows: in the next section we prove Theorem 1.1, in section § 3 we prove Theorem 1.2, finally in section § 4 we show Corollaries 1.3-1.7.

2. – Proof of Theorem 1.1.

Our task in this section is to prove Theorem 1.1. Before doing so we need to establish some preliminary results. We shall denote $L^2_\varrho(\Omega) := L^2(\Omega)$ equipped with the scalar product $(f|g) := \int_\Omega f(x)\, g(x)\, \varrho(x)\, dx$.

LEMMA 2.1. – *For $\psi \in L^2(\partial\Omega)$, $\psi \not\equiv 0$, and $\lambda \geq \lambda^*$ let $u_\lambda \in H^1(\Omega)$ solve* (1.11). *Then*

$$\|u_\lambda\|^2_{L^2_\varrho(\Omega)} \to 0 \quad \text{as } \lambda \to +\infty. \tag{2.1}$$

PROOF OF LEMMA 2.1. – Since the sequence of eigenfunctions $(\varphi_k)_{k=1}^\infty$ of (1.1) is a Hilbert basis in $L^2_\varrho(\Omega)$, we can write u_λ in the following Fourier expansion

$$u_\lambda = \sum_{k=1}^\infty (u_\lambda | \varphi_k)\, \varphi_k \quad \text{in } L^2_\varrho(\Omega). \tag{2.2}$$

After multiplying (1.11) by φ_k and integrating by parts over Ω we obtain

$$(u_\lambda | \varphi_k) = \frac{\alpha_k}{\lambda_k + \lambda}, \tag{2.3}$$

where $\alpha_k := -\langle \varphi_k, \psi \rangle$ (here and in the sequel $\langle \cdot, \cdot \rangle$ denotes the duality in $H^{\frac{1}{2}}(\partial\Omega)$, $H^{-\frac{1}{2}}(\partial\Omega)$). Next (2.2) and (2.3) yield

$$\|u_\lambda\|^2_{L^2_\varrho(\Omega)} = \sum_{k=1}^\infty \left(\frac{\alpha_k}{\lambda_k + \lambda} \right)^2. \tag{2.4}$$

Hence (2.1) follows by the uniform convergence in λ of the series in (2.4). ■

LEMMA 2.2. – *For $j = 1, 2$, $\lambda \geq \lambda^*$, let $u_{j\lambda} \in H^1(\Omega)$ solve* (1.11) *when $a := a_j$, $q := q_j$, $\varrho := \varrho_j$. Let us denote by $\gamma_{j\lambda}(\psi) := u_{j\lambda|\partial\Omega}$. Assume that* (1.12) *holds, i.e.*

$$\mathrm{Dbsd}(a_1, q_1, \varrho_1) = \mathrm{Dbsd}(a_2, q_2, \varrho_2).$$

Then

$$\gamma_{1\lambda^*} - \gamma_{2\lambda^*} = \gamma_{1\lambda} - \gamma_{2\lambda} \tag{2.5}$$

for all $\lambda \geq \lambda^*$.

PROOF OF LEMMA 2.2. – *Let* $z_{j\lambda} := u_{j\lambda} - u_{j\lambda^*}$. *Then* $z_{j\lambda} \in H^1(\Omega)$ *solve*

$$\begin{cases} -\operatorname{div}(a_j(x)\,\nabla z_{j\lambda}) + (q_j(x) + \lambda\varrho_j(x))\, z_{j\lambda} = -(\lambda - \lambda^*)\,\varrho_j(x)\, u_{j\lambda^*} & \text{in } \Omega, \\ a_j \dfrac{\partial}{\partial \boldsymbol{n}} z_{j\lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

We can write $z_{j\lambda}$ in the following Fourier expansion

$$z_{j\lambda} = \sum_{k=1}^{\infty} (z_{j\lambda} \,|\, \varphi_{jk})\, \varphi_{jk} \quad \text{in } L^2_{\varrho_j}(\Omega). \tag{2.7}$$

Multiplying (2.6) by φ_{jk}, and integrating by parts over Ω, we obtain

$$(z_{j\lambda} \,|\, \varphi_{jk}) = \frac{-(\lambda - \lambda^*)\,\alpha_k}{(\lambda_k + \lambda)^2},$$

where $\lambda_k := \lambda_{1k} = \lambda_{2k}$, $\alpha_k := -\langle \varphi_k, \psi \rangle$, and $\varphi_k := \varphi_{1k} = \varphi_{2k}$ on $\partial\Omega$.

One can verify that, thanks to the decay of its terms, the series in (2.7) converges to $z_{j\lambda}$ in $H^1(\Omega)$ (see B. Canuto & O. Kavian [4], Lemma 3.2). Therefore, the trace operator $u \mapsto u_{|\partial\Omega}$ being continuous from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$, we have

$$z_{j\lambda|\partial\Omega} = -\sum_{k=1}^{\infty} \frac{(\lambda - \lambda^*)\,\alpha_k}{(\lambda_k + \lambda)^2}\, \varphi_{k|\partial\Omega} \quad \text{in } H^{\frac{1}{2}}(\partial\Omega).$$

So one has:

$$z_{1\lambda|\partial\Omega} = z_{2\lambda|\partial\Omega} \quad \text{in } H^{\frac{1}{2}}(\partial\Omega),$$

that is

$$\gamma_{1\lambda^*}(\psi) - \gamma_{2\lambda^*}(\psi) = \gamma_{1\lambda}(\psi) - \gamma_{2\lambda}(\psi) \quad \text{in } H^{\frac{1}{2}}(\partial\Omega), \tag{2.8}$$

for all $\lambda \geq \lambda^*$, for all $\psi \in L^2(\partial\Omega)$. ■

PROOF OF THEOREM 1.1. – By Lemma 2.2 we have

$$(u_{1\lambda^*} - u_{2\lambda^*})_{|\partial\Omega} = (u_{1\lambda} - u_{2\lambda})_{|\partial\Omega} \quad \text{in } H^{\frac{1}{2}}(\partial\Omega) \tag{2.9}$$

for all $\lambda \geq \lambda^*$, for all $\psi \in L^2(\Omega)$. We claim that there exists a subsequence $\lambda^{(n)}$,

$\lambda^{(n)} \to +\infty$ as $n \to +\infty$, such that

$$u_{j\lambda^{(n)}} \rightharpoonup 0 \quad \text{in } H^{\frac{1}{2}}(\partial\Omega) \quad \text{as } n \to +\infty. \tag{2.10}$$

For a moment assume that (2.10) is true. Then (2.9) implies that

$$u_{1\lambda} - u_{2\lambda} = 0 \quad \text{in } H^{\frac{1}{2}}(\partial\Omega)$$

for all $\lambda \geqslant \lambda^*$, for all $\psi \in L^2(\partial\Omega)$. Hence Theorem 1.1 is proved.

So it remains to prove claim (2.10). We show this for $j = 1$, the case $j = 2$ being clearly analogous. Multiplying the equation in (1.11) when $j = 1$ by $u_{1\lambda}$, and integrating by parts over Ω we obtain

$$\int_\Omega a_1 |\nabla u_{1\lambda}|^2 dx + \int_\Omega (q_1 + \lambda \varrho_1) |u_{1\lambda}|^2 dx \leqslant |\langle u_{1\lambda}, \psi \rangle| \tag{2.11}$$

$$\leqslant \|u_{1\lambda}\|_{H^{\frac{1}{2}}(\partial\Omega)} \|\psi\|_{H^{-1/2}(\partial\Omega)}$$

$$\leqslant C \|u_{1\lambda}\|_{H^1(\Omega)} \|\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{2.12}$$

Next, as $q_1 + \lambda \varrho_1 \geqslant (\lambda^* - \lambda_*) \varrho_1$,

$$\int_\Omega a_1 |\nabla u_{1\lambda}|^2 dx + \int_\Omega (q_1 + \lambda \varrho_1) |u_{1\lambda}|^2 dx \geqslant \alpha \|u_{1\lambda}\|^2_{H^1(\Omega)}, \tag{2.13}$$

for all $\lambda \geqslant \lambda^*$, where we have set $\alpha := \min(1, \lambda^* - \lambda_*)$. From (2.11)-(2.13) we deduce

$$\|u_{1\lambda}\|_{H^1(\Omega)} \leqslant \frac{1}{\alpha} \|\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{2.14}$$

The embedding $H^1(\Omega) \subset L^2(\Omega)$ being compact, (2.12) yields that there exists a subsequence $\lambda^{(n)} \to +\infty$ as $n \to +\infty$ and $u \in H^1(\Omega)$, such that $u_{1\lambda^{(n)}} \rightharpoonup u$ in $H^1(\Omega)$. By Lemma 2.1 we have $\|u_{1\lambda}\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$, therefore

$$u_{1\lambda^{(n)}} \rightharpoonup 0 \quad \text{in } H^1(\Omega) \quad \text{as } n \to +\infty.$$

Finally since the trace operator $u \mapsto u_{|\partial\Omega}$ is continuous from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$, we obtain that $u_{1\lambda^{(n)}|\partial\Omega} \rightharpoonup 0$ in $H^{\frac{1}{2}}(\partial\Omega)$ as $n \to +\infty$. The claim (2.10) is proved, and the proof of Theorem 1.1 is complete. ■

3. – Proof of Theorem 1.2.

In what follows we shall use the letter c to denote various constants independent of the data and the letter C to denote constants depending on the data.

LEMMA 3.1. – *For* $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, $\varphi \not\equiv 0$, *let* $u_\lambda \in H^1(\Omega)$ *solve* (1.16) *for* $\lambda \geqslant \lambda^*$. *Then*

$$\|u_\lambda\|_{L^2_\varrho(\Omega)} \to 0 \quad \text{as } \lambda \to +\infty, \tag{3.1}$$

and

$$\overline{\lim_{\lambda \to +\infty}} \|\nabla u_\lambda\|_{L^2(\Omega)} = +\infty. \tag{3.2}$$

PROOF OF LEMMA 3.1. – The proof of (3.1) is identical to that of (2.1), therefore we can skip it.

In order to prove (3.2), suppose, by contradiction, that there exists a constant $C > 0$ such that $\|\nabla u_\lambda\|_{L^2(\Omega)} \leqslant C$ for all $\lambda \geqslant \lambda^*$. Then $\|u_\lambda\|_{H^1(\Omega)}$ is bounded, and, since $\|u_\lambda\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$, and using again as above the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$, there exists a subsequence $u_{\lambda^{(n)}} \rightharpoonup 0$ in $H^1(\Omega)$ as $\lambda^{(n)} \to +\infty$. The trace operator $u_\lambda \mapsto u_{\lambda|\partial\Omega}$ being continuous from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$, we derive that $\varphi \equiv 0$, which leads to a contradiction. ■

In the following lemma we prove that the energy $\|\nabla u_\lambda\|_{L^2(\Omega\setminus\Omega^\varepsilon)}$ of the solution u_λ of (1.16), evaluated in the domain $\Omega\setminus\Omega^\varepsilon$, Ω^ε as in (1.18), goes to zero as $\lambda \to +\infty$.

LEMMA 3.2. – *Let* $0 < 2\varepsilon < \operatorname{diam}(\Omega)$ *be fixed. Then*

$$\|\nabla u_\lambda\|_{L^2(\Omega\setminus\Omega^\varepsilon)} \to 0 \quad \text{as } \lambda \to +\infty. \tag{3.3}$$

PROOF OF LEMMA 3.2. – Let us denote by $\eta \in C^\infty(\overline{\Omega})$ a cut-off function such that $0 \leqslant \eta \leqslant 1$, and

$$\begin{cases} \eta = 0 & \text{in } \Omega^{\varepsilon/2}, \\ \eta = 1 & \text{in } \Omega\setminus\Omega^\varepsilon, \end{cases} \quad \text{and} \quad |\nabla\eta| \leqslant \frac{4}{\varepsilon}.$$

Multiplying the equation in (1.16) by $\eta^2 u_\lambda$, and integrating by parts over Ω we obtain

$$\begin{aligned} 0 &= -\int_\Omega \operatorname{div}(a\nabla u_\lambda)\, \eta^2 u_\lambda \, dx + \int_\Omega (q + \lambda\varrho)\, \eta^2 u_\lambda^2 \, dx \\ &= \int_\Omega a\eta^2 |\nabla u_\lambda|^2 dx + 2\int_\Omega a\eta u_\lambda \nabla\eta \nabla u_\lambda \, dx + \int_\Omega (q + \lambda\varrho)\, \eta^2 u_\lambda^2 \, dx. \end{aligned}$$

Hence the Cauchy-Schwarz inequality yields

$$\int_\Omega a\eta^2\,|\nabla u_\lambda|^2\,dx + \int_\Omega (q+\lambda\varrho)\,\eta^2 u_\lambda^2\,dx \leqslant 2\int_\Omega |a\eta u_\lambda\,\nabla\eta\nabla u_\lambda|\,dx \leqslant$$
$$2\|a\|_{L^\infty(\Omega)}\|\eta\nabla u_\lambda\|_{L^2(\Omega)}\|u_\lambda\,\nabla\eta\|_{L^2(\Omega)}.$$

That is, since $q+\lambda\varrho \geqslant (\lambda^* - \lambda_*)\,\varrho > 0$,

$$\|\eta\nabla u_\lambda\|_{L^2(\Omega)} \leqslant C\|u_\lambda\|_{L^2(\Omega)}$$

(since $\lambda \geqslant \lambda^*$), where $C := 8\varepsilon^{-1}\|a\|_{L^\infty(\Omega)}/\inf_\Omega a$. So we have the upper bound

$$\|\nabla u_\lambda\|_{L^2(\Omega\setminus\Omega^\varepsilon)} \leqslant C\|u_\lambda\|_{L^2(\Omega)}.$$

But, by the above Lemma 3.1, we have that $\|u_\lambda\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$. The proof of Lemma 3.2 is complete. ■

LEMMA 3.3. – *Under the assumptions of Lemma* 3.1, *let σ_* be a point of $\partial\Omega$, and D a neighborhood of σ_* relative to $\overline{\Omega}$. If $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ is such that* $\mathrm{supp}(\varphi) \subset\subset D \cap \partial\Omega$, *then*

$$\|\nabla u_\lambda\|_{L^2(\Omega\setminus D)} \to 0 \quad \text{as } \lambda \to +\infty. \tag{3.4}$$

PROOF OF LEMMA 3.3. – Let Γ be a neighborhood of σ_* relative to $\partial\Omega$, such that $\Gamma \subset\subset D \cap \partial\Omega$, and let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ be such that $\mathrm{supp}(\varphi) \subset \Gamma$. We denote by $\eta \in C^\infty(\overline{\Omega})$ a cut-off function such that $0 \leqslant \eta \leqslant 1$, with $\eta = 0$ in a neighborhood of Γ, while $\eta = 1$ in a neighborhood of $\overline{\Omega\setminus D}$, and $|\nabla\eta| \leqslant 4d^{-1}$, where $d :=$ $\mathrm{dist}(\overline{\Omega}\setminus D, \Gamma)$. Multiplying the equation in (1.16) by $\eta^2 u_\lambda$, and integrating by parts over Ω we obtain (since $\mathrm{supp}(\varphi) \subset \Gamma$, and $\eta \equiv 0$ in Γ)

$$\begin{aligned} 0 &= -\int_\Omega \mathrm{div}(a\nabla u_\lambda)\,\eta^2 w_\lambda\,dx + \int_\Omega (q+\lambda\varrho)\,\eta^2 u_\lambda^2\,dx \\ &= \int_\Omega a\eta^2\,|\nabla u_\lambda|^2\,dx + 2\int_\Omega a\eta u_\lambda\,\nabla\eta\,\nabla u_\lambda\,dx + \int_\Omega (q+\lambda\varrho)\,\eta^2 u_\lambda^2\,dx. \end{aligned}$$

Hence the Cauchy-Schwarz inequality yields

$$\int_\Omega a\eta^2\,|\nabla u_\lambda|^2\,dx + \int_\Omega (q+\lambda\varrho)\,\eta^2 u_\lambda^2\,dx \leqslant 2\int_\Omega |a\eta u_\lambda\,\nabla\eta\nabla u_\lambda|\,dx \leqslant$$
$$2\|a\|_{L^\infty(\Omega)}\|\eta\nabla u_\lambda\|_{L^2(\Omega)}\|u_\lambda\,\nabla\eta\|_{L^2(\Omega)},$$

that is

$$\|\eta\nabla u_\lambda\|_{L^2(\Omega)} \leqslant C\|u_\lambda\,\nabla\eta\|_{L^2(\Omega)},$$

where $C := 8d^{-1}\|a\|_{L^\infty(\Omega)}/\inf_\Omega a$. In particular we have

$$\|\nabla u_\lambda\|_{L^2(\Omega\setminus D)} \leqslant C\|u_\lambda\|_{L^2(\Omega)}.$$

Since $\|u_\lambda\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$, Lemma 3.3 is proved. ■

LEMMA 3.4. – *Under the assumptions of Lemma* 3.1, *the following identities hold*

$$\int_\Omega a\varphi_1\,|\nabla u_\lambda|^2\,dx + \frac{1}{2}\int_\Omega (q + (2\lambda + \lambda_1)\,\varrho)\,\varphi_1\,u_\lambda^2\,dx = \frac{-1}{2}\int_{\partial\Omega} a\varphi^2\,\frac{\partial}{\partial \boldsymbol{n}}\varphi_1\,d\sigma, \tag{3.5}$$

$$\int_\Omega a\varphi_1^2\,|\nabla u_\lambda|^2\,dx + (\lambda + \lambda_1)\int_\Omega \varrho\varphi_1^2\,u_\lambda^2\,dx = \int_\Omega a u_\lambda^2\,|\nabla\varphi_1|^2\,dx, \tag{3.6}$$

where λ_1 and φ_1 are respectively the first eigenvalue and eigenfunction of the corresponding eigenvalue problem (1.14).

PROOF OF LEMMA 3.4. – We begin by proving identity (3.5). Multiplying the equation in (1.16) by $\varphi_1 u_\lambda$, and integrating by parts over Ω we obtain

$$\begin{aligned} 0 &= -\int_\Omega \operatorname{div}(a\nabla u_\lambda)\,\varphi_1\,u_\lambda\,dx + \int_\Omega (q + \lambda\varrho)\,\varphi_1\,u_\lambda^2\,dx \\ &= \int_\Omega a\varphi_1\,|\nabla u_\lambda|^2\,dx + \frac{1}{2}\int_\Omega a\nabla\varphi_1\,\nabla(u_\lambda^2)\,dx + \int_\Omega (q + \lambda\varrho)\,\varphi_1\,u_\lambda^2\,dx. \end{aligned} \tag{3.7}$$

Now integrating by parts the second term of the right hand side of the last equality, and using the fact that we have $-\operatorname{div}(a\nabla\varphi_1) = (\lambda_1\varrho - q)\varphi_1$

$$\frac{1}{2}\int_\Omega a\nabla\varphi_1\,\nabla(u_\lambda^2)\,dx = -\frac{1}{2}\int_\Omega (q - \lambda_1\varrho)\,\varphi_1\,u_\lambda^2\,dx + \frac{1}{2}\int_{\partial\Omega} a\varphi^2\,\frac{\partial}{\partial \boldsymbol{n}}\varphi_1\,d\sigma.$$

Hence inserting this in (3.7) we obtain (3.5).

Similarly multiplying the equation in (1.16) by $\varphi_1^2 u_\lambda$, and integrating by parts over Ω, we obtain

$$\begin{aligned} 0 &= -\int_\Omega \operatorname{div}(a\nabla u_\lambda)\,\varphi_1^2\,u_\lambda\,dx + \int_\Omega (q + \lambda\varrho)\,\varphi_1^2\,u_\lambda^2\,dx \\ &= \int_\Omega a\varphi_1^2\,|\nabla u_\lambda|^2\,dx + \int_\Omega a\varphi_1\,\nabla\varphi_1\,\nabla(u_\lambda^2)\,dx + \int_\Omega (q + \lambda\varrho)\,\varphi_1^2\,u_\lambda^2\,dx. \end{aligned} \tag{3.8}$$

Now integrating by parts the second term of the right hand side of the above equality we have

$$\int_\Omega a\varphi_1 \nabla\varphi_1 \nabla u_\lambda^2\,dx = -\int_\Omega \operatorname{div}(a\nabla\varphi_1)\,\varphi_1 u_\lambda^2\,dx - \int_\Omega a u_\lambda^2\,|\nabla\varphi_1|^2\,dx.$$

Again using the equation satisfied by φ_1, we obtain (3.6). ■

The following lemma is the analogous of Lemma 2.2 for the case of Dirichlet boundary conditions.

LEMMA 3.5. – *For $j=1,2$, $\lambda \geqslant \lambda^*$, let $u_{j\lambda} \in H^1(\Omega)$ solve* (1.16) *when $a := a_j$, $q := q_j$, $\varrho := \varrho_j$. Let us denote by $\widetilde{\gamma_{j\lambda}}(\varphi) := a_j \dfrac{\partial}{\partial \boldsymbol{n}} u_{j\lambda|\partial\Omega}$. Assume that* (1.19) *holds, that is*

$$\operatorname{Nbsd}(a_1, q_1, \varrho_1) = \operatorname{Nbsd}(a_2, q_2, \varrho_2).$$

Then

$$\widetilde{\gamma_{1\lambda^*}} - \widetilde{\gamma_{2\lambda^*}} = \widetilde{\gamma_{1\lambda}} - \widetilde{\gamma_{2\lambda}} \tag{3.9}$$

for all $\lambda \geqslant \lambda^*$.

PROOF OF LEMMA 3.5. – Let $z_{j\lambda} := u_{j\lambda} - u_{j\lambda^*}$. Then $z_{j\lambda} \in H^1(\Omega)$ solves

$$\begin{cases} -\operatorname{div}(a_j(x)\,\nabla z_{j\lambda}) + (q_j(x) + \lambda\varrho_j(x))\,z_{j\lambda} = -(\lambda-\lambda^*)\,\varrho_j(x)\,u_{j\lambda^*} & \text{in } \Omega, \\ z_{j\lambda} = 0 & \text{on } \partial\Omega. \end{cases}$$

As in the proof of Lemma 2.2, we can write $z_{j\lambda}$ in the following Fourier expansion

$$z_{j\lambda} = \sum_{k=1}^{\infty} (z_{j\lambda}\,|\,\varphi_{jk})\,\varphi_{jk} \quad \text{in } L^2_{\varrho_j}(\Omega), \tag{3.10}$$

(recall that $(\varphi_{jk})_{k=1}^\infty$ are the eigenfunctions of (1.1) when $a := a_j$, $q := q_j$, $\varrho := \varrho_j$) where

$$(z_{j\lambda}\,|\,\varphi_{jk}) = \int_\Omega z_{j\lambda}\,\varphi_{jk}\,\varrho_j\,dx = \frac{-(\lambda-\lambda^*)\,\alpha_k}{(\lambda_k+\lambda)^2},$$

and $\lambda_k := \lambda_{1k} = \lambda_{2k}$, $\alpha_k := -\langle \psi_k, \varphi\rangle$, and $\psi_k := a_1 \dfrac{\partial}{\partial \boldsymbol{n}}\varphi_{1k} = a_2\dfrac{\partial}{\partial \boldsymbol{n}}\varphi_{2k}$ on $\partial\Omega$.

Let

$$X := \{u \in H_0^1(\Omega);\ \operatorname{div}(a(x)\,\nabla u) \in L^2(\Omega)\}.$$

It is well-known (see for instance J.-L. Lions [10]) that the trace operator $u \mapsto a\dfrac{\partial}{\partial \boldsymbol{n}} u_{|\partial\Omega}$ is continuous from X, endowed with the norm $\|u\|_X := \|u\|_{H^1(\Omega)} +$

$\|\mathrm{div}\,(a\nabla u)\|_{L^2(\Omega)}$, into $H^{-1/2}(\partial\Omega)$. One can verify that the series in (3.10) converges to $z_{j\lambda}$ in the norm $\|\cdot\|_X$ (see B. Canuto & O. Kavian [5], Lemma 3.2). Then we have

$$a_j \frac{\partial}{\partial \boldsymbol{n}} z_{j\lambda} = -\sum_{k=1}^{\infty} \frac{(\lambda - \lambda^*)\,\alpha_k}{(\lambda_k + \lambda)^2}\,\psi_k \quad \text{in } H^{-1/2}(\partial\Omega),$$

So we obtain

$$a_1 \frac{\partial}{\partial \boldsymbol{n}} z_{1\lambda} = a_2 \frac{\partial}{\partial \boldsymbol{n}} z_{2\lambda} \quad \text{in } H^{-1/2}(\partial\Omega),$$

that is

$$\widetilde{\gamma_{1\lambda^*}}(\varphi) - \widetilde{\gamma_{2\lambda^*}}(\varphi) = \widetilde{\gamma_{1\lambda}}(\varphi) - \widetilde{\gamma_{2\lambda}}(\varphi) \quad \text{in } H^{-1/2}(\partial\Omega),$$

for all $\lambda \geqslant \lambda^*$, for all $\varphi \in H^{\frac{1}{2}}(\Omega)$. ■

LEMMA 3.6. – *Under the hypothesis of Lemma* 3.5 *we have*

$$\|u_{1\lambda}\|_{L^2_{\varrho_1}(\Omega)} = \|u_{2\lambda}\|_{L^2_{\varrho_2}(\Omega)}, \tag{3.11}$$

and

$$\int_\Omega (a_2(|\nabla u_{1\lambda}|^2 - |\nabla u_{2\lambda}|^2) + q_2(|u_{1\lambda}|^2 - |u_{2\lambda}|^2)\,dx) \geqslant \lambda \int_\Omega (\varrho_1 - \varrho_2)\,|u_{1\lambda}|^2\,dx\,. \tag{3.12}$$

PROOF OF LEMMA 3.6. – We begin by proving (3.11). We can write $u_{j\lambda}$ in the following Fourier expansion

$$u_{j\lambda} := \sum_{k=1}^{\infty} (u_{j\lambda}\,|\,\varphi_{jk})\,\varphi_{jk} \quad \text{in } L^2_{\varrho_j}(\Omega),$$

where

$$(u_{j\lambda}\,|\,\varphi_{jk}) = \int_\Omega u_{j\lambda}\,\varphi_{jk}\,\varrho_j\,dx = \frac{\alpha_k}{\lambda_k + \lambda}$$

(we recall that $\lambda_k := \lambda_{1k} = \lambda_{2k}$, and $\alpha_k := -\langle \psi_k, \varphi\rangle$, where $\psi_k := a_1 \frac{\partial}{\partial \boldsymbol{n}} \varphi_{1k} = a_2 \frac{\partial}{\partial \boldsymbol{n}} \varphi_{2k}$ on $\partial\Omega$). Since $\|u_{j\lambda}\|^2_{L^2_{\varrho_j}(\Omega)} = \sum_{k=1}^{\infty} \left(\frac{\alpha_k}{\lambda_k + \lambda}\right)^2$, we obtain (3.11).

In order to prove (3.12) we observe that by (3.11) we have

$$\frac{1}{2}\int_\Omega (a_2|\nabla u_{1\lambda}|^2+q_2|u_{1\lambda}|^2)\,dx+\frac{\lambda}{2}\left(\int_\Omega(\varrho_2-\varrho_1)|u_{1\lambda}|^2dx+\int_\Omega\varrho_2|u_{2\lambda}|^2dx\right)$$

$$=\frac{1}{2}\int_\Omega (a_2|\nabla u_{1\lambda}|^2+q_2|u_{1\lambda}|^2)\,dx+\frac{\lambda}{2}\int_\Omega\varrho_2|u_{1\lambda}|^2dx\geqslant$$

$$\geqslant\frac{1}{2}\min_{v\in H^1_\varphi(\Omega)}\int_\Omega(a_2|\nabla v|^2+q_2|v|^2)\,dx+\lambda\int_\Omega\varrho_2|v|^2dx=$$

$$=\frac{1}{2}\int_\Omega (a_2|\nabla u_{2\lambda}|^2+q_2|u_{2\lambda}|^2)\,dx+\frac{\lambda}{2}\int_\Omega\varrho_2|u_{2\lambda}|^2dx.$$

Hence (3.12) follows. (Here we use the notation

$$H^1_\varphi(\Omega):=\{v\in H^1(\Omega)\,;\,v=\varphi\ \text{ on }\ \partial\Omega\},$$

and the fact that $u_{2\lambda}$ is the minimizer of the functional

$$v\mapsto\int_\Omega a_2\,|\nabla v|^2\,dx+\int_\Omega(q+\lambda\varrho)\,v^2\,dx$$

on $H^1_\varphi(\Omega)$). ■

We are now in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. – For $j=1,2$, $\lambda\geqslant\lambda^*$, λ^* as in (1.10), $\varphi\in H^{\frac{1}{2}}(\partial\Omega)$, $\varphi\not\equiv 0$, we recall that $u_{j\lambda}\in H^1(\Omega)$ solves

$$\text{(3.13)}\qquad\begin{cases}-\operatorname{div}(a_j(x)\,\nabla u_{j\lambda})+(q_j(x)+\lambda\varrho_j(x))\,u_{j\lambda}=0 & \text{in }\ \Omega,\\ u_{j\lambda}=\varphi & \text{on }\ \partial\Omega.\end{cases}$$

Multiplying the equation in (3.13) for $j=1$ by $u_{2\lambda}$ and integrating by parts over Ω we obtain

$$\text{(3.14)}\qquad -\langle\widetilde{\gamma_{1\lambda}}(\varphi),\,\varphi\rangle+\int_\Omega a_1\,\nabla u_{1\lambda}\,\nabla u_{2\lambda}\,dx+\int_\Omega(q_1+\lambda\varrho_1)\,u_{1\lambda}\,u_{2\lambda}\,dx=0.$$

Similarly, multiplying the equation in (3.13) for $j=2$ by $u_{1\lambda}$ and integrating by parts over Ω we obtain

$$\text{(3.15)}\qquad -\langle\widetilde{\gamma_{2\lambda}}(\varphi),\,\varphi\rangle+\int_\Omega a_2\,\nabla u_{1\lambda}\,\nabla u_{2\lambda}\,dx+\int_\Omega(q_2+\lambda\varrho_2)\,u_{1\lambda}\,u_{2\lambda}\,dx=0.$$

Subtracting (3.14) from (3.15), and using (3.9), we have

(3.16) $\langle(\widetilde{\gamma_{2\lambda^*}} - \widetilde{\gamma_{1\lambda^*}})(\varphi), \varphi\rangle =$

$$\int_\Omega (a_1 - a_2)\, \nabla u_{1\lambda} \nabla u_{2\lambda}\, dx + \int_\Omega (q_1 - q_2)\, u_{1\lambda} u_{2\lambda}\, dx + \lambda \int_\Omega (\varrho_1 - \varrho_2)\, u_{1\lambda} u_{2\lambda}\, dx.$$

We claim that

$$\int_\Omega |a_1 - a_2|\, |\nabla u_{j\lambda}|^2\, dx \to 0 \quad \text{as } \lambda \to +\infty, \tag{3.17}$$

and

$$\lambda \int_\Omega |\varrho_1 - \varrho_2|\, |u_{j\lambda}|^2\, dx \to 0 \quad \text{as } \lambda \to +\infty. \tag{3.18}$$

For a moment assume that (3.17) and (3.18) hold true, therefore, since by (3.1) we know that $\|u_{j\lambda}\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$, (3.16) and (3.9) imply that

$$\widetilde{\gamma_{1\lambda}}(\varphi) = \widetilde{\gamma_{2\lambda}}(\varphi)$$

for all $\lambda \geq \lambda^*$, for all $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, and Theorem 1.2 is proved.

So it remains to prove (3.17) and (3.18). It is a classical consequence of the maximum principle that there exist constants c_1, $c_2 > 0$ such that

$$c_1 \delta \leq \varphi_{1j} \leq c_2 \delta \quad \text{in } \Omega, \tag{3.19}$$

that is φ_{1j} and the distance function δ are of the same order near the boundary. So by hypothesis (1.20) we obtain, for some constant independent of λ,

$$\int_\Omega |a_1 - a_2|\, |\nabla u_{j\lambda}|^2\, dx \leq c \int_\Omega \varphi_{1j}^2\, |\nabla u_{j\lambda}|^2\, dx,$$

which by (3.6) yields

$$\int_\Omega |a_1 - a_2|\, |\nabla u_{j\lambda}|^2\, dx \leq C \int_\Omega |u_{j\lambda}|^2\, dx.$$

Now (3.1) implies claim (3.17). In order to prove (3.18) we can proceed in the same manner. In fact by hypothesis (1.21) and (3.19) we obtain

$$\int_\Omega |\varrho_1 - \varrho_2|\, |u_{j\lambda}|^2\, dx \leq c \int_\Omega \varphi_{1j}^2\, |u_{j\lambda}|^2\, dx,$$

which, by (3.6), yields

$$\int_\Omega |\varrho_1 - \varrho_2|\, |\nabla u_{j\lambda}|^2\, dx \leq C \int_\Omega |u_{j\lambda}|^2\, dx.$$

Invoking (3.1) we obtain (3.18). This concludes the proof of Theorem 1.2. ■

4. – Proof of Corollaries 1.3-1.7.

In order to prove Corollaries 1.3-1.7 we need some results concerning Calderón's problem. The so-called Calderón's problem [4], introduced in the 1980's, is the following. For $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, $\varphi \not\equiv 0$, let $u \in H^1(\Omega)$ solve

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

(recall that the coefficient $a \in L^\infty(\Omega)$ and there exists a positive constant α such that $a \geqslant \alpha$ a.e. in Ω). Let us denote

$$K := \{a \in L^\infty(\Omega) \text{ such that } a \geqslant \alpha \text{ a.e. in } \Omega\}$$

and Φ the nonlinear operator

$$\Phi : a \mapsto \tilde{\gamma}_a,$$

where γ_a is the Dirichlet-to-Neumann map related to (4.1). A.P. Calderón posed the following question:

is the operator Φ is injective on K?

Since then many results are obtained. We recall here the most recent ones.

In what follows we shall assume that Ω is a bounded domain of Lipschtiz class in $\mathbb{R}^N$, $N \geqslant 2$.

THEOREM 4.1 (A. I. Nachman [11] for $N \geqslant 3$, V. Isakov [8] for $N = 2$). – *For $j = 1, 2$, let $q_j \in L^p(\Omega)$, $p > \frac{N}{2}$, for $N \geqslant 3$, while for $N = 2$ let $q_j \in L^p(\Omega)$, $p > 1$, and $q_j \geqslant 0$ a.e. in Ω. Assume that for all $\psi \in L^2(\partial\Omega)$, there exists a unique $u_j \in H^1(\Omega)$ solving*

$$\begin{cases} -\Delta u_j + q_j(x)\, u_j = 0 & \textit{in } \Omega, \\ \dfrac{\partial}{\partial \boldsymbol{n}} u_j = \psi & \textit{on } \partial\Omega, \end{cases}$$

and that

$$\gamma_1 = \gamma_2,$$

where $\gamma_j(\psi) := u_{j|\partial\Omega}$. Then $q_1 = q_2$ in Ω.

THEOREM 4.2 (R.M. Brown [2] for $N \geq 3$, R.M. Brown & G. Uhlmann [3] for $N = 2$). – *For $j = 1, 2$, let a_j be two strictly positive functions in $\overline{\Omega}$ such that $a_j \in C^{1, \frac{1}{2}+\alpha}(\overline{\Omega})$, for some $0 < \alpha \leq \frac{1}{2}$, when $N \geq 3$, while, when $N = 2$, let $a_j \in W^{1,p}(\Omega)$, $p > 2$. For $\psi \in L^2(\partial\Omega)$ with $\int_{\partial\Omega} \psi d\sigma = 0$, let $u_j \in H^1(\Omega)$ solve*

$$\begin{cases} -\operatorname{div}(a_j(x)\,\nabla u_j) = 0 & in\ \Omega, \\ a_j \dfrac{\partial}{\partial \boldsymbol{n}} u_j = \psi & on \partial\Omega, \\ \displaystyle\int_\Omega u_j(x)\, dx = 0. \end{cases}$$

Suppose that

$$\gamma_1 = \gamma_2,$$

Then $a_1 = a_2$ in $\overline{\Omega}$.

Observe that Theorems 4.1, 4.2 are proved in their original version in the case of Dirichlet boundary conditions. More precisely, denote by $\tilde{\gamma}_j(\varphi) := \frac{\partial u_j}{\partial \boldsymbol{n}}$ $\left(\text{resp. } \tilde{\gamma}_j(\varphi) := a_j \frac{\partial u_j}{\partial \boldsymbol{n}}\right)$ where $-\Delta u_j + q_j u_j = 0$ (resp. $-\operatorname{div}(a_j \nabla u_j) = 0$) and $u_j = \varphi$ on the boundary $\partial\Omega$. Then under appropriate regularity conditions, such as those of the above theorems, on the coefficients q_j (resp. a_j), if $\tilde{\gamma}_1 = \tilde{\gamma}_2$, then $q_1 = q_2$ (resp. $a_1 = a_2$). However our operators γ_j are essentially the inverses of the operators $\tilde{\gamma}_j$ and also the proofs of the original theorems can be easily adapted to the case of Neumann boundary conditions.

PROOF OF COROLLARY 1.3. – By Theorem 1.1 we have

$$\gamma_{1\lambda} = \gamma_{2\lambda}$$

for all $\lambda \geq 0$ (in this case $\lambda^* = 0$). In particular

$$\gamma_1 = \gamma_2,$$

where we define $\gamma_j := \gamma_{j0}$. Invoking theorem 4.2 we have

$$a_1 = a_2 \quad \text{in } \overline{\Omega}.$$

Now we consider the following so-called Liouville transform of $u_{j\lambda}$, (see for example A. I. Nachman [11]), that is we denote by

$$v_{j\lambda} := a_j^{1/2} u_{j\lambda} \quad \text{in } \Omega.$$

It is easy to verify that $v_{j\lambda} \in H^1(\Omega)$ solve

$$\begin{cases} -\Delta v_{j\lambda} + a^{-1/2}(x)(\Delta\sqrt{a(x)} + \lambda\varrho_j(x))\, v_{j\lambda} = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial \boldsymbol{n}} v_{j\lambda} = \widetilde{\psi} & \text{on } \partial\Omega, \end{cases}$$

and that

$$v_{1\lambda} = v_{2\lambda} \quad \text{in } H^{1/2}(\partial\Omega),$$

for all $\lambda \geqslant 0$, for all $\widetilde{\psi} \in L^2(\Omega)$, where $a := a_1 = a_2$ in $\overline{\Omega}$. Therefore using Theorem 4.1 (when $N = 2$, we may choose λ such that $a^{-1/2}(\Delta\sqrt{a} + \lambda\varrho_j) \geqslant 0$ a.e. in Ω) we have

$$\varrho_1 = \varrho_2 \quad \text{in } \Omega\,.$$

The proof of Corollary 1.3 is complete. ■

The proof of Corollary 1.4 is very similar to that of Corollary 1.3, and is left to the reader.

Before proceeding further in the proofs of Corollaries 1.5-1.7 we need a lemma in which we prove that if we assume that the Nbsd(a_j, q_j, ϱ_j) are the same and $|\varrho_1 - \varrho_2| \leqslant c\delta$ in Ω, then the coefficients a_1 and a_2 coincide on $\partial\Omega$. More precisely we prove the following

LEMMA 4.3. – *Under the assumptions of Theorem* 1.2, *assume that* $a_j \in L^\infty(\Omega) \cap C(\Omega^\varepsilon)$, Ω^ε *as in* (1.18), *for some* $\varepsilon > 0$. *If*

$$\text{Nbsd}\,(a_1, q_1, \varrho_1) = \text{Nbsd}\,(a_1, q_1, \varrho_1),$$

and

$$|\varrho_1 - \varrho_2| \leqslant c\delta \quad \textit{in } \Omega, \tag{4.2}$$

then

$$a_1 = a_2 \quad \textit{on } \partial\Omega.$$

PROOF OF LEMMA 4.3. – By contradiction let σ_* be a point of $\partial\Omega$ such that $a_1(\sigma_*) \neq a_2(\sigma_*)$. Without loss of generality we can suppose that $a_1(\sigma_*) > a_2(\sigma_*)$, and that there exists a neighborhood D of σ_* relative to $\overline{\Omega}$ and a positive constant $c > 0$ such that

$$a_1 \geqslant a_2 + c \quad \text{in } D\,. \tag{4.3}$$

Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ be as in Lemma 3.3. Computing $\langle \gamma_{j\lambda}(\varphi), \varphi \rangle$, after an integration by parts over Ω, one obtains the identity

$$\langle \gamma_{j\lambda}(\varphi), \varphi \rangle = \int_{\Omega} a_j |\nabla u_{j\lambda}|^2 dx + \int_{\Omega} (q_j + \lambda \varrho_j) |u_{j\lambda}|^2 dx.$$

Therefore using (3.9) and (3.11) we have, for all $\lambda \geqslant \lambda^*$,

$$\int_{\Omega} a_1 |\nabla u_{1\lambda^*}|^2 dx - \int_{\Omega} a_2 |\nabla u_{2\lambda^*}|^2 dx + \int_{\Omega} q_1 |u_{1\lambda^*}|^2 dx - \int_{\Omega} q_2 |u_{2\lambda^*}|^2 dx =$$

$$\int_{\Omega} a_1 |\nabla u_{1\lambda}|^2 dx - \int_{\Omega} a_2 |\nabla u_{2\lambda}|^2 dx + \int_{\Omega} q_1 |u_{1\lambda}|^2 dx - \int_{\Omega} q_2 |u_{2\lambda}|^2 dx =$$

$$\int_{\Omega} (a_1 - a_2) |\nabla u_{1\lambda}|^2 dx + \int_{\Omega} a_2 (|\nabla u_{1\lambda}|^2 - |\nabla u_{2\lambda}|^2)\, dx +$$

$$\int_{\Omega} q_2 (|u_{1\lambda}|^2 - |u_{2\lambda}|^2)\, dx + \int_{\Omega} (q_1 - q_2) |u_{1\lambda}|^2 dx.$$

Recalling that by (3.12), (4.2), and (3.5) we have

$$\int_{\Omega} a_2 (|\nabla u_{1\lambda}|^2 - |\nabla u_{2\lambda}|^2)\, dx + \int_{\Omega} q_2 (|u_{1\lambda}|^2 - |u_{2\lambda}|^2)\, dx \geqslant \lambda \int_{\Omega} (\varrho_1 - \varrho_2) |u_{1\lambda}|^2 dx,$$

$$\geqslant -C,$$

we obtain

$$\int_{\Omega} a_1 |\nabla u_{1\lambda^*}|^2 dx - \int_{\Omega} a_2 |\nabla u_{2\lambda^*}|^2 dx + \int_{\Omega} q_1 |u_{1\lambda^*}|^2 dx - \int_{\Omega} q_2 |u_{2\lambda^*}|^2 dx \geqslant \tag{4.4}$$

$$\int_{\Omega} (a_1 - a_2) |\nabla u_{1\lambda}|^2 dx + \int_{\Omega} (q_1 - q_2) |u_{1\lambda}|^2 dx - C =$$

$$\int_{\Omega \setminus D} (a_1 - a_2) |\nabla u_{1\lambda}|^2 dx + \int_{D} (a_1 - a_2) |\nabla u_{1\lambda}|^2 dx - C + o(1),$$

where in the last step we use the fact that $\|u_{1\lambda}\|_{L^2(\Omega)} \to 0$ as $\lambda \to +\infty$. Concerning the first integral on the right hand side of (4.4), we observe that by (3.4), for $a := a_1$, $q := q_1$, $\varrho := \varrho_1$, we have

$$\|\nabla u_{1\lambda}\|_{L^2(\Omega \setminus D)} \to 0 \quad \text{as } \lambda \to +\infty.$$

Since by (3.2) we know that $\overline{\lim}_{\lambda \to +\infty} \|\nabla u_{1\lambda}\|_{L^2(\Omega)} = +\infty$, we derive

$$\overline{\lim}_{\lambda \to +\infty} \|\nabla u_{1\lambda}\|_{L^2(D)} = +\infty,$$

and since by (4.3) we have

$$\int_D (a_1 - a_2)\, |\nabla u_{1\lambda}|^2\, dx \geqslant c \int_D |\nabla u_{1\lambda}|^2\, dx,$$

passing to the lim sup in (4.4), we get a contradiction. Therefore $a_1 = a_2$ on $\partial\Omega$. ∎

Now we may prove Corollaries 1.6.

PROOF OF COROLLARY 1.6. – By Lemma 4.3 we have

$$a_1 = a_2 \quad \text{on } \partial\Omega\,.$$

Since by assumption we have $\nabla a_1 = \nabla a_2$ on $\partial\Omega$, we can suppose that there exists a positive constant $c > 0$ such that

$$|a_1 - a_2| \leqslant c\delta^2 \quad \text{in } \Omega\,.$$

Therefore by Theorem 1.2 it follows that

$$\widetilde{\gamma_{1\lambda}} = \widetilde{\gamma_{2\lambda}}$$

for all $\lambda \geqslant \lambda^*$. As before we consider the Liouville transform of $u_{j\lambda}$, that is we denote by

$$v_{j\lambda} := a_j^{1/2} u_{j\lambda} \quad \text{in } \Omega\,.$$

Hence $v_{j\lambda} \in H^1(\Omega)$ solves

$$\begin{cases} -\Delta v_{j\lambda} + a_j^{-1/2}(x)(\Delta\sqrt{a_j(x)} + q(x) + \lambda\varrho_j(x))\, v_{j\lambda} = 0 & \text{in } \Omega\,, \\ v_{j\lambda} = \tilde{\varphi} & \text{on } \partial\Omega\,, \end{cases}$$

and

$$\frac{\partial}{\partial \boldsymbol{n}} v_{1\lambda} = \frac{\partial}{\partial \boldsymbol{n}} v_{2\lambda} \quad \text{in } H^{-1/2}(\partial\Omega),$$

for all $\lambda \geqslant \lambda^*$, for all $\tilde{\varphi} \in H^{\frac{1}{2}}(\Omega)$. Invoking Theorem 4.1 (with Dirichlet boundary conditions) — by choosing, when $N = 2$, $\lambda \geqslant \lambda^*$ large enough such that $a_j^{-1/2}(\Delta\sqrt{a_j} + \lambda\varrho_j) \geqslant 0$ a.e. in Ω — we have

$$a_1^{-1/2}(\Delta\sqrt{a_1} + q + \lambda\varrho_1) = a_2^{-1/2}(\Delta\sqrt{a_2} + q + \lambda\varrho_2) \quad \text{in } \Omega\,.$$

So, deriving in λ, we have

$$a_1^{-1/2}\varrho_1 = a_2^{-1/2}\varrho_2 \quad \text{in } \Omega\,,$$

hence

$$a_1^{-1/2}(\Delta\sqrt{a_1}+q) = a_2^{-1/2}(\Delta\sqrt{a_2}+q) := p \quad \text{in } \Omega .$$

Now we observe that a_j solve the following Cauchy problem

$$\begin{cases} -\Delta\sqrt{a_j} + p(x)\sqrt{a_j} = q(x) & \text{in } \Omega , \\ \sqrt{a_j} = \varphi & \text{on } \partial\Omega , \\ \dfrac{\partial}{\partial \boldsymbol{n}}\sqrt{a_j} = \psi & \text{on } \partial\Omega , \end{cases}$$

where we have set $\varphi := \sqrt{a_1} = \sqrt{a_2}$ and $\psi := \frac{\partial}{\partial \boldsymbol{n}}\sqrt{a_1} = \frac{\partial}{\partial \boldsymbol{n}}\sqrt{a_2}$. Therefore by well-known results on unique continuation for elliptic operators (see for instance N. Garofalo & F. H. Lin [6]), it follows that

$$a_1 = a_2 \quad \text{in } \overline{\Omega}$$

The proof of Corollary 1.6 is complete. ■

Corollaries 1.7, 1.8 can be proved in much a similar way, and leave the details to the reader.

REMARK. – The technical assumptions $\varrho_1 = \varrho_2$ and $\nabla a_1 - \nabla a_2 = \nabla\varrho_1 - \nabla\varrho_2 = 0$ on the boundary $\partial\Omega$, are not much restrictive in those applications where the density ϱ and the conductivity a are to be determined, via boundary measurements, in the interior of the domain Ω, far from the boundary. It is not too unreasonable, in these applications, to assume that those coeffecients are known up to order one. Nevertheless in a mathematical point of view we must admit that these technical assumptions are not satisfactory and might be removed. Unfortunately at this point we are not able to remove these assumptions. ■

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